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HAYMAN T DIRECTIONS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we prove the existence of Hayman T directions of meromorphic functions. To achieve our purpose, we establish a fundamental inequality about the Ahlfors-Shimizu characteristic for an angle and explain that the inequality is best possible in terms of the existence of Julia directions and Hayman T directions.

1. Introduction and Results

We mean by a transcendental meromorphic function a function meromorphic on the whole complex plane which is not a rational function, in other words, ∞ is an essential singular point. We assume the reader is familiar with the Nevanlinna theory of meromorphic functions and basic notations such as Nevanlinna characteristic $T(r, f)$, integrated counting function $N(r, f)$ and proximity function $m(r, f)$. In this paper, we discuss the argument distribution of transcendental meromorphic functions. It is one of the main subjects in the theory of value distribution. In argument distribution, we consider the problem of whether a result which holds in the complex plane is still true in an angular domain. Corresponding to Picard's theorem in the complex plane is the Julia direction (a ray from the origin to infinity), that is, in arbitrary angular domain containing the direction, the Picard's theorem still holds. The existence of Julia directions was first proved by G. Julia in 1919. Since the work of G. Julia, the argument distribution has developed approximately one century. And G. Valiron proved in 1928 the existence of Borel directions which corresponds to Borel's theorem. According to the Picard type theorem produced by the Hayman inequality dealing with derivatives, every transcendental meromorphic function takes every value infinitely often or its derivative of each order takes every non-zero value infinitely often. Yang Lo in [14] introduced a singular direction named Hayman direction. A direction is called a Hayman direction of a meromorphic function if in

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every angular domain containing the direction the Picard type theorem of Hayman still holds for the function. Yang [14] gave out a growth condition for the existence of a Hayman direction: if a transcendental meromorphic function satisfies

$$\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{(\log r)^3}=\infty, \tag{1.1}$$

then $f(z)$ has at least one Hayman direction. However, it is well-known that a sufficient condition for the existence of a Julia direction is

$$\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{(\log r)^2}=\infty \tag{1.2}$$

and it is sharp. Actually, Ostrowski [11] proved that the function

$$f(z)=\prod_{n=0}^{\infty}\left(\frac{q^n-z}{q^n+z}\right)$$

for a fixed number $q>1$ has no Julia direction, while (1.2) does not hold. The following problem was proposed by D. Drasin in 1984 which was collected in [1] (See also Problem 11 in Yang [15]):

Drasin's Problem. *Does a transcendental meromorphic function have a Hayman direction if it satisfies* (1.2)?

A proof of Drasin's Problem seems to be given by Zhu [20], but we think the proof is incomplete, which we shall explain in the next section. Actually, this has been pointed out by Fenton and Rossi [6]. Chen [2] obtained a refined result which stated that if $f(z)$ satisfies (1.1), then $f(z)$ has a Hayman direction $\arg z=\theta$ such that for arbitrary small $\varepsilon>0$, any positive integer k and any complex numbers a and $b\neq 0$, we have

$$\limsup_{r\to\infty}\frac{n(r,Z_\varepsilon(\theta),f=a)+n(r,Z_\varepsilon(\theta),f^{(k)}=b)}{(\log r)^2}=\infty. \tag{1.3}$$

Here and throughout the paper we denote by $Z_\varepsilon(\theta)$ the angle $\{z:|\arg z-\theta|<\varepsilon\}$ and by $n(r,Z_\varepsilon(\theta),f=a)$ the number of the roots of $f(z)=a$ in $Z_\varepsilon(\theta)\cap\{|z|<r\}$ counted according to multiplicity. The estimation given in (1.3) for the numbers of roots of $f(z)=a$ and $f^{(k)}(z)=b$ in $Z_\varepsilon(\theta)\cap\{|z|<r\}$ is best possible under the condition (1.1). However, if $(\log r)^3$ in (1.1) is replaced with $(\log r)^4$, then the best estimation should be (1.3) with $(\log r)^2$ replaced by $(\log r)^3$. In this paper, we come to seek an unified expression for the above considerations, which will be realized in terms of Hayman T directions.

In order to make our statement clear, let us begin with some basic notation. Given an angle $\Omega = \{z : \alpha \leq \arg z \leq \beta\}$, let $f(z)$ be a meromorphic function in Ω. Define

$$N(r, \Omega, f = a) = \int_1^r \frac{n(t, \Omega, f = a)}{t} dt$$

where $n(t, \Omega, f = a)$ is the number of the roots of $f(z) = a$ in $\Omega \cap \{1 < |z| < t\}$ counted according to multiplicity.

In view of the second fundamental theorem of Nevanlinna, the first author in [19] introduced a new singular direction, which is named T direction. The reason about the name is we use the Nevanlinna's characteristic $T(r, f)$ of the meromrophic function as comparison body. A direction $\arg z = \theta$ is called a T direction of a meromorphic function $f(z)$ if for arbitrary small $\varepsilon > 0$, we have

$$\limsup_{r \longrightarrow \infty} \frac{N(r, Z_\varepsilon(\theta), f = a)}{T(r, f)} > 0 \tag{1.4}$$

for all but at most two values of a in the extended complex plane $\widehat{\mathbb{C}}$. It is proved in [7] and [17] that a meromorphic function $f(z)$ has at least one T direction, if $f(z)$ satisfies (1.2). According to the Hayman inequality (see [8]) on the estimation of $T(r, f)$ in terms of only two integrated counting functions for the roots of $f(z) = a$ and $f^{(k)}(z) = b$ with $b \neq 0$, we proposed in [7] a singular direction named Hayman T direction as follows.

Definition 1.1. Let $f(z)$ be a transcendental meromorphic function. A direction $\arg z = \theta$ is called a Hayman T direction of $f(z)$ if for arbitrary small $\varepsilon > 0$, any positive integer k and any complex numbers a and $b \neq 0$, we have

$$\limsup_{r \longrightarrow \infty} \frac{N(r, Z_\varepsilon(\theta), f = a) + N(r, Z_\varepsilon(\theta), f^{(k)} = b)}{T(r, f)} > 0. \tag{1.5}$$

The purpose of this paper is to discuss the existence of Hayman T directions. The following is our main result.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function satisfying (1.1). Then $f(z)$ has a Hayman T direction which is a T direction as well.*

We also establish the following result which is an improvement of other known results about the Hayman directions.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function satisfying (1.1). Then for any sequence of positive numbers $\{r_n\}$ such that*

$$\limsup_{n \to \infty} \frac{T(r_n, f)}{(\log r_n)^3} = \infty,$$

$f(z)$ has a ray $\arg z = \theta$ such that for arbitrary small $\varepsilon > 0$ and for all but at most two values of $c \in \widehat{\mathbb{C}}$, we have

$$\limsup_{n\longrightarrow\infty} \frac{N(2r_n, Z_\varepsilon(\theta), f = c)}{T(r_n, f)} > 0 \tag{1.6}$$

and for any positive integer k and any complex numbers a and $b \neq 0$, we have

$$\limsup_{n\longrightarrow\infty} \frac{N(2r_n, Z_\varepsilon(\theta), f = a) + N(2r_n, Z_\varepsilon(\theta), f^{(k)} = b)}{T(r_n, f)} > 0. \tag{1.7}$$

From the definition of the integrated counting function for an angle it follows that

$$N(r, Z_\varepsilon(\theta), *) \leq n(r, Z_\varepsilon(\theta), *) \log r$$

for $r > 1$ and then Chen's result mentioned above is a direct consequence of Theorem 1.2. For a transcendental meromorphic function with finite positive order λ, Yang and Zhang [16] proved that each Borel direction of the function is a Hayman direction of order λ. We easily also obtain Yang and Zhang's result in view of Theorem 1.2. Likewise, we can consider the case of infinite order from Theorem 1.2. Actually, Theorem 1.2 gives an unified expression for results about singular directions related to the Hayman inequality, as we mentioned previously.

Finally, we propose a problem of Drasin's type:

Does a transcendental meromorphic function have a Hayman T direction, if it satisfies (1.2)?

2. Proofs of Theorem 1.2 and Theorem 1.1

First of all we establish a fundamental inequality on the estimation of the Ahlfors-Shimizu characteristic in terms of the integrated counting functions for the roots of $f(z) = a$ and $f^{(k)}(z) = b$ in an angle. The inequality is of independent significance. We know that the difference between the characteristics of Ahlfors-Shimizu and Nevanlinna in the complex plane is bounded. Hence the Hayman inequality is still true for Ahlfors-Shimizu characteristic in the complex plane. However the case of an angular domain is not simple. Notice that Ahlfors' theory of covering surfaces does not deal with the derivatives of covering mappings. Therefore, the inequality for Ahlfors-Shimizu characteristic in an angle does not seem to be directly able to result from the theory. Recall the definition of Ahlfors-Shimizu characteristic in an angle (see [13]). Let $f(z)$ be a meromorphic function on an angle $\Omega = \{z : \alpha \leq \arg z \leq \beta\}$. Set $\Omega(r) = \Omega \cap \{z : 1 < |z| < r\}$. Define

$$\mathcal{S}(r, \Omega, f) = \frac{1}{\pi} \int\int_{\Omega(r)} \left(\frac{|f'(z)|}{1 + |f(z)|^2} \right)^2 d\sigma$$

and

$$\mathcal{T}(r,\Omega,f)=\int_1^r \frac{\mathcal{S}(t,\Omega,f)}{t}dt.$$

There are several basic formulae which are used often in the sequel:

$$\mathcal{S}(r,\Omega,f)=\frac{1}{\pi}\int\int_{|z|\leq\infty} n(r,\Omega,f=z)d\omega(z), \tag{2.1}$$

$$\frac{1}{\pi}\int\int_{|z|\leq\infty} d\omega(z)=1 \text{ and } \frac{1}{\pi}\int\int_{|z|\leq\infty} \log\frac{1}{|x,z|}d\omega(z)=\frac{1}{2} \tag{2.2}$$

where $d\omega(z)$ is the area element on the Riemann sphere and $|x,z|$ is the chordal distance between x and z.

We need the following result due to Chen and Guo [4] which essentially comes from the Hayman inequality and the estimation of primary values appeared in the inequality.

Lemma 2.1. *Let $f(z)$ be meromorphic in $|z|<R$ and let*

$$N=n(R,f=a)+n(R,f^{(k)}=b)$$

for two complex numbers a and b with $b\neq 0$. Then there exists a point z_0, $|z_0|<\frac{R}{8}$ such that for any $c\in\widehat{\mathbb{C}}$, we have

$$n\left(\frac{R}{256},f=c\right)<C_k\{N+1+\log R+\log\frac{1}{|f(z_0),c|}\}, \tag{2.3}$$

where C_k is a positive constant depending only on k.

Lemma 2.1 can be obtained by applying Lemma 8 of Chen and Guo [4] to the function $(bR^k)^{-1}(f(Rz)-a)$ in the unit disk.

In view of Lemma 2.1, we establish the following

Theorem 2.3. *Let $f(z)$ be meromorphic in an angle $\Omega=\{z:\alpha\leq\arg z\leq\beta\}$ Then for arbitrary small $\varepsilon>0$, any positive integer k and any two complex numbers a and $b\neq 0$, we have*

$$\mathcal{T}(r,\Omega_\varepsilon,f)\leq K\{N(2r,\Omega,f=a)+N(2r,\Omega,f^{(k)}=b)\}+O((\log r)^3) \tag{2.4}$$

for a positive constant K depending only on k, where $\Omega_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$.

Proof. For $r>1$, we use finitely many disks A_n to cover the sector $\Omega_\varepsilon(r)$ such that the disks B_n produced by enlarging A_n 256 times are in $\Omega(2r)$. We may require that the number of A_n is $O(\log r)$ and the overlap number of B_n over a point does not exceed an absolute positive constant. Certainly, the radius of B_n does not exceed $2r$. Applying Lemma 2.1 to B_n, for each n we have

$$n(A_n, f=c) < C_k \left\{ n(B_n, f=a) + n(B_n, f^{(k)}=b) + 1 + \log(2r) + \log\frac{1}{|f(z_n), c|} \right\}$$

for all $c \in \widehat{\mathbb{C}}$ and some $z_n \in B_n$. Then it is easy to see that

$$\begin{aligned} & n(r, \Omega_\varepsilon, f=c) \leq \sum_n n(A_n, f=c) \\ \leq\ & C_k \sum_n \{n(B_n, f=a) + n(B_n, f^{(k)}=b) + 1 + \log(2r) + \log\frac{1}{|f(z_n), c|}\} \\ \leq\ & K\{n(2r, \Omega, f=a) + n(2r, \Omega, f^{(k)}=b)\} + O(\log r)^2 + C_k \sum_n \log\frac{1}{|f(z_n), c|}. \end{aligned}$$

Integrating in c over $\widehat{\mathbb{C}}$ both sides of the above inequality, in view of (2.1) and (2.2), we have

$$\mathcal{S}(r, \Omega_\varepsilon, f) \leq K\{n(2r, \Omega, f=a) + n(2r, \Omega, f^{(k)}=b)\} + O((\log r)^2).$$

Noticing that the above inequality holds for all $r > 1$, we divide by r and then integrate both sides of it from 1 to r. In view of the inequality

$$\int_1^r \frac{n(2r, \Omega, *)}{r} dr = \int_2^{2r} \frac{n(r, \Omega, *)}{r} dr \leq N(2r, \Omega, *),$$

we obtain the desired inequality (2.4). Theorem 2.3 follows.

We claim that generally, the term $O((\log r)^3)$ in (2.4) cannot be replaced by a quantity $\phi(r)$ such that $\liminf_{r\to\infty} \phi(r)(\log r)^{-3} = 0$. Let us prove that. Consider the function

$$g(z) = \prod_{n=0}^{\infty} \left(\frac{e^{\sqrt{n}} - z}{e^{\sqrt{n}} + z} \right).$$

Rossi [12] proved that $T(r, g) = (1/3 + o(1))(\log r)^3$ and $g(z)$ has exactly two Julia directions (the positive and negative imaginary axes $\arg z = \pm\frac{\pi}{2}$) and these two directions, however, are not Julia directions of $g'(z)$. Since $g(z)$ does not take zero and infinity in any small angular domains containing them, these two Julia directions of $g(z)$ are not Hayman directions and then $g(z)$ has no Julia directions which are Hayman directions. In [18] we have pointed out that $g(z)$ has T directions exactly on the rays $\arg z = \pm\frac{\pi}{2}$ and then $g(z)$ has no T directions which are Hayman T directions.

Suppose that there exists a $\phi(r)$ with $\liminf_{r\to\infty} \phi(r)(\log r)^{-3} = 0$ such that (2.4) holds for $g(z)$ with $\phi(r)$ in place of $O(\log r)^3$. Then we can find an unbounded sequence of positive numbers $\{r_n\}$ such that $\phi(r_n) = o(T(r_n, g))$. As we do in the proof of Theorem 1.1 below, accordingly we can prove that $g(z)$ has a T direction which is a Hayman T direction. A contradiction is derived and then the desired claim follows.

From Rossi's example, it is easy to see that the proof of Drasin's Problem given by Zhu [20] is incomplete. From the proof we know that the Hayman direction he found is also a Julia direction. As we mentioned as above, under (1.2) the function under consideration may have no Hayman direction which is a Julia direction. Actually, in Zhu's proof, three points $\alpha_v (v = 1, 2, 3)$ he chose for application of Ahlfors theory of covering surfaces depend on $|z_k|$ which tends to ∞ as k goes to ∞. Therefore, h in the proof is not an absolute constant and actually it relies on $|z_k|$. Hence, Drasin's Problem is still open.

In view of the same argument as above, we can assert that the term "$O(\log r)^3$" in (2.4) cannot be reduced either even if we add one more integrated counting function for points of other value. Therefore, the inequality in Lemma 5 of [10] is incorrect. Actually, the derivative of f of k order in the proof of the inequality should be the derivative of composition of f and $r_{j+1}e^{i\theta_0}\zeta(w)$.

To prove Theorem 1.2, we need a result of Tsuji, that is, Theorem VII.3 of [13].

Lemma 2.2. *Let $f(z)$ be meromorphic in an angular domain Ω. Then for arbitrary small $\varepsilon > 0$ and three distinct points a_j $(j = 1, 2, 3)$ on $\widehat{\mathbb{C}}$, we have*

$$\mathcal{T}(r, \Omega_\varepsilon, f) \le 3 \sum_{j=1}^{3} N(2r, \Omega, f = a_j) + O((\log r)^2) \tag{2.5}$$

for $r > 1$.

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. Since the difference between the Nevanlinna and Ahlfors-Shimizu characteristics on the whole complex plane is bounded, it is easy to see that there exists a ray $\arg z = \theta$ such that

$$\limsup_{n\to\infty} \frac{\mathcal{T}(r_n, Z_{\varepsilon/2}(\theta), f)}{T(r_n, f)} > 0 \tag{2.6}$$

for arbitrary small $\varepsilon > 0$. Thus it follows from (2.5) that (1.6) holds and from (2.4) that (1.7) holds. The proof of Theorem 1.2 is completed.

In order to treat the case when the function is of infinite order in the proof of Theorem 1.1, we need the following two results.

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function and let a and $b \neq 0$ be two complex numbers. Then for an angle Ω and $\varepsilon > 0$, we have*

$$\mathcal{T}(r, \Omega_\varepsilon, f) \le K \left(N(r, \Omega) + r^\omega \int_1^r \frac{N(t, \Omega)}{t^{\omega+1}} dt + r^\omega \log(rT(r, f)) \right) \tag{2.7}$$

for $r \notin E$, where E is a subset of the positive real axis with linear measure at most two which only depends on $T(r,f)$, $N(r,\Omega) = N(r,\Omega, f=a) + N(r,\Omega, f^{(k)} = b)$, $\omega = \frac{\pi}{\beta-\alpha}$ and K is a positive constant.

Lemma 2.3 follows from Theorem 2.4.7, the Hayman inequality (2.2.10) for Nevanlinna characteristic for an angle and Lemma 2.2.2 in Zheng [17]. The following is Lemma 1.1.3 of Zheng [17].

Lemma 2.4. *Let $T(r)$ be an increasing and non-negative function with infinite order and F be a set of positive real numbers having finite logarithmic measure. Then given a sequence $\{s_n\}$ of positive real numbers, there exist an unbounded sequence $\{r_n\}$ of positive real numbers outside F such that*

$$\frac{T(t)}{t^{s_n}} \le e\frac{T(r_n)}{r_n^{s_n}},\ 1 \le t \le r_n. \tag{2.8}$$

Proof of Theorem 1.1. We need to treat three cases.

Case I. The order $\lambda(f) = 0$. Set $W = \{r > 0 : T(2r,f) \le 2T(r,f)\}$, and in view of Lemma 4 of Hayman [9], we have $\overline{\log dens} W = 1$. We choose a sequence $\{r_n\}$ of positive numbers such that $(\log r_n)^3 = o(T(r_n,f))$. A simple calculation implies that for all sufficiently large n, $W \bigcap (r_n, r_n^2) \ne \emptyset$, and hence we can take a $r_n' \in W \bigcap (r_n, r_n^2)$. Thus we have $T(2r_n' f) \le 2T(r_n', f)$ and

$$(\log r_n')^3 \le 8(\log r_n)^3 = o(T(r_n,f)) = o(T(r_n',f))$$

and equivalently $(\log r_n')^3 = o(T(r_n',f))$. In view of Theorem 1.2 for $\{r_n'\}$, we have a ray which is a T direction as well as a Hayman T direction.

Case II. The order $\lambda(f) > 0$ and the lower order $\mu(f) < \infty$. In view of a result of Edrei [5], we can find a sequence $\{r_n\}$ of Polya peaks such that for some $\sigma > 0$,

$$T(2r_n,f) \le 2^\sigma T(r_n,f) \text{ and } \lim_{n\to\infty} \frac{\log T(r_n,f)}{\log r_n} \ge \sigma > 0.$$

Applying Theorem 1.2 implies that there exists a ray $\arg z = \theta$ such that (1.6) and (1.7) hold with $T(r_n,f)$ replaced by $T(2r_n,f)$. Then the ray $\arg z = \theta$ is a T direction as well as a Hayman T direction of $f(z)$.

Case III. The lower order $\mu(f) = \infty$. In view of Lemma 2.4, we have a sequence $\{r_n\}$ of positive numbers outside E as in Lemma 2.3 such that (2.8) holds for a sequence $\{s_n\}$ of positive numbers tending to ∞. For the sequence $\{r_n\}$, there exists a ray $\arg z = \theta$ such that (2.6) holds for arbitrarily small $\varepsilon > 0$. In view of Theorem 3.1.1 of [17], the ray $\arg z = \theta$ is a T direction of $f(z)$. Now we prove that the ray is also a Hayman T direction.

Suppose that the ray $\arg z = \theta$ is not a Hayman T direction and then for some $\varepsilon_0 > 0$ and two complex numbers a and $b \neq 0$, we have

$$N(r, Z_{\varepsilon_0}(\theta)) = N(r, Z_{\varepsilon_0}(\theta), f = a) + N(r, Z_{\varepsilon_0}(\theta), f^{(k)} = b) = o(T(r, f)).$$

We shall use the inequality (2.7) for $Z_{\varepsilon_0}(\theta)$ to derive a contradiction. For each n with $s_n > \omega = \frac{\pi}{2\varepsilon_0}$, in view of (2.8) we have

$$\begin{aligned} \int_1^{r_n} \frac{N(t, Z_{\varepsilon_0}(\theta))}{t^{\omega+1}} dt &= o\left(\int_1^{r_n} \frac{T(t, f)}{t^{\omega+1}} dt\right) \\ &= o\left(e \frac{T(r_n, f)}{r_n^{s_n}} \int_1^{r_n} t^{s_n - \omega - 1} dt\right) \\ &= o\left(\frac{e}{s_n - \omega} \frac{T(r_n, f)}{r_n^{\omega}}\right) \\ &= o\left(\frac{T(r_n, f)}{r_n^{\omega}}\right). \end{aligned}$$

Then applying the inequality (2.7) for $Z_{\varepsilon_0}(\theta)$ yields

$$\mathcal{T}(r_n, Z_{\varepsilon_0/2}(\theta)) \leq o(T(r_n, f)) + O(r_n^{\omega} \log(r_n T(r_n, f))).$$

Since $T(r_n, f) \geq e^{-1} r_n^{s_n} T(1, f)$, we have $O({r_n}^{\omega} \log r_n T(r_n, f)) = o(T(r_n, f))$ and then we obtain $\mathcal{T}(r_n, Z_{\varepsilon_0/2}(\theta)) = o(T(r_n, f))$. This contradicts (2.6) for $Z_{\varepsilon_0/2}(\theta)$ and $\{r_n\}$.

Thus we complete the proof of Theorem 1.1.

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