TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 4, pp. 1667-1675, August 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

# AN EXTENSION OF A GENERALIZED EQUILIBRIUM PROBLEM

Sangho Kum[1] and Mu-Ming Wong[2,*]

**Abstract.** In this note, we consider a multi-valued version of a generalized system (GS) called the multi-valued generalized system (MGS). Using the Fan-Browder fixed point theorem and Brouwer's fixed point theorem as basic tools, we provide existence theorems on (MGS) with and without monotonicity, respectively.

## 1. Introduction

The equilibrium problem (EP) has been intensively studied, beginning with Blume and Oettli [1] where they proposed it as a generalization of optimization and variational inequality problem. It turns out that this problem includes, as special cases, other problems such as the fixed point and coincidence point problem, the complementarity problem, the Nash equilibrium problem, etc. Because of the general form of this problem, it was investigated under other terminologies, e.g., see [1]. We observe that existence, extensions and applications of equilibrium problems have been extensively investigated in the literature. See, e.g., [2-22] and the references therein.

Recently, Kazmi and Khan [23] introduced a kind of EP called *generalized system* (GS) which extends the strong vector variational inequality (SVVI) studied in Fang and Huang [24] in real Banach spaces. From Brouwer's fixed point theorem, they first derived an existence theorem of solutions of (GS) without monotonicity. Then they presented a corresponding result with monotonicity using Fan's KKM Lemma [25] in real Banach spaces. However, they dealt with only the single-valued case of the bi-operator $F$.

Received January 5, 2010, accepted April 6, 2010.
Communicated by Jen-Chih Yao.
2000 *Mathematics Subject Classification*: Primary 49J40, 47H10, 54H25.
*Key words and phrases*: Equilibrium problem, Vector variational inequality, Generalized system, Pseudomonotonicity.
[1]This work was supported by the research grant of the Chungbuk National University in 2009.
[2]This research was partially supported by a grant from NSC 97-2115-M-033-006.
*Corresponding author.

In this note, we consider a multi-valued version of (GS) due to Kazmi and Khan [23] which is called the multi-valued generalized system (in short, MGS). From both theoretical and practical points of view, it is natural and useful to extend a single-valued case to a corresponding multi-valued one. Using the Fan-Browder fixed point theorem [26] and Brouwer's fixed point theorem as basic tools, we provide existence theorems on (MGS) with and without monotonicity in general Hausdorff topological vector spaces.

## 2. Preliminaries

Let us take a brief look at the standard definition of continuous multi-valued functions. Let $X, Y$ be non-empty topological spaces and $T : X \to 2^Y$ be a multifunction. A multifunction $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$. A multifunction $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. $T$ is said to be *continuous* if $T$ is both lower semicontinuous and upper semicontinuous. It is also known that $T : X \to 2^Y$ is lower semicontinuous if and only if for each closed set $V$ in $Y$, the set $\{x \in X \mid T(x) \subset V\}$ is closed in $X$.

Let $X$, $Y$ be Hausdorff topological vector spaces and let $K$ be a nonempty convex subset of $X$. Let $C$ be a pointed closed convex cone in $Y$ with $\operatorname{int} C \neq \emptyset$. Let $F : K \times K \rightrightarrows Y$ be a nonempty multivalued bi-operator such that $F(x,x) \supseteq \{0\}$ for each $x \in K$. Then *a multivalued generalized sytem* (MGS) is defined to be the problem of finding $x \in K$ such that

$$(MGS) \qquad F(x,y) \nsubseteq -C \setminus \{0\} \ \text{ for all } y \in K.$$

$F$ is said to be *$C$-strongly pseudomonotone* if it satisfies

$$\forall x, y \in K, \quad F(x,y) \nsubseteq -C \setminus \{0\} \ \Rightarrow \ F(y,x) \subseteq -C.$$

A multivalued mapping $G : K \rightrightarrows Y$ is said to be *$C$-convex* if $\forall x, y \in K,\ \forall \lambda \in [0,1]$,

$$G(\lambda x + (1-\lambda) y) \subseteq \lambda G(x) + (1-\lambda) G(y) - C.$$

The mapping $G$ is said to be *generalized hemicontinuous* (in short, g.h.c.) if $\forall x, y \in K,\ \forall \lambda \in [0,1]$,

$$\lambda \mapsto G(x + \lambda(y-x)) \text{ is upper semicontinuous at } 0^+.$$

We denote by $L(X, Y)$ the space of all continuous linear operators from $X$ to $Y$. A multifunction $T : K \rightrightarrows L(X, Y)$ is said to be *$C$-strongly pseudomonotone* if it satisfies

$$\forall x, y \in K, \ \langle Tx, y - x\rangle \nsubseteq -C \setminus \{0\} \ \Rightarrow \ \langle Ty, x - y\rangle \subseteq -C,$$

where $\langle Tx, y - x\rangle = \{\langle s, y - x\rangle \mid s \in Tx\}$.

## 3. Main Results

We first present an existence of solutions of (MSG) under monotonicity. To do this, the following lemma is necessary.

**Lemma 1.** Let $F : K \times K \rightrightarrows Y$ be g.h.c. in the first variable and $C$-convex in the second variable. Assume that $F$ is $C$-strongly pseudomonotone. Then the following two problems are equivalent:

(i) Find $x \in K$ such that $F(x, y) \nsubseteq -C \setminus \{0\}, \quad \forall y \in K$.

(ii) Find $x \in K$ such that $F(y, x) \subseteq -C, \quad \forall y \in K$.

*Proof.* (i) $\Rightarrow$ (ii). This is clear by the $C$-strong pseudomonotonicity of $F$.

(ii) $\Rightarrow$ (i). Let $x \in K$ be such that $F(y, x) \subseteq -C, \ \forall y \in K$. For any $y \in K, \ \lambda \in (0, 1)$, set $y_\lambda = x + \lambda(y - x)$. Then $y_\lambda \in K$, hence

$$F(y_\lambda, x) \subseteq -C. \tag{1}$$

Since $F$ is $C$-convex in the second variable, we have

$$0 \in F(y_\lambda, y_\lambda) \subseteq \lambda F(y_\lambda, y) + (1 - \lambda)F(y_\lambda, x) - C. \tag{2}$$

By $(1)$ and the fact that $C$ is a convex cone, we get

$$\lambda F(y_\lambda, y) \cap C \neq \emptyset, \ \text{hence } F(y_\lambda, y) \cap C \neq \emptyset. \tag{3}$$

Since $F$ is g.h.c. in the first variable and $y_\lambda \to x$, we have $F(x, y) \cap C \neq \emptyset$. If not, $F(x, y) \subseteq C^c = V$ an open set. By the generalized hemicontinuity of $F$ in its first variable, there exists $\lambda_0 \in (0, 1]$ such that $\forall \lambda \in [0, \lambda_0), \ F(y_\lambda, y) \subseteq V$, i.e., $F(y_\lambda, y) \cap C = \emptyset$, which is a contradiction to $(3)$. Therefore we obtain that

$$F(x, y) \nsubseteq -C \setminus \{0\},$$

because $C$ is a pointed convex cone. This completes the proof. ■

**Theorem 1.** Let $K$ be a nonempty compact convex subset of $X$. Let $F : K \times K \rightrightarrows Y$ be g.h.c. in the first variable, $C$-convex and l.s.c. in the second variable. Assume that $F$ is $C$-strongly pseudomonotone. Then (MGS) is solvable.

*Proof.* We define two multifunctions $A, B : K \rightrightarrows K$ as follow:

$$A(x) = \{y \in K \mid F(y, x) \nsubseteq -C\}$$

$$B(x) = \{y \in K \mid F(x, y) \subseteq -C \setminus \{0\}\}.$$

The proof is proceeded in the following way.

(i) It is obvious that for each $x \in K$, $A(x) \subseteq B(x)$ by the $C$-strong pseudomonotonicity of $F$.

(ii) $B(x)$ is convex. Indeed, if $y_1, y_2 \in B(x)$, we have

$$F(x, y_i) \subseteq -C \setminus \{0\} \quad \text{for } i = 1, 2.$$

By the $C$-convexity of $F$ in the second variable, for any $\lambda \in (0, 1)$,

$$\begin{aligned} F(x, \lambda y_1 + (1-\lambda) y_2) &\subseteq \lambda F(x, y_1) + (1-\lambda) F(x, y_2) - C \\ &\subseteq (-C \setminus \{0\}) + (-C \setminus \{0\}) - C \\ &\subseteq (-C \setminus \{0\}) - C = -C \setminus \{0\}, \end{aligned}$$

which implies that $B(x)$ is convex.

(iii) $B$ has no fixed point because $0 \in F(x, x),\ \forall x \in K$.

(iv) For each $y \in K$, $A^{-1}(y)$ is open in $K$. In fact, let $\{x_\lambda\}$ be a net in $(A^{-1}(y))^c$ which converges to $x \in K$. Then $y \notin A(x_\lambda)$, i.e.,

$$F(y, x_\lambda) \subseteq -C. \tag{4}$$

Since $F$ is l.s.c. in its second variable and $x_\lambda \to x$, we see that $F(y, x) \subseteq -C$. If not, $F(y, x) \cap (-C)^c \neq \emptyset$. As $(-C)^c$ is open, there exists $\lambda_0$ such that $\forall \lambda \geq \lambda_0,\ F(y, x_\lambda) \cap (-C)^c \neq \emptyset$ by means of the lower semicontinuity of $F$, which contradicts $(4)$. Thus, $x \in (A^{-1}(y))^c$ and $(A^{-1}(y))^c$ is closed. Consequently, $A^{-1}(y)$ is open in $K$.

(v) By the Fan-Browder fixed point theorem, there is an $x_0 \in K$ such that $A(x_0) = \emptyset$, that is,

$$F(y, x_0) \subseteq -C, \quad \forall y \in K.$$

Therefore the result follows from Lemma 1. ■

As a direct consequence of Theorem 1, we get the following.

**Theorem 2.** Let $T : K \rightrightarrows L(X, Y)$ be a $C$-strongly pseudomonotone and generalized hemicontinuous multifunction with nonempty compact values where

$L(X, Y)$ is equipped with the topology of bounded convergence. Then the following vector variational inequality has a solution $x_0$ such that

$$\langle Tx_0, x - x_0\rangle \nsubseteq -C \setminus \{0\},\ \forall x \in K.$$

*Proof.* Define $F : K \times K \rightrightarrows Y$ by

$$F(x, y) = \langle Tx, y - x\rangle.$$

Clearly $F(x, x) = \{0\}$. Then all the hypotheses of Theorem 1 hold as follows.

(i) $F$ is g.h.c. in the first variable. Indeed, fix $y_0 \in K$. Then $\forall x, z \in K,\ \forall\lambda \in [0, 1]$, consider the diagram below.

$$\lambda \overset{G_1}{\to} (T(x+\lambda(z-x)), y_0-(x+\lambda(z-x))) \overset{G_2}{\to} \langle T(x+\lambda(z-x)), y_0-(x+\lambda(z-x))\rangle.$$

Since $T$ is g.h.c. and compact-valued, so the product map $G_1$ of two u.s.c. multifunctions at $0^+$ with compact values is u.s.c. at $0^+$, too. Moreover, the evaluation map $G_2(s, y) = \langle s, y\rangle$ os continuous on the product space $L(X, Y) \times L$ where $L$ is the compact line segment $L = \{y_0-(x+\lambda(z-x)) \in X \mid \lambda \in [0, 1]\}$ [27, Lemma B]. This is because $L(X, Y)$ is endowed with the topology of bounded convergence. Hence the composition map $G_2 \circ G_1$ is u.s.c. at $0^+$. This means that $F(x, y) = \langle Tx, y - x\rangle$ is g.h.c. in its first variable, as desired.

(ii) $F$ is $C$-convex in the second variable. In fact, fix $x_0 \in K$. For $\forall y_1, y_2 \in K,\ \forall\lambda \in [0, 1]$, we have

$$\begin{aligned} F(x_0, \lambda y_1 + (1-\lambda)y_2) &= \langle Tx_0, \lambda(y_1 - x_0) + (1-\lambda)(y_2 - x_0)\rangle \\ &\subseteq \lambda\langle Tx_0, y_1 - x_0\rangle + (1-\lambda)\langle Tx_0, y_2 - x_0\rangle \\ &\subseteq \lambda\langle Tx_0, y_1 - x_0\rangle + (1-\lambda)\langle Tx_0, y_2 - x_0\rangle - C \\ &= \lambda F(x_0, y_1) + (1-\lambda)F(x_0, y_2) - C \end{aligned}$$

(iii) $F$ is l.s.c. in the second variable. Fix $x_0 \in K$. Let $y_0 \in K$ and $V$ be an open set in $Y$ such that $F(x_0, y_0) = \langle Tx_0, y_0 - x_0\rangle \cap V \neq \emptyset$. So there is an $s_0 \in Tx_0$ with $\langle s_0, y_0 - x_0\rangle \in V$. Since the linear operator $s_0$ is continuous, there exists an open neighborhood $U$ of $y_0$ such that $\forall y \in U$, $\langle s_0, y - x_0\rangle \in V$ because $V$ is open. Thus $F(x_0, y) = \langle Tx_0, y - x_0\rangle \cap V \neq \emptyset,\ \forall y \in U$.

(iv) The $C$-strong pseudomonotonicity of $F$ directly comes from that of $T$.

By Theorem 1, there exists $x_0 \in K$ such that

$$F(x_0, x) \nsubseteq -C \setminus \{0\} \ \Leftrightarrow \langle Tx_0, x - x_0\rangle \nsubseteq -C \setminus \{0\},\ \forall x \in K.$$

This completes the proof. ∎

**Remark.** In Theorems 1 and 2, the topologies on $K$ and $Y$ can be replaced by the weak topologies with the help of [28, Lemma 2.4]. Hence Theorem 2 can be regarded as a multi-valued generalization of Fang and Huang [24, Theorem 2.3] in Hausdorff topological vector spaces (not necessarily Banach spaces) $X$ and $Y$. Of course, Theorem 1 is a multi-valued version of Kazmi and Khan [23, Theorem 2.3].

Now we are in a position to show an existence of solutions of (MSG) without monotonicity. The following lemma is the first step for this.

**Lemma 2.** A multifunction $G : K \rightrightarrows F$ is $C$-convex if and only if for every $n \geq 2$, whenever $x_1, \ldots, x_n \in K$ are given and for any $\lambda_i \in [0,1], i = 1, \ldots, n,$ with $\sum_{i=1}^n \lambda_i = 1$, we have

$$G\left(\sum_{i=1}^{n} \lambda_i x_i\right) \subseteq \lambda_1 G(x_1) + \cdots + \lambda_n G(x_n) - C. \tag{5}$$

*Proof.* The sufficiency is clear. For the necessity, we shall use the induction argument on $n$. When $n = 2$, the condition $(5)$ is exactly the same as the definition of $C$-convexity. Assume that the condition $(5)$ holds for all $k \leq n-1 \ \ (n \geq 3)$. Let $\{x_1, \ldots, x_n\} \subset K$ be given, and $\lambda_i \in [0,1], i = 1, \ldots, n,$ with $\sum_{i=1}^n \lambda_i = 1$ be arbitrarily given. Without loss of generality, we may assume $\sum_{i=1}^{n-1} \lambda_i > 0$ by reindexing $i$. Then, for a given set $\{x_1, \ldots, x_{n-1}\}$, the induction assumption assures that

$$G\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j} x_i\right) \subseteq \frac{\lambda_1}{\sum_{j=1}^{n-1} \lambda_j} G(x_1) + \cdots + \frac{\lambda_{n-1}}{\sum_{j=1}^{n-1} \lambda_j} G(x_{n-1}) - C. \tag{6}$$

Again applying $C$-convexity to $\left\{\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j} x_i,\, x_n\right\}$ in $K$, we see that

$$\begin{aligned} G\left(\sum_{i=1}^{n} \lambda_i x_i\right) &= G\left(\left(\sum_{j=1}^{n-1} \lambda_j\right)\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j} x_i\right) + \lambda_n x_n\right) \\ &\subseteq \left(\sum_{j=1}^{n-1} \lambda_j\right) G\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_{j=1}^{n-1} \lambda_j} x_i\right) + \lambda_n G(x_n) - C. \end{aligned}$$

Using the inclusion $(6)$, we have

$$\begin{aligned} G\left(\sum_{i=1}^{n} \lambda_i x_i\right) &\subseteq \left[\lambda_1 G(x_1) + \cdots + \lambda_{n-1} G(x_{n-1}) - C\right] + \lambda_n G(x_n) - C \\ &= \lambda_1 G(x_1) + \cdots + \lambda_{n-1} G(x_{n-1}) + \lambda_n G(x_n) - C \end{aligned}$$

because $C$ is a convex cone. Therefore, by induction, for every $n \geq 2$, we can obtain the desired conclusion. ∎

Using the same argument in Kazmi and Khan [23, Theorem 2.1], we obtain the following.

**Theorem 3.** Let $K$ be a nonempty compact convex subset of $X$. Let $F : K \times K \rightrightarrows Y$ be $C$-convex in the second variable. Assume that for each $y \in K$, the set $\{x \in K \mid F(x, y) \subseteq -C \setminus \{0\}\}$ is open. Then (MGS) is solvable.

*Proof.* Suppose the contrary. Then for each $x \in K$, there exists $y \in K$ such that

$$F(x, y) \subseteq -C \setminus \{0\}. \tag{7}$$

For each $y \in K$, define

$$N_y = \{x \in K \mid F(x, y) \subseteq -C \setminus \{0\}\}. \tag{8}$$

So $\{N_y\}_{y \in K}$ is an open cover of $K$ by means of the assumption and (7). Thus there is a finite subcover $\{N_{y_i}\}_{i=1}^n$ and a partition of unity $\{\beta_i\}_{i=1}^n$ subordinated to it such that

$$K = \bigcup_{i=1}^n N_{y_i},\ \beta_i(x) \geq 0,\ \sum_{i=1}^n \beta_i(x) = 1\ \forall x \in K,\ \text{and}\ \left\{\begin{matrix} \beta_i(x) = 0 \text{ if } x \notin N_{y_i} \\ \beta_i(x) > 0 \text{ if } x \in N_{y_i} \end{matrix}\right\}$$

We define a continuous function $h : K \to K$ by $h(x) = \sum_{i=1}^n \beta_i(x) y_i$. By Brouwer's fixed point theorem, there exists $x_0 \in \text{co}\{y_1, ..., y_n\}$ such that $h(x_0) = x_0$ where $\text{co}\{y_1, ..., y_n\}$ denotes the convex hull of $\{y_1, ..., y_n\}$. Define a multifunction $G : K \rightrightarrows Y$ by $G(x) = F(x, h(x))$ for all $x \in K$. Since $F$ is $C$-convex in the second variable, by Lemma 2, we have

$$G(x) = F(x, h(x)) = F(x, \sum_{i=1}^n \beta_i(x) y_i) \subseteq \sum_{i=1}^n \beta_i(x) F(x, y_i) - C. \tag{9}$$

From (8) and (9), it is easily checked that $G(x) = F(x, h(x)) \subseteq -C \setminus \{0\},\ \forall x \in K$. Hence

$$G(x_0) = F(x_0, h(x_0)) = F(x_0, x_0) \subseteq -C \setminus \{0\},$$

which contradicts $F(x_0, x_0) \ni 0$.

## References

1. E. Blume and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123-145.

2. Q. H. Ansari, L. J. Lin and L. B. Su, Systems of simultaneous generalized vector quasiequilibrium problems and their applications, *Journal of Optimization Theory and Applications*, **127** (2005), 27-44.

3. I. V. Konnov and J. C. Yao, On the generalized vector variational inequalities, *Journal of Mathematical Analysis and Applications*, **206** (1997), 42-58.

4. I. V. Konnov and J. C. Yao, Existence of solutions for generalized vector equilibrium problems, *Journal of Mathematical Analysis and Applications*, **233** (1999), 328-335.

5. Q. H. Ansari and J. C. Yao, An existence result for the generalized vector equilibrium problems, *Applied Mathematics Letters*, **12** (1999), 53-56.

6. Q. H. Ansari, S. Schaible and J. C. Yao, The system of vector equilibrium problems and its applications, *Journal of Optimization Theory and Applications*, **107(3)** (2000), 547-557.

7. Q. H. Ansari, I. V. Konnov and J. C. Yao, Existence of solutions and variational principles for vector equilibrium problems, *Journal of Optimization Theory and Applications*, **110(3)** (2001), 481-492.

8. Q. H. Ansari, I. V. Konnov and J. C. Yao, Characterizations of solutions for vector equilibrium problems, *Journal of Optimization Theory and Applications*, **113(3)** (2002), 435-447.

9. Q. H. Ansari, I. V. Konnov and J. C. Yao, On generalized vector equilibrium problems, *Nonlinear Analysis*, **47** (2001), 543-554.

10. Q. H. Ansari, S. Schaible and J. C. Yao, System of generalized vector equilibrium problems with applications, *Journal of Global Optimization*, **22** (2002), 3-16.

11. O. Chadli, N. C. Wong and J. C. Yao, Equilibrium problems with applications to eigenvalue problems, *Journal of Optimization Theory and Applications*, **117(2)** (2003), 245-266 .

12. O. Chadli, S. Schaible and J. C. Yao, Regularized equilibrium problems with an application to noncoercive hemivariational inequalities, *Journal of Optimization Theory and Applications*, **121** (2004), 571-596.

13. O. Chadli, Z. H. Liu and J. C. Yao, Applications of Equilibrium Problems to a Class of Noncoercive Variational Inequalities, *Journal of Optimization Theory and Applications*, **132** (2007), 89-110.

14. L. C. Zeng, S. Y. Wu and J. C. Yao, Generalized KKM Theorem with Applications to Generalized Minimax Inequalities and Generalized Equilibrium Problems, *Taiwanese Journal of Mathematics*, **10** (2006), 1497-1514.

15. K. Kimura and J. C. Yao, Sensitivity Analysis of Vector Equilibrium Problems, *Taiwanese Journal of Mathematics*, **12** (2008), 649-669.

16. B. T. Kien, N. C. Wong and J. C. Yao, Generalized Vector Variational Inequalities with Star-Pseudomonotone and Discontinuous Operators, *Nonlinear Analysis Series A: Theory, Methods & Applications,* **68** (2008), 2859-2871.

17. K. Kimura and J. C. Yao, Sensitivity analysis of solution mappings of parametric generalized quasi vector equilibrium problems, *Taiwanese Journal of Mathematics*, **12** (2008), 2233-2268.

18. L. C. Ceng, S. Schaible and J. C. Yao, Existence of Solutions for Generalized Vector Variational-like Inequalities, *Journal of Optimization Theory and Applications*, **137** (2008), 121-133.

19. X. H. Gong and J. C. Yao, Lower Semicontinuity of the Set of Efficient Solutions for Generalized Systems, *Journal of Optimization Theory an Applications*, **138** (2008), 197-205.

20. L. C. Ceng, G. Y. Chen, X. X. Huang and J. C. Yao, Existence Theorems for Generalized Vector Variational Inequalities with Pseudomonotonicity and Their Applications, *Taiwanese Journal of Mathematics*, **12** (2008), 151-172.

21. J. W. Peng and J. C. Yao, Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings, *Taiwanese Journal of Mathematics*, **13** (2009), 1537-1582.

22. L. Q. Anh, P. Q. Khanh, D. T. M. Van and J. C. Yao, Well-posedness for vector quasiequilibria, *Taiwanese Journal of Mathematics*, **13** (2009), 713-737.

23. K. R. Kazmi and S. A. Khan, Existence of solutions to a generalized system, *Journal of Optimization Theory and Applications*, **142** (2009), 355-361.

24. Y.-P. Fang and N.-J. Huang, Strong vector variational inequalities in Banach spaces, *Applied Mathematics Letters*, **19** (2006), 362-368.

25. K. Fan, A Generalization of Tychonoff's Fixed Point Theorem, *Math. Ann.*, **142** (1961), 305-310.

26. F. E. Browder, The Fixed-Point Theory of Multivalued Mappings in Topological Vector Space, *Math. Ann.*, **177** (1968), 283-301.

27. S. H. Kum, A Generalization of Generalized Quasi-Variational Inequalities, *Journal of Mathematical Analysis and Applications*, **182** (1994), 158-164.

28. G. M. Lee and S. H. Kum, On implicit vector variational inequalities, *Journal of Optimization Theory and Applications*, **104** (2000), 409-425.

Sangho Kum
Department of Mathematics Education
Chungbuk National University
Cheongju 361763
Korea
E-mail: shkum@cbnu.ac.kr

Mu-Ming Wong
Department of Applied Mathematics
Chung Yuan Christian University
Chung Li 32023, Taiwan
E-mail: mmwong@cycu.edu.tw