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L^2-CURVATURE BOUND FOR PSEUDOHOLOMORPHIC CURVES IN SYMPLECTISATIONS

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Abstract. We provide an L^2-bound for pseudoholomorphic curves in symplectisation of contact manifolds which is the first step for compactness theorem in symplectic field theory introduced by Eliashberg, Givental and Hofer.

1. Introduction

It is now well-known that compactness theorem plays a fundamental rule in the theory of pseudoholomorphic curves. The Gromov-Witten invariant can be defined through the intersection of a suitable compactified moduli space of pseudoholomorphic curves. If the ambient manifold is closed, compactness result was first proved by Parker and Wolfson [8], through a detailed bubbling-off analysis. The method was more systematically treated by Ruan and Tian in [9]. A key step in the bubbling-off which leads to the structure of degeneration of pseudoholomorphic curves is an L^2 estimate of some geometric quantity of the pseudoholomorphic curves. In the colsed case, the energy of the pseudoholomorphic curve gives a natural L^2-bound, on which the bubbling-off analysis is based. The situatin is different if the ambient manifold is non-compact or with boundary. In this paper we provide an L^2-bound for the curvature when the ambient manifold is symplectisation of contact manifold. The idea of proof is a construction of Hofer et al in [2, 3].

2. Periodic Orbits and Local Cordinates Near the Ends

First, we will introduce necessary preliminaries that is needed for our computatioins in later section. We remark that the notions introduced here are quite general so that symplectic manifold with contact boundary is a special case.

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Let $(\tilde{M},\xi)$ be a $(2n+1)$-dimensional contact manifold with contact structure ξ. The manifold is called so-oriented if we can choose an one form λ, called the contact form, so that $\xi = ker\lambda$. By Frobenius' theorem the contact condition is the maximal non-integrability of the tangent hyperplane field ξ. This is equivalent to say that $(d\lambda)^n \wedge \lambda$ is a volume form on $\tilde{M}$ and $d\lambda|_\xi$ gives a symplectic structure making $(\xi, d\lambda|_\xi)$ a symplectic vector bundle over $\tilde{M}$. There is a unique nonvanishing vector field on $T\tilde{M}$ such that $\iota_X\lambda = 1$ and $\iota_X d\lambda = 0$; $X = X_\lambda$ is called the Reeb vector field of λ. With this, the tangent bundle $T\tilde{M}$ splits

$$T\tilde{M} = \xi \oplus \mathbb{R}X.$$

For each contact manifold $(\tilde{M},\xi)$, we can associate a manifold $M = (T\tilde{M}/\xi)^\star \backslash\tilde{M}$, called the symplectization of $(\tilde{M},\xi)$. A choice of the contact form λ then defines a splitting $M = \tilde{M} \times (\mathbb{R}\backslash 0)$.

The sympletization of a contact manifold is an example of a symplectic manifold with cylindrical ends. A possibly non-compact symplectic manifold (W,ω) is called a directed symplectic cobordism if the following holds. It has ends of the form $E^+ = \tilde{M}^+ \times [0,\infty)$ and $E^- = \tilde{M}^- \times (-\infty, 0]$ such that $\tilde{M}^\pm$ are contact manifolds and $\omega\mid_{\tilde{M}^\pm} = d(e^t\lambda^\pm)$, where $\lambda^\pm$ are the contact forms on $\tilde{M}^\pm$. We also denote a directed symplectic cobordism by $\overrightarrow{\tilde{M}^-\tilde{M}^+}$. In particular the ends can be of finite length and if it has infinite $\mathbb{R}$-length, we will call it a complete directed symplectic cobordism.

The case which is of particular interest to us is the following. Let (M,ω) be a symplectic manifold and $\tilde{M} \subset M$ be a contact hypersurface dividing M into two parts W_-, W_+. Then near $\tilde{M}$, ω is of the form $\omega = d\beta$ for some one form β on M, and $\lambda = \beta\mid_{\tilde{M}}$ defines a contact form on $\tilde{M}$. We can construct two new complete directed symplectic cobordisms from the above situation by setting

$$(W_-^\infty, \omega_-^\infty) = (W_-,\omega)\bigcup(\tilde{M}\times[0,\infty], d(e^t\lambda))$$

and

$$(W_+^\infty, \omega_+^\infty) = (\tilde{M}\times(-\infty,0], d(e^t\lambda))\bigcup(W_+,\omega).$$

Now let $(\Sigma_n, \tilde{u}_n)$ be a sequence of J-holomorphic maps. Here Σ_n is a Riemann surface of genus g with s^+ positive punctures $\mathbf{x}^+ = \{x_1^+, \cdots, x_{s^+}^+\}$, and s^- negative punctures $\mathbf{x}^- = \{x_1^-, \cdots, x_{s^-}^-\}$, j_n is a given complex structures on Σ_n, and $\tilde{u}_n : \Sigma_n \longrightarrow M$ is a J-holomorphic map with finite energy. Here the finite energy condition is that near the ends of M

$$E(\tilde{u}_n) = \sup_{f\in S}\int_{\Sigma_n} \tilde{u}_n^* d(f\lambda^\pm) < \infty \tag{1}$$

for each n where $S = \{f \in C^\infty(R,[0,1]) \mid f' \geq 0\}$.

For a co-oriented contact manifold $(\tilde{M}^{2n+1}, \xi, \lambda)$ with contact structure ξ and contact form λ so that $\xi = \ker\lambda$, let X be the Reeb vector field corresponding to λ, i.e. $\iota_X\lambda = 1$, $\iota_X d\lambda = 0$. As we have mentioned that $\xi = ker\lambda$ defines a symplectic vector bundle over $\tilde{M}$ with symplectic structure $d\lambda \mid \xi$. Since $\iota_X\lambda = 1$ and $\iota_X d\lambda = 0$, the tangent bundle $T\tilde{M}$ splits into $T\tilde{M} = \mathbb{R}X \oplus \xi$. Let $\pi : T\tilde{M} \longrightarrow \xi$ be the projection, then for any $h \in T\tilde{M}$,

$$h = \lambda(h)X + \pi(h).$$

Finally choose $\tilde{J}$ to be a $d\lambda \mid \xi$ compatible almost complex structure on ξ, so that

$$g_{\tilde{J}}(h,k) := d\lambda \mid \xi(h, \tilde{J}k)$$

defines a Riemannian metric on ξ.

We will consider the following first order elliptic system for maps:

$$u = (a, \tilde{u}) : \mathbb{C} \longrightarrow \mathbb{R} \times \tilde{M}$$

defined by

$$\begin{aligned} \pi\tilde{u}_s + \tilde{J}(u)\pi\tilde{u}_t &= 0 \\ u^{\star}\lambda \circ I &= da \end{aligned} \tag{2}$$

where $z = s + it \in \mathbb{C}$. Introduce an almost complex structure J on $\mathbb{R} \times \tilde{M}$, by

$$J_{(a,m)}(h,k) = (-\lambda_m(k), hX(m) + \tilde{J}(m)\pi k)$$

where $(h,k) \in T_{(a,m)}(\mathbb{R} \times \tilde{M})$. Then it is easy to see that $J^2 = -1$ and hence it is really an almost complex structure on $\mathbb{R} \times \tilde{M}$. For later use we will also consider the cylindrical coordinates $s + it$ which is given by the biholomorphic map

$$\phi : \mathbb{R} \times S^1 \longrightarrow \mathbb{C} - \{0\}$$

with $\phi(s + it) = e^{2\pi(s+it)}$. The equation (2) is then equivalent to the following equation.

$$\begin{aligned} u &= (a, \tilde{u}) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times \tilde{M} \\ u_s + J(u)u_t &= 0 \end{aligned} \tag{3}$$

As explained above, to get interesting solutions we have to impose the finite energy condition, and compactness theorem is to consider sequence of maps satisfying (3) and its convergence property. For this we recall some asymptotic properties of maps satisfying due to Hofer et al. In the famous work [2], Hofer studied the asymptotic behavior of a J-holomorphic map from an infinite cylinder to the symplectization of a contact manifold. He showed that for a finite energy map $u = (a, \tilde{u}) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times \tilde{M}$, there is a sequence $R_k \longrightarrow \infty$ such that $\lim_{k \to \infty} u(R_k e^{2\pi i t}) = x(Tt)$ in $C^\infty(\mathbb{R})$. Here T is a period of the Reeb vector field and $x(t)$ is a T-periodic solutioon of the Reeb vector field. Furthermore if the solution is non-degenerate then $\lim_{R \to \infty} u(Re^{2\pi i t}) = x(Tt)$ in $C^\infty(\mathbb{R})$. As a consequence of this result, we may choose the so-called Darboux coordinate near the periodic solution $x(t)$ and the computation then becomes a local one.

The above theorem of Hofer is then extended in [3], they showed that for a finite energy J-holomorphic map $u = (a(s,t), \theta(s,t), z(s,t))$ from an infinite cylinder, there is an exponential decay for the components $a(s,t)$,$\theta(s,t)$. Furthermore there is an explicit expression for the component $z(s,t)$ see Theorem 2.8 in [3]. In the following computation we will make use of the above two results. Now we begin the computation of the curvature.

3. L^2 Estimate of the Curvature

Let $u = (a, \tilde{u}) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times \tilde{M}$ be a finite energy plane, we want to compute near the asymptotic limit of u the curvature of $\mathbb{R} \times \tilde{M}$ restricting to the image of u, namely we want to compute $R(u_s, u_t, u_s, u_t)$ where $R(\cdot,\cdot,\cdot,\cdot)$ is the curvature operator and (s,t) is the coordinate on $\mathbb{R} \times S^1$.

For this purpose, according to Hofer , it is enough to study $u = (a, \tilde{u}) : [s, \infty] \times S^1 \longrightarrow R \times \tilde{M}$ with s large enough ($s \geq s_0$ say) in a tubular neighborhood of the T-periodic solution $x(t)$ of the Reeb vector field, $\dot{x}(t) = X(x(t))$.

Choose the so-called Darboux coordinate near $x(t)$ by

$$S^1 \times R^{2n}, f\lambda_0$$

where the positive solution is on $S^1 \times \{0\}$, f is a positive smooth function and $\lambda = f\lambda_0$ is a contact form with

$$\lambda_0 = d\theta + \sum_{i=1}^{n} x_i dy_i$$

being the standard contact form of $S^1 \times R^{2n}$. Here we set

$(\theta, x_1, y_1, \ldots x_n, y_n)$ to be the coordinate of $\tilde{M}$ in the tubular neighborhood of $x(t)$, and $(a, \theta, x_1, y_1, \ldots x_n, y_n)$ to be the coordinates of $\mathbb{R} \times \tilde{M}$.

With this coordinate, we compute

$$d\lambda = \sum_{i=1}^{n}(x_i f_\theta - f_{y_i}) d\theta \wedge dy_i + \sum_{i=1}^{n}(-f_{x_i}) d\theta \wedge dx_i + \sum_{i=1}^{n}(x_i f_{x_i} + f) dx_i \wedge dy_i$$

$$+\sum_{j\neq i} x_j(f_{x_i}dx_i + f_{y_i}dy_i)\wedge dy_j$$

By considering the equations $\iota_X\lambda = 1$ and $\iota_X d\lambda = 1$ for the Reeb vector field X, we find

$$X = \frac{1}{f^2}\left[\left(f + \sum_{i=1}^{n}(x_i f_{x_i})\frac{\partial}{\partial\theta} - \sum_{i=1}^{n}((x_i f_\theta - f_{y_i})\frac{\partial}{\partial x_i} + f_{x_i}\frac{\partial}{\partial y_i})\right)\right].$$

For later use, we will also write $X = (X_\theta, X_1, X_1', \ldots, X_n, X_n')$, namely

$$\begin{aligned} X_i &= -\frac{x_i f_\theta - f_{y_i}}{f^2} \\ X_i' &= -\frac{f_{x_i}}{f^2} \\ X_\theta &= \frac{f + \sum_{i=1}^{n} x_i f_{x_i}}{f^2} \end{aligned}$$

Now let $\xi = ker\lambda$. To go further, we will introduce a new coordinate system in the tubular neighborhood of $x(t)$ such that in this coordinate, $d\lambda \mid \xi$ takes a simple form. Set

$$e_i = \frac{\partial}{\partial x_i}, e_i' = -x_i\frac{\partial}{\partial\theta} + \frac{\partial}{\partial y_i}$$

i=1,2,. . . ,n, then we find

$$d\lambda \mid \xi(e_i, e_j') = \begin{cases} f & \text{if i = j} \\ 0 & \text{otherwise} \end{cases}$$

and $d\lambda \mid \xi(e_i, e_j) = d\lambda \mid \xi(e_i', e_j') = 0$, i.e. in $\{e_i, e_i'\}$, $i = 1, 2, \ldots, n$

$$d\lambda \mid \xi = f\begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ & & 0 & 1 & & 0 & & & \\ & & -1 & 0 & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & 0 & & & & & \cdot & & \\ & & & & & & & 0 & 1 \\ & & & & & & & -1 & 0 \end{pmatrix}$$

For simplicity, we take the compatible almost complex structure $\tilde{J}$ on ξ to be

$$\tilde{J}e_i = e_i'$$

and

$$\tilde{J}e_i' = -e_i$$

i.e. in terms of $\{e_i, e_i'\}$, $i = 1, 2, \ldots, n$

$$\tilde{J} = \begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ & & 0 & 1 & & 0 & & & \\ & & -1 & 0 & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & 0 & & & & & \cdot & & \\ & & & & & & & 0 & 1 \\ & & & & & & & -1 & 0 \end{pmatrix}$$

We also have a simple expression for the metric on $T\tilde{M} = \mathbb{R}X \oplus \xi$ in terms of coordinates $X, e_1, e_1', \ldots, e_n, e_n'$. Recall that the metric is defined to be

$$\langle h, k \rangle = \lambda(h)\lambda(k) + g_{\tilde{J}}(\pi h, \pi k)$$

for $h, k \in T\tilde{M}$, where $g_{\tilde{J}}(\cdot, \cdot) = d\lambda \mid \xi(\cdot, \tilde{J}\cdot)$. Now the matrix for $\langle \cdot, \cdot \rangle$ is simply

$$\begin{pmatrix} 1 & & & & & & & & \\ & f & & & & & & & \\ & & f & & & 0 & & & \\ & & & f & & & & & \\ & & & & \cdot & & & & \\ & & & & & \cdot & & & \\ & 0 & & & & & \cdot & & \\ & & & & & & & f & \\ & & & & & & & & f \end{pmatrix}$$

Now we compute $R(u_s, u_t, u_s, u_t)$ mentioned above. For a finite energy cylinder $u = (a, \tilde{u}) : \mathbb{R} \times S^1 \longrightarrow \mathbb{R} \times \tilde{M}$, where

$u(s,t) = (a(s,t), \theta(s,t), x_1(s,t), y_1(s,t), \cdots, x_n(s,t), y_n(s,t))$ and $\mathbb{R} \times \tilde{M}$ has the complete metric $< (\alpha, h), (\beta, h) >= \alpha\beta + \lambda(h)\lambda(k) + g_{\tilde{J}}(\pi h, \pi k)$, we first represent $\tilde{u}_s$, $\tilde{u}_t$ in terms of the coordinates $X, e_1, e_1', \cdots, e_n, e_n'$. Recall first that the equation (3) is equivalent to

$$\begin{aligned} a_s - \lambda(\tilde{u}_t) &= 0 \\ a_t + \lambda(\tilde{u}_s) &= 0 \\ \pi\tilde{u}_s + \tilde{J}(\tilde{u})\pi\tilde{u}_t &= 0. \end{aligned} \tag{4}$$

Write

$$\begin{aligned} \tilde{u}_s &= \lambda(\tilde{u}_s)X + \sum_{i=1}^{n} A_i e_i + \sum_{i=1}^{n} C_i e_i' \\ \tilde{u}_t &= \lambda(\tilde{u}_t)X + \sum_{i=1}^{n} B_i e_i + \sum_{i=1}^{n} D_i e_i', \end{aligned} \tag{5}$$

then

$$\begin{aligned} A_i &= x_{i,s} + a_t X_i \\ C_i &= y_{i,s} + a_t X_i' \\ B_i &= x_{i,t} - a_s X_i \\ D_i &= y_{i,t} - a_s X_i' \end{aligned} \tag{6}$$

for $i = 1, \cdots, n$. Now (4) is equivalent to

$$\begin{aligned} (x_{i,s} + a_t X_i) + (y_{i,t} - a_s X_i') = 0 \\ -(x_{i,t} - a_s X_i) + (y_{i,s} + a_t X_i') = 0 \end{aligned} \tag{7}$$

$i = 1, \cdots, n$.

We begin with the computation of the curvature tensor of $\mathbb{R} \times \tilde{M}$ with the complete metric $< (\alpha, h), (\beta, h) >= \alpha\beta + \lambda(h)\lambda(k) + g_{\tilde{J}}(\pi h, \pi k)$. Note that most of the tensors are zero except the following: $R_{\alpha i \alpha i}$, $R_{\alpha i' \alpha i'}$, $R_{\alpha i j i}$, $R_{\alpha i' j i'}$, $R_{\alpha i j' i}$, $R_{\alpha i' j' i'}$, R_{ijij}, $R_{i'ji'j}$, $R_{ij'ij'}$, $R_{i'j'i'j'}$, R_{ijik}, $R_{i'ji'k}$, $R_{ij'ik}$, $R_{i'j'i'k}$, $R_{ijik'}$, $R_{i'ji'k'}$, $R_{ij'ik'}$, $R_{i'j'i'k'}$. Here we use the following convention of indices : the index α always means the direction of X, the index i belongs to $\{e_1, \cdots, e_n\}$ and the index i' belongs to $\{e_1', \cdots, e_n'\}$. We then find

$$\begin{aligned} R_{\alpha i \alpha i} &= R_{\alpha i' \alpha i'} = f\left(X\left(\frac{Xf}{2f}\right) + \left(\frac{Xf}{2f}\right)^2\right) \\ R_{\alpha i j i} &= R_{\alpha i' j i'} = fX\left(\frac{e_j f}{2f}\right) \\ R_{\alpha i j' i} &= R_{\alpha i' j' i'} = fX\left(\frac{e_j' f}{2f}\right) \\ R_{ijij} &= f\left(e_i\left(\frac{e_i f}{2f}\right) + e_j\left(\frac{e_j f}{2f}\right)\right) \\ R_{i'ji'j} &= f\left(e_i'\left(\frac{e_i' f}{2f}\right) + e_j\left(\frac{e_j f}{2f}\right)\right) \\ R_{ij'ij'} &= f\left(e_i\left(\frac{e_i f}{2f}\right) + e_j'\left(\frac{e_j' f}{2f}\right)\right) \end{aligned} \tag{8}$$

$$
\begin{aligned}
R_{i'j'i'j'} &= f\left(e'_i\left(\frac{e'_i f}{2f}\right) + e'_j\left(\frac{e'_j f}{2f}\right)\right)\\
R_{ijik} &= R_{i'ji'k} = f\left(e_j\left(\frac{e_k f}{2f}\right) - 2\frac{e_j f}{2f}\frac{e_k f}{2f}\right)\\
R_{ij'ik} &= R_{i'j'i'k} = f\left(e'_j\left(\frac{e_k f}{2f}\right) - 2\frac{e'_j f}{2f}\frac{e_k f}{2f}\right)\\
R_{ijik'} &= R_{i'ji'k'} = f\left(e_j\left(\frac{e'_k f}{2f}\right) - 2\frac{e_j f}{2f}\frac{e'_k f}{2f}\right)\\
R_{ij'ik'} &= R_{i'j'i'k'} = f\left(e'_j\left(\frac{e'_k f}{2f}\right) - 2\frac{e'_j f}{2f}\frac{e'_k f}{2f}\right)
\end{aligned}
$$

Now we can compute $R(u_s, u_t, u_s, u_t)$. According to the above results about the curvature tensors, we have to care of the following four kinds of terms $R_{\alpha k\alpha k}$, $R_{\alpha klk}$, R_{klkl}, and R_{klkm} where the index α still denotes the X direction and k, l, and m belong to $\{e_1, \cdots, e_n\}$ or $\{e'_1, \cdots, e'_n\}$. After some computation, we write the summatiions corresponding to the above four kinds of terms to be K, L, M and N and compute them separately. We have

$$
\begin{aligned}
K &= \sum_{j=1}^{n} f\left[X\left(\frac{Xf}{2f}\right) + \left(\frac{Xf}{2f}\right)^2\right]\\
&\quad \cdot \{(-a_t)B_j(-a_t)B_j - A_j(a_s)(-a_t)B_j - (-a_t)B_jA_j(a_s) + A_ja_sA_ja_s\\
&\quad + (-a_t)D_j(-a_t)D_j - C_j(a_s)(-a_t)D_j - (-a_t)D_jC_j(a_s) + C_ja_sC_ja_s\}.
\end{aligned}
$$

Here we have used (6). Take into account of (7), we can further simplify K to get

$$K = \sum_{j=1}^{n}\left(A_j^2 + C_j^2\right)\left(a_s^2 + a_t^2\right) f\left[X\left(\frac{Xf}{2f}\right) + \left(\frac{Xf}{2f}\right)^2\right].$$

To simplify expression further, we introduce the notation

$$
\begin{aligned}
\mid \tilde{\pi}\tilde{u}_s \mid^2 &= \sum_{j=1}^{n}(A_j^2 + C_j^2)\\
&= \sum_{j=1}^{n}[(x_{j,s} + a_tX_j)^2 + (y_{j,s} + a_tX'_j)^2].
\end{aligned}
\tag{9}
$$

In terms of this, K can be expressed simply as

$$K = \mid \tilde{\pi}\tilde{u}_s \mid^2 (a_s^2 + a_t^2) f\left[X\left(\frac{Xf}{2f}\right) + \left(\frac{Xf}{2f}\right)^2\right]. \tag{10}$$

Now we can do the similar computation for L, M and N. For L, we have

$$\begin{aligned}
L \;=\; & 2\sum_{i,j=1}^{n} fX\left(\frac{e_j f}{2f}\right)\{(-a_t)\,B_iA_jB_i - A_ia_sA_jB_i + a_tB_iA_iB_j + A_ia_sA_iB_j \\
& + (-a_t)\,D_iA_jD_i - C_ia_sA_jD_i + a_tD_iC_iB_j + C_ia_sC_iB_j\} \\
& +2\sum_{i,j=1}^{n} fX\left(\frac{e'_j f}{2f}\right)\{(-a_t)\,B_iC_jB_i - A_ia_sC_jB_i + a_tB_iA_iD_j + A_ia_sA_iD_j \\
& + (-a_t)\,D_iC_jD_i - C_ia_sC_jD_i + a_tD_iC_iD_j + C_ia_sC_iD_j\}
\end{aligned}$$

Using (6) and (9), we may further simplify to get

$$L=2\mid \tilde{\pi}\tilde{u}_s \mid^2\sum_{j=1}^{n}\left[(a_sC_j-a_tA_j)\,fX\left(\frac{e_jf}{2f}\right)-(a_sA_j+a_tC_j)\,fX\left(\frac{e'_jf}{2f}\right)\right]. \tag{11}$$

Using the same idea, we may compute the remaining terms M and N. For M,

$$\begin{aligned}
M \;=\; & \sum_{i,j=1}^{n} f\left(e_i\left(\frac{e_i f}{2f}\right)+e_j\left(\frac{e_j f}{2f}\right)\right) \\
& \quad \{A_iB_jA_iB_j - A_jB_iA_iB_j - A_iB_jA_jB_i + A_jB_iA_jB_i\} \\
& + \sum_{i,j=1}^{n} f\left(e'_i\left(\frac{e'_i f}{2f}\right)+e_j\left(\frac{e_j f}{2f}\right)\right) \\
& \quad \{C_iB_jC_iB_j - A_jD_iC_iB_j - C_iB_jA_jD_i + A_jD_iA_jD_i\} \\
& + \sum_{i,j=1}^{n} f\left(e_i\left(\frac{e_i f}{2f}\right)+e'_j\left(\frac{e'_j f}{2f}\right)\right) \\
& \quad \{A_iD_jA_iD_j - C_jB_iA_iD_j - A_iD_jC_jB_i + C_jB_iC_jB_i\} \\
& + \sum_{i,j=1}^{n} f\left(e'_i\left(\frac{e'_i f}{2f}\right)+e'_j\left(\frac{e'_j f}{2f}\right)\right) \\
& \quad \{C_iD_jC_iD_j - C_jD_iC_iD_j - C_iD_jC_jD_i + C_jD_iC_jD_i\}
\end{aligned}$$

Using (6) and (9) to get

$$M = 2\mid \tilde{\pi}\tilde{u}_s \mid^2 \sum_{j=1}^{n}\left(A_j^2+C_j^2\right) f\left(e_j\left(\frac{e_jf}{2f}\right)+e'_j\left(\frac{e'_jf}{2f}\right)\right). \tag{12}$$

Finally for N,

$$\begin{aligned}N = {} & 2\sum_{i,j\neq k} f\left(e_j\left(\frac{e_kf}{2f}\right)-2\left(\frac{e_jf}{2f}\right)\left(\frac{e_kf}{2f}\right)\right)\{A_i^2B_jB_k-A_iB_i\left(A_jB_k+A_kB_j\right)\\ & +B_i^2A_jA_k+C_i^2B_jB_k-C_iD_i\left(A_jB_k+A_kB_j\right)+D_i^2A_jA_k\}\\ & +2\sum_{i,j\neq k} f\left(e_j'\left(\frac{e_kf}{2f}\right)-2\left(\frac{e_j'f}{2f}\right)\left(\frac{e_kf}{2f}\right)\right)\{A_i^2D_jB_k-A_iB_i\left(C_jB_k+A_kD_j\right)\\ & +B_i^2C_jA_k+C_i^2D_jB_k-C_iD_i\left(C_jB_k+A_kD_j\right)+D_i^2C_jA_k\}\\ & +2\sum_{i,j\neq k} f\left(e_j\left(\frac{e_k'f}{2f}\right)-2\left(\frac{e_jf}{2f}\right)\left(\frac{e_k'f}{2f}\right)\right)\{A_i^2B_jD_k-A_iB_i\left(A_jD_k+C_kB_j\right)\\ & +B_i^2A_jC_k+C_i^2B_jD_k-C_iD_i\left(A_jD_k+C_kB_j\right)+D_i^2A_jC_k\}\\ & +2\sum_{i,j\neq k} f\left(e_j'\left(\frac{e_k'f}{2f}\right)-2\left(\frac{e_j'f}{2f}\right)\left(\frac{e_k'f}{2f}\right)\right)\{A_i^2D_jD_k-A_iB_i\left(C_jD_k+C_kD_j\right)\\ & +B_i^2C_jC_k+C_i^2D_jD_k-C_iD_i\left(C_jD_k+C_kD_j\right)+D_i^2C_jC_k\}.\end{aligned}$$

Using (6) and (9), we get

$$\begin{aligned} & N = 2 \mid \tilde{\pi}\tilde{u}_s \mid^2 \sum_{j\neq k}\left(A_jA_k + C_jC_k\right) \\ & f\left(e_j\left(\frac{e_kf}{2f}\right)-2\left(\frac{e_jf}{2f}\right)\left(\frac{e_kf}{2f}\right)+e_j'\left(\frac{e_k'f}{2f}\right)-2\left(\frac{e_j'f}{2f}\right)\left(\frac{e_k'f}{2f}\right)\right)\end{aligned} \tag{13}$$

Now by Theorem 2.8 in [3], each term A_i, B_i, C_i and D_i in the expression (6) is of exponential decay as $s \longrightarrow \infty$. Hence the leading coefficient $\mid \tilde{\pi}\tilde{u}_s \mid^2$ in each term K, L, M and N is of exponential decay as $s \longrightarrow \infty$. Namely by taking some large positive s_0, we have for any J-holomorphic finite energy cylinder $u = (a, \tilde{u}) : \mathbb{R}^+ \times S^1 \longrightarrow \tilde{M} \times \mathbb{R}^+$ with asymptotic limit $x(t)$ a fixed T-periodic solution of the Reeb vector field,

$$\mid \tilde{\pi}\tilde{u}_s \mid^2 \leq C_1 e^{-C_2 s} \tag{14}$$

where $s \geq s_0$ and C_1, C_2 are constants depending only on the geometry of $\tilde{M}\times\mathbb{R}^+$.

From this we see that for any J-holomorphic finite energy cylinder with asymptotic limit $x(t)$, the integral of $R(u_s, u_t, u_s, u_t)$ over $[s_0, \infty) \times S^1$ for large s_0 is bounded by a universal constant C_T depending only on the geometry of $\tilde{M} \times \mathbb{R}^+$. Now we have proved the main theorem of the paper.

Theorem 1. *Let $u = (a, \tilde{u}) : \mathbb{R}^+ \times S^1 \longrightarrow \tilde{M}\times\mathbb{R}^+$ be a finite energy cylinder with asymptotic limit $x(t)$ a T-periodic solution of the Reeb vector field. We have the expression for the curvature $R = R(u_s, u_t, u_s, u_t)$.*

$$
\begin{aligned}
R &= K + L + M + N \\
&= \mid \tilde{\pi}\tilde{u}_s \mid^2 \{(a_s^2 + a_t^2) f \left[X\left(\frac{Xf}{2f}\right) + \left(\frac{Xf}{2f}\right)^2 \right] \\
&\quad + 2\sum_{j=1}^{n} \left[(a_s C_j - a_t A_j) f X \left(\frac{e_j f}{2f}\right) - (a_s A_j + a_t C_j) f X \left(\frac{e'_j f}{2f}\right) \right] \\
&\quad + 2\sum_{j=1}^{n} \left(A_j^2 + C_j^2\right) f \left(e_j \left(\frac{e_j f}{2f}\right) + e'_j \left(\frac{e'_j f}{2f}\right) \right) \\
&\quad + 2\sum_{j\neq k} (A_j A_k + C_j C_k) f (e_j \left(\frac{e_k f}{2f}\right) - 2\left(\frac{e_j f}{2f}\right)\left(\frac{e_k f}{2f}\right) \\
&\quad + e'_j \left(\frac{e'_k f}{2f}\right) - 2\left(\frac{e'_j f}{2f}\right)\left(\frac{e'_k f}{2f}\right)
\end{aligned} \tag{15}
$$

Fruthermore, there exists an $s_0 \gg 1$ and a constant C_T such that

$$
\int_{[s_0,\infty)\times S^1} R(u_s, u_t, u_s, u_t)\, d\mu \le C_T. \tag{16}
$$

Finally using the Gauss-Bonnet theorem in bundle version we have

$$
\int_\Sigma \mid B \mid^2 = \int_\Sigma R(u_s, u_t, u_s, u_t) - \chi(\Sigma).
$$

Here B is the square of the second fundamental form for embedded surfaces in a Riemannian manifold. Note that the origional formula has a term about integral of the mean curvature on the right hand side. Since in our case the J-holomorphic map is naturally a minimal surface that term vanishes, and $\chi(\Sigma)$ is the Euler class of a section of a bundle. Now apply theorem 2, we finally get an L^2-estimate for the square of the second fundamental form B.

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