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ON LOCAL INTEGRATED C-COSINE FUNCTION AND WEAK SOLUTION OF SECOND ORDER ABSTRACT CAUCHY PROBLEM

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Abstract. Let α be a nonnegative number, $\mathrm{C} : \mathrm{X} \to \mathrm{X}$ a bounded linear injection on a Banach space X and $A : \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ a closed linear operator in X which satisfies $\mathrm{C}^{-1}A\mathrm{C} = A$ and may not be densely defined. We prove some equivalence relations between the generation of a local α-times integrated C-cosine function on X with generator A and the uniqueness existence of weak solutions of the abstract Cauchy problem:

$$\mathrm{ACP}_2(\mathrm{A}, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\ u(0) = x, u'(0) = y, \end{cases}$$

where $x, y \in \mathrm{X}$ are given and f is an X-valued function defined on a subset of $\mathbb{R}$.

1. INTRODUCTION

Let X be a Banach space over $\mathbb{F}$ with norm $\|\cdot\|$ and dual space X^*, and let $\mathrm{B}(\mathrm{X})$ denote the set of all bounded linear operators from X into itself. For each $0 < T_0 \le \infty$, we consider the following abstract Cauchy problem:

$$\mathrm{ACP}_2(\mathrm{A}, f, x, y) \qquad \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0), \\ u(0) = x, u'(0) = y, \end{cases}$$

where $x, y \in \mathrm{X}$ are given, $A : \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ is a closed linear operator and f is an X-valued function defined on a subset of $\mathbb{R}$ containing $(0, T_0)$. A function u is called a strong solution of $\mathrm{ACP}_2(\mathrm{A}, f, x, y)$, if $u \in \mathrm{C}^2((0, T_0), \mathrm{X}) \cap \mathrm{C}^1([0, T_0), \mathrm{X}) \cap \mathrm{C}((0, T_0), [\mathrm{D}(A)])$ and satisfies $\mathrm{ACP}_2(\mathrm{A}, f, x, y)$. Here $[\mathrm{D}(A)]$ denotes the Banach

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space $D(A)$ equipped with the graph norm $|x|_A = \|x\| + \|Ax\|$ for $x \in D(A)$. For each $\alpha > 0$ and $C \in B(X)$, a family $C(\cdot)(= \{C(t) \,|\, 0 \le t < T_0\})$ in $B(X)$ is called a local α-times integrated C-cosine function on X if it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$2C(t)C(s)x = \frac{1}{\Gamma(\alpha)}\Big\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1}C(r)Cxdr + \int_{|t-s|}^t (s-t+r)^{\alpha-1}C(r)Cxdr + \int_{|t-s|}^s (t-s+r)^{\alpha-1}C(r)Cxdr + \int_0^{|t-s|}(|t-s|+r)^{\alpha-1}C(r)Cxdr\Big\} \tag{1.1}$$

for all $0 \le t, s, t+s < T_0$ and $x \in X$; or called a local (0-times integrated) C-cosine function on X if it is strongly continuous, $C(\cdot)C = CC(\cdot)$, and satisfies

$$2C(t)C(s)x = C(t+s)Cx + C(|t-s|)Cx \quad \text{for all } 0 \le t, s, t+s < T_0 \text{ and } x \in X, \tag{1.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $C(\cdot)$ is nondegenerate, if $x = 0$ whenever $C(t)x = 0$ for all $0 \le t < T_0$. In this case, its (integral) generator $A : D(A) \subset X \to X$ is a closed linear operator in X defined by $D(A) = \{x \,|\, x \in X$ and there exists a $y_x \in X$ such that $C(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t \int_0^s C(r)y_x drds$ for all $0 \le t < T_0\}$ and $Ax = y_x$ for all $x \in D(A)$. In general, a local α-times integrated C-cosine function on X is also called an α-times integrated C-cosine function on X if $T_0 = \infty$; or called a local α-times integrated cosine function on X if $C = I$ the identity operator on X. The relation between the existence of an exponentially bounded α-times integrated C-cosine function with generator A and the unique existence of strong solutions of $ACP_2(A, f, x, y)$ have been considered as in [4, 5, 9, 12, 14, 15] if $\alpha \in \mathbb{N} \cup \{0\}$. When $\alpha = 0$ and A is densely defined, some results concerning the relation between the existence of a C-cosine function with generator A and the unique existence of weak solutions of $ACP_2(A, f, x, y)$ are also investigated in [9], and in [7] for the case $C = I$. Just as in the case $\alpha \in \mathbb{N} \cup \{0\}$, some equivalence relations between the existence of an α-times integrated C-cosine function on X and the unique existence of strong solutions for $ACP_2(A, f, x, y)$ are also obtained in [10,11] for which the resolvent set $\rho(A)$ of A may be nonempty and $D(A)$ may be dense in X. Several examples concerning α-times integrated cosine functions with densely defined generators are given as in [8], and in [16] when integrated cosine functions are exponentially bounded. Unfortunately, the generator of a local C-cosine function or a local α-times integrated cosine function may not be densely defined except for the case $\alpha = 0$ and $\mathrm{C} = I$. In this case, the adjoint of a closed linear operator $A : \mathrm{D}(A) \subset \mathrm{X} \to \mathrm{X}$ is the multi-valued function $A^* : \mathrm{D}(A^*) \subset \mathrm{X}^* \to 2^{\mathrm{X}^*}$ defined by $\mathrm{D}(A^*) = \{x^* \in \mathrm{X}^* \mid$ there exists a $y^*_{x^*} \in \mathrm{X}^*$ such that $< x^*, Ax >=< y^*_{x^*}, x >$ for all $x \in \mathrm{D}(A)\}$ and $A^*x^* = \{y^*_{x^*} \in \mathrm{X}^* \mid < x^*, Ax >=< y^*_{x^*}, x >$ for all $x \in \mathrm{D}(A)\}$. In particular, we write $A^*x^* = y^*_{x^*}$ for $x^* \in \mathrm{D}(A^*)$ if A is densely defined, where 2^{X^*} denotes the power set of X^* and either $< x^*, x >$ or $< x, x^* >$ denotes the value of x^* at x for all $x \in \mathrm{X}$ and $x^* \in \mathrm{X}^*$. Moreover, a function u is called a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{f}, \mathrm{x}, \mathrm{y})$, if $< u(\cdot), x^* >\in W^{2,1}_{loc}([0, T_0))$, $< u(t), x^* > |_{t=0} =< x, x^* >$, $\frac{d}{dt} < u(t), x^* > |_{t=0} =< y, x^* >$ and $\frac{d^2}{dt^2} < u(t), x^* >=< u(t), y^* > + < f(t), x^* >$ for a.e. $0 \le t < T_0$ whenever $x^* \in \mathrm{D}(A^*)$ and $y^* \in \mathrm{A}^*x^*$. Here $W^{2,1}_{loc}([0, T_0)) = \{v \mid v : [0, T_0) \to \mathbb{F}$ is continuously differentiable, v' is differentiable a.e. on $[0, T_0)$ and v'' is locally Lebesque integrable on $[0, T_0)\}$. The purpose of this paper is to obtain some generalization theorems concerning local α-times integrated C-cosine functions for $\alpha \ge 0$ when their generators may not be densely defined. We first investigate an important result (see Lemma 2.1 below) which has been deduced by Ball in [3] when A is densely defined. Under the assumption $\mathrm{C}^{-1}A\mathrm{C} = A$. We show that A generates a nondegenerate local $(\alpha+1)$-times (respectively, α-times) integrated C-cosine function on X if and only if $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x + \mathrm{j}_{\alpha-1} * \mathrm{C}g(\cdot), 0, 0)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$(respectively, in $\mathrm{C}^1([0, T_0), \mathrm{X})$) for all $x \in \mathrm{X}$ and $g \in \mathrm{L}^1_{loc}([0, T_0), \mathrm{X})$ if and only if $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$ (respectively, in $\mathrm{C}^1([0, T_0), \mathrm{X})$) for all $x \in \mathrm{X}$, where $\alpha > 0$ (see Theorems 2.4 and 2.5 below). Here $\mathrm{j}_\beta(t) = \frac{t^\beta}{\Gamma(\beta+1)}$for $\beta > -1$ and $t > 0$. Applying these results, we then show that A generates a nondegenerate local 1-times (respectively, 0-times) integrated C-cosine function on X if and only if $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}g(\cdot), 0, \mathrm{C}x)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$(respectively, in $\mathrm{C}^1([0, T_0), \mathrm{X})$) for all $x \in \mathrm{X}$ and $g \in \mathrm{L}^1_{loc}([0, T_0), \mathrm{X})$ if and only if $\mathrm{ACP}_2(\mathrm{A}, 0, 0, \mathrm{C}x)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$(respectively, in $\mathrm{C}^1([0, T_0), \mathrm{X})$) for all $x \in \mathrm{X}$ (see Theorems 2.6 and 2.7 below), which can be applied to show that A generates a nondegenerate local (0-times integrated) C-cosine function on X if and only if $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}g(\cdot), \mathrm{C}x, \mathrm{C}y)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$ for all $x, y \in \mathrm{X}$ and $g \in \mathrm{L}^1_{loc}([0, T_0), \mathrm{X})$ if and only if $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}g(\cdot), \mathrm{C}x, 0)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$ for all $x \in \mathrm{X}$ and $g \in \mathrm{L}^1_{loc}([0, T_0), \mathrm{X})$ if and only if $\mathrm{ACP}_2(\mathrm{A}, 0, \mathrm{C}x, \mathrm{C}y)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$ for all $x, y \in \mathrm{X}$ if and only if $\mathrm{ACP}_2(\mathrm{A}, 0, \mathrm{C}x, 0)$ has a unique weak solution in $\mathrm{C}([0, T_0), \mathrm{X})$ for all $x \in \mathrm{X}$ (see Theorem 2.8 below). Our results are still new even when $\alpha = 0$. An illustrative example concerning these theorems is also presented in the final part of this paper.

2. Existence Theorems

In this section, we always assume that $\mathrm{C} \in \mathrm{B}(\mathrm{X})$ is an injection. We first investigate an important lemma which is used in the proofs of the following theorems, and has been obtained by Ball in [3] when A is densely defined.

Lemma 2.1. *Let $A : D(A) \subset X \to X$ be a closed linear operator. Assume that $x_0, y_0 \in X$ and $< y^*, x_0 > = < x^*, y_0 >$ for all $x^* \in D(A^*)$ and $y^* \in A^*x^*$. Then $x_0 \in D(A)$ and $Ax_0 = y_0$.*

Proof. If not, then there exist $x^*, y^* \in X^*$ such that $y^*(x_0) + x^*(y_0) \neq 0$ and $y^*(x) + x^*(Ax) = 0$ for all $x \in D(A)$, and so $< -y^*, x > = < x^*, Ax >$ for all $x \in D(A)$. Hence $x^* \in D(A^*)$ and $-y^* \in A^*x^*$. By hypothesis, we have $< -y^*, x_0 > = < x^*, y_0 >$ or equivalently, $y^*(x_0) + x^*(y_0) = 0$. We obtain a contradiction. Consequently, $x_0 \in D(A)$ and $Ax_0 = y_0$. ■

By slightly modifying the proofs of [11. Proposition 1.5] and [11. Lemma 1.6], the next proposition and lemma are also attained, and so their proofs are omitted.

Proposition 2.2. *Let A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X. Then*

(2.1) *$C(0) = C$ on X if $\alpha = 0$, and $C(0) = 0$ (the zero operator) on X if $\alpha > 0$;*

(2.2) *C is injective and $C^{-1}AC = A$;*

(2.3) *$C(t)x \in D(A)$ and $AC(t)x = C(t)Ax$ for all $x \in D(A)$ and $0 \leq t < T_0$;*

(2.4) *$\int_0^t \int_0^s C(r)xdrds \in D(A)$ and $A \int_0^t \int_0^s C(r)xdrds = C(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $0 \leq t < T_0$;*

(2.5) *$R(C(t)) \subset \overline{D(A)}$ for all $0 \leq t < T_0$.*

Lemma 2.3. *Let A be the generator of a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X, and let $0 < t_0 < T_0$ be fixed. Assume that $u \in C([0, t_0), X)$ satisfies $u(t) = A \int_0^t (t - s)u(s)ds$ for all $0 \leq t < t_0$. Then $u \equiv 0$ on $[0, t_0)$.*

Theorem 2.4. *Let $\alpha > 0$, and $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent :*

(*i*) *A generates a nondegenerate local $(\alpha+1)$-times integrated C-cosine function $S(\cdot)$ on X;*

(*ii*) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(A, j_{\alpha-1}(\cdot)Cx + j_{\alpha-1} * Cg(\cdot), 0, 0)$ has a unique weak solution in $C([0, T_0), X)$;*

(*iii*) *For each* $x \in X$, $\mathrm{ACP}_2(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$ *has a unique weak solution in* $C([0, T_0), X)$.

Here $L^1_{loc}([0, T_0), X)$ *denotes the set of all locally Bochner integrable functions from* $[0, T_0)$ *into X and* $j_\beta * g(t) = \int_0^t j_\beta(t-s)g(s)ds$ *for all* $0 \leq t < T_0$ *and* $g \in L^1_{loc}([0, T_0), X)$. *Moreover,*

(*i*) $\|S(t)\| \leq Ke^{\omega t}$ *for all* $t \geq 0$ *and for some* $K, \omega \geq 0$ *if and only if for each* $x \in X$, $\|u(t, Cx)\| \leq Ke^{\omega t}\|x\|$ *for all* $t \geq 0$;

(*ii*) $\|S(t+h) - S(t)\| \leq Khe^{\omega(t+h)}$ *for all* $t, h \geq 0$ *and for some* $K, \omega \geq 0$ *if and only if for each* $x \in X$, $\|u(t+h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t+h)}\|x\|$ *for all* $t, h \geq 0$;

(*iii*) *For each* $0 < t_0 < T_0$, $\|S(t+h)-S(t)\| \leq K_{t_0}h$ *for all* $0 \leq t, h \leq t+h \leq t_0$ *and for some* $K_{t_0} > 0$ *if and only if for each* $x \in X$ *and* $0 < t_0 < T_0$, $\|u(t+h, Cx) - u(t, Cx)\| \leq K_{t_0}h\|x\|$ *for all* $0 \leq t, h \leq t+h \leq t_0$ *and for some* $K_{t_0} > 0$.

Proof. (i)⇒(ii). Indeed, if A is the generator of a nondegenerate local $(\alpha+1)$-times integrated C-cosine function $\mathrm{S}(\cdot)$ on X and $x \in$ X is given. Then for $x^* \in$ $\mathrm{D}(\mathrm{A}^*)$ and $y^* \in \mathrm{A}^*x^*$, we have $< S(t)x, x^* >= \int_0^t\int_0^s < S(r)x, y^* > drds + \mathrm{j}_{\alpha+1}(t) < \mathrm{C}x, x^* >$ for all $0 \leq t < T_0$, and so

$$\frac{d}{dt} < S(t)x, x^* >= \int_0^t < S(s)x, y^* > ds + \mathrm{j}_\alpha(t) < \mathrm{C}x, x^* >$$

for all $0 \leq t < T_0$. Hence

$$\frac{d^2}{dt^2} < S(t)x, x^* >=< S(t)x, y^* > +\mathrm{j}_{\alpha-1}(t) < \mathrm{C}x, x^* >$$

for all $0 < t < T_0$. Now if $g \in \mathrm{C}([0, T_0), \mathrm{X})$ is given, then

$$\begin{aligned} &< \int_0^t S(t-s)g(s)ds, x^* > \\ &= \int_0^t < S(t-s)g(s)ds, x^* > ds \\ &= \int_0^t < \tilde{\tilde{S}}(t-s)g(s), y^* > ds + \int_0^t < \mathrm{j}_{\alpha+1}(t-s)\mathrm{C}g(s), x^* > ds \end{aligned}$$

for all $0 \leq t < T_0$. Here $\tilde{\tilde{S}}(t)y = \int_0^t\int_0^s S(r)ydrds$ for all $0 \leq t < T_0$ and $y \in$ X. By differentiation, we have

$$\begin{aligned} &\frac{d}{dt} < \int_0^t S(t-s)g(s)ds, x^* > \\ &= \int_0^t < \tilde{S}(t-s)g(s), y^* > ds + \int_0^t < \mathrm{j}_\alpha(t-s)\mathrm{C}g(s), x^* > ds \end{aligned}$$

for all $0 \le t < T_0$, and

$$\frac{d^2}{dt^2} < \int_0^t S(t-s)g(s)ds, x^* >$$
$$= \int_0^t < S(t-s)g(s), y^* > ds + \int_0^t < \mathrm{j}_{\alpha-1}(t-s)\mathrm{C}g(s), x^* > ds$$

for all $t \in (0, T_0)$, where $\widetilde{S}(t)y = \int_0^t S(r)ydr$ for all $0 \le t < T_0$ and $y \in \mathrm{X}$. Next we set $u(\cdot) = S(\cdot)x + S * g(\cdot)$, then $u \in \mathrm{C}([0, T_0), \mathrm{X})$, $u(0) = 0$ and $\frac{d}{dt} < u(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u(t), x^* >=< u(t), y^* > + < \mathrm{j}_{\alpha-1}(t)\mathrm{C}x + \mathrm{j}_{\alpha-1} * \mathrm{C}g(t), x^* >$ for $t \in (0, T_0)$, which implies that $u \in \mathrm{C}([0, T_0), \mathrm{X})$ is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x + \mathrm{j}_{\alpha-1} * \mathrm{C}g(\cdot), 0, 0)$ satisfying $u(0) = 0$. Finally, we turn to the case that g is only an $\mathrm{L}^1_{loc}([0, T_0), \mathrm{X})$ function and $\{g_m\}_{m=1}^{\infty}$ is a sequence in $\mathrm{C}([0, T_0), \mathrm{X})$ such that $g_m \to g$ in $\mathrm{L}^1([0, t_0], \mathrm{X})$ for all $0 < t_0 < T_0$. We define

$$u(\cdot) = S(\cdot)x + S * g(\cdot)$$

and

$$u_m(\cdot) = S(\cdot)x + S * g_m(\cdot)$$

for $m \in \mathbb{N}$, then $\|u_m(t) - u(t)\| \le \int_0^{t_0} \sup_{\tau \in [0, t_0]} \|S(\tau)\| \|g_m(s) - g(s)\| ds$ for all $0 \le t \le t_0 < T_0$, and so $u_m(\cdot) \to u(\cdot)$ uniformly on compact subsets of $[0, T_0)$. Hence $u(\cdot)$ is continuous on $[0, T_0)$. The previous argument shows that $u_m(0) = 0$, $\frac{d}{dt} < u_m(t), x^* > |_{t=0}$=0 and $\frac{d^2}{dt^2} < u_m(t), x^* >=< u_m(t), y^* > + < \mathrm{j}_{\alpha-1}(t)\mathrm{C}x, x^* > + < \mathrm{j}_{\alpha-1} * \mathrm{C}g_m(t), x^* >$ for $t \in (0, T_0)$. By integration, we have

$$\frac{d}{dt} < u_m(t), x^* > = \int_0^t < u_m(s), y^* > ds + < \mathrm{j}_\alpha(t)\mathrm{C}x, x^* >$$
$$+ < \mathrm{j}_\alpha * \mathrm{C}g_m(t), x^* >$$

and

$$< u_m(t), x^* > = \int_0^t \int_0^s < u_m(r), y^* > drds + < \mathrm{j}_{\alpha+1}(t)\mathrm{C}x, x^* >$$
$$+ < \mathrm{j}_{\alpha+1} * \mathrm{C}g_m(t), x^* >$$

for all $0 \le t < T_0$. Letting $m \to \infty$, we get that

$$\int_0^{\cdot} < u_m(s), y^* > ds + < \mathrm{j}_\alpha(\cdot)\mathrm{C}x, x^* > + < \mathrm{j}_\alpha * \mathrm{C}g_m(\cdot), x^* >$$
$$\to \int_0^{\cdot} < u(s), y^* > ds + < \mathrm{j}_\alpha(\cdot)\mathrm{C}x, x^* > + < \mathrm{j}_\alpha * \mathrm{C}g(\cdot), x^* >$$

uniformly on compact subsets of $[0, T_0)$ and

$$< u(t), x^* > = \int_0^t \int_0^s < u(r), y^* > drds + < \mathrm{j}_{\alpha+1}(t)\mathrm{C}x, x^* >$$
$$+ < \mathrm{j}_{\alpha+1} * \mathrm{C}g(t), x^* >$$

for all $0 \le t < T_0$. In particular, $u(0) = 0$, $\frac{d}{dt} < u(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u(t), x^* > = < u(t), y^* > + < \mathrm{j}_{\alpha-1}(t)\mathrm{C}x, x^* > + < \mathrm{j}_{\alpha-1} * \mathrm{C}g(t), x^* >$ for $t \in (0, T_0)$, which implies that $u \in \mathrm{C}([0, T_0), \mathrm{X})$ is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot) \mathrm{C}x + \mathrm{j}_{\alpha-1} * \mathrm{C}g(\cdot), 0, 0)$ satisfying $u(0) = 0$. To prove the uniqueness, let v be another weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x + \mathrm{j}_{\alpha-1} * \mathrm{C}g(\cdot), 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ and $w(\cdot) = u(\cdot) - v(\cdot)$ on $[0, T_0)$. Applying the continuity of w, we get that

$$< w(t), x^* > = \left\langle \int_0^t \int_0^s w(r)drds, y^* \right\rangle$$
$$\text{for all } 0 \le t < T_0,\ x^* \in \mathrm{D}(\mathrm{A}^*) \text{ and } y^* \in \mathrm{A}^* x^*,$$

which together with Lemma 2.1 implies that $\int_0^t \int_0^s w(r)drds \in \mathrm{D}(\mathrm{A})$ and $\mathrm{A} \int_0^t \int_0^s w(r)drds = w(t)$ for all $0 \le t < T_0$. It follows from Lemma 2.3 that we have $w = 0$ on $[0, T_0)$ or equivalently, $u = v$ on $[0, T_0)$.

(iii)⇒(i). Indeed, if the unique weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ is denoted by $w(\cdot, \mathrm{C}x)$ for all $x \in \mathrm{X}$. We define the map $S(t) : \mathrm{X} \to \mathrm{X}$ by $S(t)x = w(t, \mathrm{C}x)$ for all $x \in \mathrm{X}$ and $0 \le t < T_0$. Clearly, $S(\cdot)x : [0, T_0) \to \mathrm{X}$ is continuous for all $x \in \mathrm{X}$. It follows from the uniqueness of weak solutions of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ and Lemma 2.1 that $S(t)$ is linear for all $0 \le t < T_0$, $S(\cdot)(= \{S(t) \,|\, 0 \le t < T_0\})$ commutes with C and is nondegenerate. Next we shall show that $S(\cdot) \subset \mathrm{B}(\mathrm{X})$. By the closed graph theorem, we need only to show that the linear map $\eta : \mathrm{X} \to \mathrm{C}([0, T_0), \mathrm{X})$ defined by $\eta(x) = S(\cdot)x$ for $x \in \mathrm{X}$, is a continuous function from the Banach space X into the Frechet space $\mathrm{C}([0, T_0), \mathrm{X})$ with the quasi-norm $|\cdot|$ defined by $|v| = \sum_{k=1}^{\infty} \frac{\|v\|_k}{2^k(1+\|v\|_k)}$ for $v \in \mathrm{C}([0, T_0), \mathrm{X})$, where $\|v\|_k = \max_{t \in [0,k]} \|v(t)\|$. Indeed, if $\{x_m\}_{m=1}^{\infty}$ is a sequence in X such that $x_m \to x$ in X and $\eta(x_m) \to u(\cdot)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ as $m \to \infty$. Then for $x^* \in \mathrm{D}(\mathrm{A}^*)$ and $y^* \in \mathrm{A}^* x^*$, we have

$$\int_0^t \int_0^s < S(r)x_m, y^* > drds$$
$$= \int_0^t \int_0^s \frac{d^2}{dt^2} < S(r)x_m, x^* > drds - \int_0^t \int_0^s < \mathrm{j}_{\alpha-1}(r)\mathrm{C}x_m, x^* > drds$$
$$= < S(t)x_m, x^* > - \mathrm{j}_{\alpha+1}(t) < \mathrm{C}x_m, x^* >$$

for all $0 \le t < T_0$, and so

$$< u(t), x^* >= \mathrm{j}_{\alpha+1}(t) < \mathrm{C}x, x^* > + \int_0^t \int_0^s < u(r), y^* > drds$$

for all $0 \le t < T_0$. Hence $< u(\cdot), x^* >\in W^{2,1}_{loc}([0, T_0))$, $< u(t), x^* > |_{t=0} = \frac{d}{dt} < u(t), x^* > |_{t=0} = 0$ and $\frac{d^2}{dt^2} < u(t), x^* >= \mathrm{j}_{\alpha-1}(t) < \mathrm{C}x, x^* > + < u(t), y^* >$ for a.e. $0 \le t < T_0$, which implies that u is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$. The uniqueness of weak solutions in $\mathrm{C}([0, T_0), \mathrm{X})$ implies that $u(\cdot) = S(\cdot)x = \eta(x)$. Consequently, η is closed. In order, we shall show that $S(\cdot)$ is a local $(\alpha + 1)$-times integrated C-cosine function on X. Indeed, if $x \in \mathrm{X}$ and $0 \le s < T_0$ are given. We first assume that $\alpha \ge 1$ and define

$$\begin{aligned} v_s(t) &= \frac{1}{\Gamma(\alpha+1)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^\alpha S(r)\mathrm{C}xdr \\ &+ \int_{|t-s|}^t (s-t+r)^\alpha S(r)\mathrm{C}xdr + \int_{|t-s|}^s (t-s+r)^\alpha S(r)\mathrm{C}xdr \\ &+ \int_0^{|t-s|} (|t-s|+r)^\alpha S(r)\mathrm{C}xdr\} \end{aligned}$$

for all $0 \le t \le t + s < T_0$. Then for $x^* \in \mathrm{D}(\mathrm{A}^*)$ and $y^* \in \mathrm{A}^*x^*$, we obtain from [11, Lemma 2.1] that

$$\begin{aligned} &< v_s(t), y^* > \\ &= \frac{1}{\Gamma(\alpha+1)}\{[\int_0^{t+s} - \int_0^t \\ &- \int_0^s](t+s-r)^\alpha[\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > - < \mathrm{j}_{\alpha-1}(r)\mathrm{C}^2x, x^* >]dr \\ &+ \int_{|t-s|}^t (s-t+r)^\alpha[\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > - < \mathrm{j}_{\alpha-1}(r)\mathrm{C}^2x, x^* >]dr \\ &+ \int_{|t-s|}^s (t-s+r)^\alpha[\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > - < \mathrm{j}_{\alpha-1}(r)\mathrm{C}^2x, x^* >]dr \\ &+ \int_0^{|t-s|} (|t-s|+r)^\alpha[\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > - < \mathrm{j}_{\alpha-1}(r)\mathrm{C}^2x, x^* >]dr\} \\ &= \frac{1}{\Gamma(\alpha+1)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^\alpha \frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > dr \\ &+ \int_{|t-s|}^t (s-t+r)^\alpha \frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > dr \end{aligned}$$

$$+\int_{|t-s|}^{s}(t-s+r)^{\alpha}\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > dr$$
$$+\int_{0}^{|t-s|}(|t-s|+r)^{\alpha}\frac{d^2}{dr^2} < S(r)\mathrm{C}x, x^* > dr\}$$

for all $0 \leq t \leq t+s < T_0$. Using integration by parts twice, we also have, for $0 \leq t \leq t+s < T_0$.

$$\begin{aligned}
&< v_s(t), y^* > \\
&= \frac{1}{\Gamma(\alpha-1)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr \\
&+ \int_{|t-s|}^{t}(s-t+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr + \int_{|t-s|}^{s}(t-s+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr \\
&+ \int_0^{|t-s|}(|t-s|+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr\} - 2\mathrm{j}_{\alpha-1}(s) < S(t)\mathrm{C}x, x^* > \\
&-2\mathrm{j}_{\alpha-1}(t) < S(s)\mathrm{C}x, x^* >,
\end{aligned}$$

$\frac{d}{dt} < v_s(t), x^* > |_{t=0} = 0 = < v_s(t), x^* > |_{t=0}$ and

$$\begin{aligned}
&\frac{d^2}{dt^2} < v_s(t), x^* > \\
&= \frac{1}{\Gamma(\alpha-1)}\{[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr \\
&\quad + \int_{|t-s|}^{t}(s-t+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr + \int_{|t-s|}^{s}(t-s+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr \\
&\quad + \int_0^{|t-s|}(|t-s|+r)^{\alpha-2} < S(r)\mathrm{C}x, x^* > dr\} - 2\mathrm{j}_{\alpha-1}(s) < S(t)\mathrm{C}x, x^* > \\
&= < v_s(t), y^* > + 2 < \mathrm{j}_{\alpha-1}(t)\mathrm{CS}(s)x, x^* >
\end{aligned}$$

when $\alpha > 1$. Similarly, we can show that for $0 \leq t \leq t+s < T_0$

$$\begin{aligned}
< v_s(t), y^* > &= < S(t+s)\mathrm{C}x, x^* > + < S(|t-s|)\mathrm{C}x, x^* > \\
&\quad -2 < S(s)\mathrm{C}x, x^* > -2 < S(t)\mathrm{C}x, x^* >,
\end{aligned}$$

$\frac{d}{dt} < v_s(t), x^* > |_{t=0} = 0 = < v_s(t), x^* > |_{t=0}$ and

$$\begin{aligned}
\frac{d^2}{dt^2} < v_s(t), x^* > &= < S(t+s)\mathrm{C}x, x^* > + < S(|t-s|)\mathrm{C}x, x^* > \\
&\quad -2 < S(t)\mathrm{C}x, x^* > \\
&= < v_s(t), y^* > + 2 < \mathrm{j}_{\alpha-1}(t)\mathrm{CS}(s)x, x^* >
\end{aligned}$$

when $\alpha = 1$. Applying the uniqueness of weak solutions of $\mathrm{ACP}_2(\mathrm{A}, 2\mathrm{j}_{\alpha-1}(\cdot)$ $\mathrm{C}S(s)x, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$, we get that $v_s(t) = 2S(t)S(s)x$ for all $0 \le t \le t + s < T_0$. Consequently, $S(\cdot)$ is a nondegenerate local $(\alpha+1)$-times integrated C-cosine function on X when $\alpha \ge 1$. We now turn to the case $0 < \alpha < 1$. By hypothesis, $\int_0^{\cdot} w(s, \mathrm{C}x)ds$ is a unique weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_\alpha(\cdot)\mathrm{C}x, 0, 0)$ in $\mathrm{C}^1([0, T_0), \mathrm{X})$ for all $x \in \mathrm{X}$. Just as in the proof of the case $\alpha > 1$, we can show that $\widetilde{S}(\cdot)$ is a nondegenerate local $(\alpha+2)$-times integrated C-cosine function on X. Here $\widetilde{S}(t)x = \int_0^t S(s)xds$ for all $0 \le t < T_0$ and $x \in \mathrm{X}$. An easy computation shows that $S(\cdot)$ is a nondegenerate local $(\alpha+1)$-times integrated C-cosine function on X. Finally, we shall show that A is its generator. Indeed, if B denotes the generator of $S(\cdot)$ and $x \in \mathrm{D(B)}$ is given. Then for $x^* \in \mathrm{D(A^*)}$ and $y^* \in \mathrm{A}^*x^*$, we have

$$\begin{aligned} < S(t)x, y^* > &= \frac{d^2}{dt^2} < S(t)x, x^* > - < \mathrm{j}_{\alpha-1}(t)\mathrm{C}x, x^* > \\ &= < S(t)\mathrm{B}x, x^* > \end{aligned}$$

for a.e. $0 \le t < T_0$ because $S(\cdot)x$ is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$. The strong continuity of $S(\cdot)$ implies that $< S(t)x, y^* > = < S(t)\mathrm{B}x, x^* >$ for all $0 \le t < T_0$. Applying Lemma 2.1, we get that $S(t)x \in \mathrm{D(A)}$ and $\mathrm{A}S(t)x = S(t)\mathrm{B}x$ for all $0 \le t < T_0$. Since $S(\cdot)\mathrm{B}x$ is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{CB}x, 0, 0)$, we also have

$$\begin{aligned} &< \int_0^t \int_0^s S(r)\mathrm{B}xdrds, y^* > \\ = &\int_0^t \int_0^s < S(r)\mathrm{B}x, y^* > drds \\ = &\int_0^t \int_0^s [\frac{d^2}{dr^2} < S(r)\mathrm{B}x, x^* > - < \mathrm{j}_{\alpha-1}(r)\mathrm{CB}x, x^* >]drds \\ = &< S(t)\mathrm{B}x, x^* > - < \mathrm{j}_{\alpha+1}(t)\mathrm{CB}x, x^* > \end{aligned}$$

for all $x^* \in \mathrm{D(A^*)}$, $y^* \in \mathrm{A}^*x^*$ and $0 \le t < T_0$. Applying Lemma 2.1 again, we get that
$\int_0^t \int_0^s S(r)\mathrm{B}xdrds \in \mathrm{D(A)}$ and $\mathrm{A}\int_0^t \int_0^s S(r)\mathrm{B}xdrds = S(t)\mathrm{B}x - \mathrm{j}_{\alpha+1}(t)\mathrm{CB}x$ for all $0 \le t < T_0$, and so $-\mathrm{j}_{\alpha+1}(t)\mathrm{C}x = \int_0^t \int_0^s S(r)\mathrm{B}xdrds - S(t)x \in \mathrm{D(A)}$ and

$$\begin{aligned} -\mathrm{j}_{\alpha+1}(t)\mathrm{AC}x &= \mathrm{A}\int_0^t \int_0^s S(r)\mathrm{B}xdrds - \mathrm{A}S(t)x \\ &= [S(t)\mathrm{B}x - \mathrm{j}_{\alpha+1}(t)\mathrm{CB}x] - S(t)\mathrm{B}x \\ &= -\mathrm{j}_{\alpha+1}(t)\mathrm{CB}x \end{aligned}$$

for all $0 \leq t < T_0$. Hence $x \in \mathrm{D}(\mathrm{C}^{-1}\mathrm{AC})$ and $\mathrm{C}^{-1}\mathrm{AC}x = \mathrm{B}x$. Having shown that $\mathrm{B} \subset \mathrm{C}^{-1}\mathrm{AC}$. We next show that $\mathrm{A} \subset \mathrm{B}$. Indeed, if $x \in \mathrm{D}(\mathrm{A})$ is given, then $\int_0^t \int_0^s S(r)xdrds$, $\int_0^t \int_0^s S(r)\mathrm{A}xdrds \in \mathrm{D}(\mathrm{A})$,

$$S(t)x = \mathrm{j}_{\alpha+1}(t)\mathrm{C}x + \mathrm{A}\int_0^t \int_0^s S(r)xdrds \tag{2.6}$$

and

$$S(t)\mathrm{A}x = \mathrm{j}_{\alpha+1}(t)\mathrm{CA}x + \mathrm{A}\int_0^t \int_0^s S(r)\mathrm{A}xdrds \tag{2.7}$$

for all $0 \leq t < T_0$. It is easy to see from (2.6) and (2.7) that the function $t \rightarrow \int_0^t \int_0^s S(r)\mathrm{A}xdrds - \mathrm{A}\int_0^t \int_0^s S(r)xdrds$ is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, 0, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$, and hence it must be the zero function on $[0, T_0)$ or equivalently, $\int_0^t \int_0^s S(r)\mathrm{A}xdrds = \mathrm{A}\int_0^t \int_0^s S(r)xdrds$ for all $0 \leq t < T_0$, which together with (2.6) implies that $x \in \mathrm{D}(\mathrm{B})$ and $\mathrm{B}x = \mathrm{A}x$. Consequently, $\mathrm{A} = \mathrm{B}$.

Theorem 2.5. *Let $\alpha > 0$, and $A : D(A) \subset X \rightarrow X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent :*

- (i) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(A, j_\alpha(\cdot)Cx + j_\alpha * Cg(\cdot), 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*
- (ii) *For each $x \in X$, $\mathrm{ACP}_2(A, j_\alpha(\cdot)Cx, 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*
- (iii) *A generates a nondegenerate local α-times integrated C-cosine function $C(\cdot)$ on X;*
- (iv) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(A, j_{\alpha-1}(\cdot)Cx + j_{\alpha-1} * Cg(\cdot), 0, 0)$ has a unique weak solution in $C^1([0, T_0), X)$;*
- (v) *For each $x \in X$,$\mathrm{ACP}_2(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$ has a unique weak solution in $C^1([0, T_0), X)$.*

Moreover, $\|C(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$ and for some $K, \omega \geq 0$ if and only if for each $x \in X$, the unique weak solution $u(\cdot, Cx)$ of $\mathrm{ACP}_2(\mathrm{A}, j_{\alpha-1}(\cdot)\mathrm{C}x, 0, 0)$ satisfies $\|u(t + h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t+h)}\|x\|$ for all $t, h \geq 0$.

Proof. The equivalence relations (i)-(iii) follow from [11, Theorem 2.3]. To show that (iii)$\Rightarrow$(iv). Indeed, if $\mathrm{C}(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X with generator A, then $S(\cdot)$ is a nondegenerate local $(\alpha+1)$-times integrated C-cosine function on X with generator A and satisfies $S(\cdot)x \in \mathrm{C}^1([0, T_0), \mathrm{X})$ for all $x \in \mathrm{X}$, where $S(t)x = \int_0^t \mathrm{C}(r)xdr$. It follows from Theorem 2.4 that $S(\cdot)x + S * g(\cdot)$ is the unique weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{j}_{\alpha-1}(\cdot)\mathrm{C}x +$

$j_{\alpha-1} * Cg(\cdot), 0, 0)$ in $C^1([0, T_0), X)$ for all $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$. Finally, we shall show that (v)⇒(iii). Indeed, if $u(\cdot, Cx)$ denotes the unique weak solution of $ACP_2(A, j_{\alpha-1}(\cdot)Cx, 0, 0)$ in $C^1([0, T_0), X)$ and $S(t) : X \to X$ is defined by $S(t)x = u(t, Cx)$ for all $0 \le t < T_0$ and $x \in X$. Applying Theorem 2.4, we get that $S(\cdot)$ is a nondegenerate local $(\alpha + 1)$-times integrated C-cosine function on X with generator A, which implies that $C(\cdot)$ is a nondegenerate local α-times integrated C-cosine function on X with generator A, where $C(t)x = \frac{d}{dt}S(t)x$ for all $0 \le t < T_0$ and $x \in X$.

Applying Theorem 2.5, the next theorem concerning local 1-times integrated C-cosine functions is also obtained.

Theorem 2.6. *Let $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent :*

- (i) *A generates a nondegenerate local 1-times integrated C-cosine function $C(\cdot)$ on X;*
- (ii) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $ACP_2(A, Cg(\cdot), 0, Cx)$ has a unique weak solution in $C([0, T_0), X)$;*
- (iii) *For each $x \in X$, $ACP_2(A, 0, 0, Cx)$ has a unique weak solution $u(\cdot, Cx)$ in $C([0, T_0), X)$.*

Moreover,

- (i) *$\|C(t)\| \le Ke^{\omega t}$ for all $t \ge 0$ and for some $K, \omega \ge 0$ if and only if for each $x \in X$, $\|u(t, Cx)\| \le Ke^{\omega t}\|x\|$ for all $t \ge 0$;*
- (ii) *$\|C(t + h) - C(t)\| \le Khe^{\omega(t+h)}$ for all $t, h \ge 0$ and for some $K, \omega \ge 0$ if and only if for each $x \in X$, $\|u(t + h, Cx) - u(t, Cx)\| \le Khe^{\omega(t+h)}\|x\|$ for all $t, h \ge 0$;*
- (iii) *For each $0 < t_0 < T_0$, $\|C(t+h)-C(t)\| \le K_{t_0}h$ for all $0 \le t, h \le t+h \le t_0$ and for some $K_{t_0} > 0$ if and only if for each $x \in X$ and $0 < t_0 < T_0$, $\|u(t + h, Cx) - u(t, Cx)\| \le K_{t_0}h\|x\|$ for all $0 \le t, h \le t + h \le t_0$ and for some $K_{t_0} > 0$.*

Proof. We first show that (i)⇒(ii). Indeed, if A generates a nondegenerate local 1-times integrated C-cosine function on X. Then for each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, we obtain from Theorem 2.5 that $ACP_2(A, Cx + j_0 * Cg(\cdot), 0, 0)$ has a unique weak solution u in $C^1([0, T_0), X)$ which satisfies $u(0) = 0$, so that for each $x^* \in D(A^*)$ and $y^* \in A^*x^*$, we have $< u'(t), x^* > |_{t=0} = \frac{d}{dt} < u(t), x^* > |_{t=0} = 0, < u'(\cdot), x^* >\in W^{2,1}_{loc}([0, T_0))$ and

$$\frac{d^2}{dt^2} < u'(t), x^* > = \frac{d^3}{dt^3} < u(t), x^* >$$
$$= \frac{d}{dt}[< u(t), y^* > + < \mathrm{C}x + \mathrm{j}_0 * \mathrm{C}g(t), x^* >]$$
$$=< u'(t), y^* > + < \mathrm{C}g(t), x^* >$$

for a.e. $0 \leq t < T_0$. Clearly, $\frac{d}{dt} < u'(t), x^* >= \frac{d^2}{dt^2} < u(t), x^* >=< u(t), y^* > + < \mathrm{C}x + \mathrm{j}_0 * \mathrm{C}g(t), x^* >$ for all $0 \leq t < T_0$. In particular, $\frac{d}{dt} < u'(t), x^* > |_{t=0} =< \mathrm{C}x, x^* >$. It follows that u' is a weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}g, 0, \mathrm{C}x)$ in $\mathrm{C}([0, T_0), \mathrm{X})$. The uniqueness of weak solutions of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}g, 0, \mathrm{C}x)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ follows from the uniqueness of weak solutions of $\mathrm{ACP}_2(\mathrm{A}, 0, 0, 0)$ in $\mathrm{C}([0, T_0), \mathrm{X})$. In order, we show that (iii)⇒(i). Indeed, if $u(\cdot, x)$ denotes the unique weak solution of $\mathrm{ACP}_2(\mathrm{A}, 0, 0, \mathrm{C}x)$ in $\mathrm{C}([0, T_0), \mathrm{X})$ for all $x \in \mathrm{X}$, then $v = \mathrm{j}_0 * u$ is the unique weak solution of $\mathrm{ACP}_2(\mathrm{A}, \mathrm{C}x, 0, 0)$ in $\mathrm{C}^1([0, T_0), \mathrm{X})$. Applying Theorem 2.5, we get that A generates a nondegenerate local 1-times integrated C-cosine function $\mathrm{C}(\cdot)$ on X which is defined by $\mathrm{C}(t)x = u(t, x)$ for all $0 \leq t < T_0$ and $x \in X$.

By slightly modifying the proof of Theorem 2.5, we can apply Theorem 2.6 to prove the next theorem concerning local (0-times integrated) C-cosine functions.

Theorem 2.7. *Let $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent :*

- (i) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cx + j_0 * Cg(\cdot), 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*
- (ii) *For each $x \in X$, $\mathrm{ACP}_2(\mathrm{A}, Cx, 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*
- (iii) *A generates a nondegenerate local (0-times integrated) C-cosine function $C(\cdot)$ on X;*
- (iv) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cg(\cdot), 0, Cx)$ has a unique weak solution in $C^1([0, T_0), X)$;*
- (v) *For each $x \in X$, $\mathrm{ACP}_2(\mathrm{A}, 0, 0, Cx)$ has a unique weak solution $u(\cdot, Cx)$ in $C^1([0, T_0), X)$.*

Moreover, $\|C(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$ and for some $K, \omega \geq 0$ if and only if for each $x \in X$, $\|u(t + h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t+h)}\|x\|$ for all $t, h \geq 0$.

Similarly, we can apply Theorem 2.7 to prove the next theorem concerning local (0-times integrated) C-cosine functions which has been obtained in [9] when A is densely defined.

Theorem 2.8. *Let $A : D(A) \subset X \to X$ be a closed linear operator such that $C^{-1}AC = A$. Then the following are equivalent :*

(*i*) *For each $x, y \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cx+j_1(\cdot)Cy+j_0 * Cg(\cdot), 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(*ii*) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cx + j_0 * Cg(\cdot), 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(*iii*) *For each $x, y \in X$, $\mathrm{ACP}_2(\mathrm{A}, Cx + j_1(\cdot)Cy, 0, 0)$ has a unique strong solution in $C^2([0, T_0), X) \cap C([0, T_0), [D(A)])$;*

(*iv*) *A generates a nondegenerate local (0-times integrated) C-cosine function $C(\cdot)$ on X;*

(*v*) *For each $x, y \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cg(\cdot), Cx, Cy)$ has a unique weak solution in $C([0, T_0), X)$;*

(*vi*) *For each $x \in X$ and $g \in L^1_{loc}([0, T_0), X)$, $\mathrm{ACP}_2(\mathrm{A}, Cg(\cdot), Cx, 0)$ has a unique weak solution in $C([0, T_0), X)$;*

(*vii*) *For each $x, y \in X$, $\mathrm{ACP}_2(\mathrm{A}, 0, Cx, Cy)$ has a unique weak solution in $C([0, T_0), X)$;*

(*viii*) *For each $x \in X$, $\mathrm{ACP}_2(\mathrm{A}, 0, Cx, 0)$ has a unique weak solution $u(\cdot, Cx)$ in $C([0, T_0), X)$.*

Moreover,

(*i*) *$\|C(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$ and for some $K, \omega \geq 0$ if and only if for each $x \in X$, $\|u(t, Cx)\| \leq Ke^{\omega t}\|x\|$ for all $t \geq 0$;*

(*ii*) *$\|C(t + h) - C(t)\| \leq Khe^{\omega(t+h)}$ for all $t, h \geq 0$ and for some $K, \omega \geq 0$ if and only if for each $x \in X$, $\|u(t + h, Cx) - u(t, Cx)\| \leq Khe^{\omega(t+h)}\|x\|$ for all $t, h \geq 0$;*

(*iii*) *For each $0 < t_0 < T_0$, $\|C(t+h)-C(t)\| \leq K_{t_0}h$ for all $0 \leq t, h \leq t+h \leq t_0$ and for some $K_{t_0} > 0$ if and only if for each $x \in X$ and $0 < t_0 < T_0$, $\|u(t + h, Cx) - u(t, Cx)\| \leq K_{t_0}h\|x\|$ for all $0 \leq t, h \leq t + h \leq t_0$ and for some $K_{t_0} > 0$.*

We end this paper with a simple illustrative example. Let $X = C_b(\mathbb{R})$(or $L^\infty(\mathbb{R})$), and A be the maximal differential operator in X defined by $Au = \sum_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $UC_b(\mathbb{R})$ (or $C_0(\mathbb{R})) = \overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [2, Theorem 6.7] that A generates an exponentially bounded, norm continuous 1-times integrated cosine function C$(\cdot)$ on X which is defined by $(C(t)f)(x) = \frac{1}{\sqrt{2\pi}}(\widetilde{\phi}_t * f)(x)$

for all $f \in X$ and $t \geq 0$ if the real-valued polynomial $p(x) = \sum_{j=0}^{k} a_j(ix)^j$ satisfies $\sup_{x\in\mathbb{R}} p(x) < \infty$. Here $\widetilde{\phi_t}$ denotes the inverse Fourier transform of ϕ_t with $\phi_t(x) = \int_0^t cosh(\sqrt{p(x)}s)ds$. Applying Theorem 2.6, we get that for each $f \in X$ and continuous function g on $[0, T_0) \times \mathbb{R}$ with $\int_0^t \sup_{x\in\mathbb{R}} |g(s, x)|ds < \infty$ for all $0 \leq t < T_0$, the function u on $[0, T_0) \times \mathbb{R}$ defined by $u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{\phi_t}(x - y)f(y)dy + \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \widetilde{\phi_{t-s}}(x - y)g(s, y)dyds$ for all $0 \leq t < T_0$ and $x \in \mathbb{R}$, is the unique weak solution of

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} = \displaystyle\sum_{j=0}^{k} a_j(\frac{\partial}{\partial x})^j u(t,x) + g(t,x) \text{ for } t \in (0, T_0) \text{ and a.e. } x \in \mathbb{R}, \\ u(0,x) = 0 \text{ and } \dfrac{\partial u}{\partial t}(0,x) = f(x) \quad \text{ for a.e. } x \in \mathbb{R} \end{cases}$$

in $C([0, T_0), X)$.

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