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# ON NEW HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITIES

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**Abstract.** Inverses of some new inequalities similar to Hilbert's inequality are established. Our results provide new estimates on these types of inequalities.

## 1. Introduction

In recent years several authors [1-9] have given considerable attention to Hilbert's inequalities and Hilbert's type inequalities and their various generalizations. In particular, in 1988, B. G. Pachpatte [1] proved some new integral inequalities similar to Hilbert's inequality [10, p. 226] , The main purpose of this paper is to establish their inverses.

## 2. Main Results

In [1], Pachpatte established the following Hilbert type integral inequality.

**Theorem A** *Let* $h \geq 1, l \geq 1$ *and let* $f(\sigma) \geq 0, g(\tau) \geq$ *for* $\sigma \in (0, x), \tau \in (0, y)$, *where* $x$ *and* $y$ *are positive real numbers and define* $F(s) = \int_0^s f(\sigma)d\sigma$ *and* $G(t) = \int_0^t g(\tau)d\tau$, *for* $s \in (0, x), t \in (0, y)$. *Then*

$$\int_0^x \int_0^y \frac{F^h(s)G^l(t)}{s+t} dsdt \leq \frac{1}{2} hl(xy)^{1/2} \left( \int_0^x \Big(x-s\Big)\Big(F^{h-1}(s)f(s)\Big)^2 ds \right)^{1/2}$$

$$\times \left( \int_0^y \Big(y-t\Big)\Big(G^{l-1}(t)g(t)\Big)^2 dt \right)^{1/2}. \tag{1}$$

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In this section, we first establish a general inverse of inequality (1) as follows.

**Theorem 1.** *Let* $h_i \geq 1$ *and let* $f_i(\sigma_i, \tau_i) \in C^2[(0, x_i) \times (0, y_i), (0, \infty)], i = 1, \ldots, n,$ *where* $x_i$ *and* $y_i$ *are positive real numbers and define* $F_i(s_i, t_i) = \int_0^{s_i} \int_0^{t_i} f_i(\sigma_i, \tau_i) d\sigma_i d\tau_i$, for $s_i \in (0, x_i), t_i \in (0, y_i)$. *Then for* $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$ ,$0 < \beta_i < 1$ *and* $\sum_{i=1}^n \frac{1}{\alpha_i} = \frac{1}{\alpha}$,

$$
\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n} \frac{\prod_{i=1}^n F_i^{h_i}(s_i, t_i)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i t_i\right)^{1/\alpha}} ds_1 dt_1 \cdots ds_n dt_n \geq \prod_{i=1}^n h_i (x_i y_i)^{1/\alpha_i} \\
&\times \left\{ \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i) \right. \\
&\left. \left[ H(h_i, s_i, t_i) + F_i^{h_i - 1}(s_i, t_i) \cdot f(s_i, t_i) \right]^{\beta_i} ds_i dt_i \right\}^{1/\beta_i},
\end{aligned} \tag{2}
$$

*where*

$$
H(h_i, s_i, t_i) = (h_i - 1) F_i^{h_i - 2}(s_i, t_i) \cdot \frac{\partial F_i}{\partial s_i}(s_i, t_i).
$$

*Proof.* From the hypotheses and in view of inverse Hölder integral inequality [11], it is easy to observe that

$$
\begin{aligned}
&F_i^{h_i}(s_i, t_i) \\
&= \int_0^{s_i}\int_0^{t_i} h_i \left( H(h_i, \sigma_i, \tau_i) + F_i^{h_i - 1}(\sigma_i, \tau_i) \cdot \frac{\partial^2 F_i}{\partial \sigma_i \partial \tau_i}(\sigma_i, \tau_i) \right) d\sigma_i d\tau_i \\
&= h_i \int_0^{s_i}\int_0^{t_i} \left( H(h_i, \sigma_i, \tau_i) + F_i^{h_i - 1}(\sigma_i, \tau_i) \cdot f(\sigma_i, \tau_i) \right) d\sigma_i d\tau_i \\
&\geq h_i (s_i t_i)^{1/\alpha_i} \left\{ \int_0^{s_i}\int_0^{t_i} \left( H(h_i, \sigma_i, \tau_i) + F_i^{h_i - 1}(\sigma_i, \tau_i) \cdot f(\sigma_i, \tau_i) \right)^{\beta_i} d\sigma_i d\tau_i \right\}^{1/\beta_i},
\end{aligned}
$$

where

$$
H(h_i, \sigma_i, \tau_i) = (h_i - 1) F_i^{h_i - 2}(\sigma_i, \tau_i) \cdot \frac{\partial F_i}{\partial \sigma_i}(\sigma_i, \tau_i).
$$

Let us note the following means inequality

$$
\prod_{i=1}^n m_i^{1/\alpha_i} \geq \left( \alpha \sum_{i=1}^n \frac{1}{\alpha_i} m_i \right)^{1/\alpha}.
$$

We obtain that

$$
\tag{3}
\begin{aligned}
&\frac{\prod_{i=1}^{n} F_i^{h_i}(s_i,t_i)}{\left(\alpha\displaystyle\sum_{i=1}^{n}\frac{1}{\alpha_i}s_it_i\right)^{1/\alpha}}\\
&\geq \prod_{i=1}^{n} h_i\left\{\int_0^{s_i}\int_0^{t_i}\Big(H(h_i,\sigma_i,\tau_i)+F_i^{h_i-1}(\sigma_i,\tau_i)\cdot f(\sigma_i,\tau_i)\Big)^{\beta_i}d\sigma_i d\tau_i\right\}^{1/\beta_i}.
\end{aligned}
$$

Integrating both sides of (3) over $s_i,t_i$ from $0$ to $x_i,y_i (i=1,2,\ldots,n)$ and in view of Hölder integral inequality and Fubini′s theorem, we observe that

$$
\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^{n} F_i^{h_i}(s_i,t_i)}{\left(\alpha\displaystyle\sum_{i=1}^{n}\frac{1}{\alpha_i}s_it_i\right)^{1/\alpha}}ds_1dt_1\cdots ds_ndt_n \geq \prod_{i=1}^{n} h_i\\
&\times\int_0^{x_i}\int_0^{y_i}\left\{\int_0^{s_i}\int_0^{t_i}\Big(H(h_i,\sigma_i,\tau_i)+F_i^{h_i-1}(\sigma_i,\tau_i)\cdot f(\sigma_i,\tau_i)\Big)^{\beta_i}d\sigma_i d\tau_i\right\}^{1/\beta_i}ds_idt_i\\
&\geq \prod_{i=1}^{n} h_i(x_iy_i)^{1/\alpha_i}\\
&\times\left\{\int_0^{x_i}\int_0^{y_i}\left\{\int_0^{s_i}\int_0^{t_i}\Big(H(h_i,\sigma_i,\tau_i)+F_i^{h_i-1}(\sigma_i,\tau_i)\cdot f(\sigma_i,\tau_i)\Big)^{\beta_i}d\sigma_i d\tau_i\right\}ds_idt_i\right\}^{1/\beta_i}\\
&= \prod_{i=1}^{n} h_i(x_iy_i)^{1/\alpha_i}\\
&\times\left\{\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\Big(H(h_i,s_i,t_i)+F_i^{h_i-1}(s_i,t_i)\cdot f(s_i,t_i)\Big)^{\beta_i}ds_idt_i\right\}^{1/\beta_i}.
\end{aligned}
$$

The proof is complete.

**Remark 1.** Let $f_i(\sigma_i,\tau_i)$ and $F_i(s_i,t_i)=\int_0^{s_i}\int_0^{t_i}f_i(\sigma_i,\tau_i)d\sigma_id\tau_i$ change to $f_i(\sigma_i)$ and $F_i(s_i)=\int_0^{s_i}f_i(\sigma_i)d\sigma_i$, respectively and with suitable changes, we have

**Corollary 1.** *Let* $h_i\geq 1$ *and let* $f_i(\sigma_i)\in C^1[(0,x_i),(0,\infty)], i=1,\ldots,n,$ *where* $x_i$ *are positive real numbers and define* $F_i(s_i)=\int_0^{s_i}f_i(\sigma_i)d\sigma_i$, for $s_i\in(0,x_i)$. *Then for* $\frac{1}{\alpha_i}+\frac{1}{\beta_i}=1$ ,$0<\beta_i<1$ *and* $\sum_{i=1}^{n}\frac{1}{\alpha_i}=\frac{1}{\alpha}$,

$$
\tag{4}
\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}F_i^{h_i}(s_i)}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}s_i\right)^{1/\alpha}}ds_1\cdots ds_n\\
&\geq\prod_{i=1}^{n}x_i^{1/\alpha_i}h_i\left(\int_0^{x_i}(x_i-s_i)\left(F_i^{h_i-1}(s_i)f_i(s_i)\right)^{\beta_i}ds_i\right)^{1/\beta_i}.
\end{aligned}
$$

**Remark 2.** Taking $n = 2, \beta_i = \frac{1}{2}$ to (4), (4) changes to

$$\begin{aligned}
&\int_0^{x_1}\int_0^{x_2} \frac{F_1^{h_1}(s_1)F_2^{h_2}(s_2)}{(s_1+s_2)^{-2}} ds_1 ds_2 \\
&\geq 4h_1h_2(x_1x_2)^{-1}\left(\int_0^{x_1}\Big(x_1-s_1\Big)\Big(F_1^{h_1-1}(s_1)f_1(s_1)\Big)^{1/2} ds_1\right)^2 \\
&\times\left(\int_0^{x_2}\Big(x_2-s_2\Big)\Big(F_2^{h_2-1}(s_2)f_2(s_2)\Big)^{1/2} ds_2\right)^2.
\end{aligned}$$

This is an inverse of the following inequality in Theorem A which was proved by Pachpatte [1].

$$\begin{aligned}
&\int_0^x\int_0^y \frac{F^h(s)G^l(t)}{s+t} dsdt \leq \frac{1}{2}hl(xy)^{1/2}\left(\int_0^x\Big(x-s\Big)\Big(F^{h-1}(s)f(s)\Big)^2 ds\right)^{1/2} \\
&\times\left(\int_0^y\Big(y-t\Big)\Big(G^{l-1}(t)g(t)\Big)^2 dt\right)^{1/2}.
\end{aligned}$$

On the other hand, for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$, (4) becomes

$$\begin{aligned}
&\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{(s_1+\cdots+s_n)^{-n/(n-1)}} ds_1\cdots ds_n \\
&\geq n^{n/(n-1)}\prod_{i=1}^n x_i^{-1/(n-1)}h_i\left(\int_0^{x_i}(x_i-s_i)\left(F_i^{h_i-1}(s_i)f_i(s_i)\right)^{(n-1)/n} ds_i\right)^{n/(n-1)}.
\end{aligned}$$

In [1], Pachpatte also established the following Hilbert type integral inequality.

**Theorem B.** *Let $f, g, F, G$ be as in Theorem* A. *Let $p(\sigma)$ and $q(\tau)$ be two positive real functions defined for $\sigma \in (0, x), \tau \in (0, y)$ and define $P(s) = \int_0^s p(\sigma)d\sigma$ and $Q(t) = \int_0^t q(\tau)d\tau$, for $s \in (0, x), t \in (0, y)$, where $x, y$ are positive real numbers. Let $\phi$ and $\psi$ be two real-valued, nonnegative, convex, and submultiplicative functions defined on $R+ = [0, \infty)$. Then*

$$\begin{aligned}
&\int_0^x\int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s+t} dsdt \\
&\leq L(x,y)\left(\int_0^x(x-s)\left(p(s)\phi\Big(\frac{f(s)}{p(s)}\Big)\right)^2 ds\right)^{1/2} \qquad (5)\\
&\times\left(\int_0^y(y-t)\left(q(t)\psi\Big(\frac{g(t)}{q(t)}\Big)\right)^2 dt\right)^{1/2},
\end{aligned}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\phi\Big(P(s)\Big)}{P(s)}\right)^2ds\right)^{1/2}\left(\int_0^y\left(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\right)^2dt\right)^{1/2}.$$

In this section, we establish a general inverse of inequality (5) as follows.

**Theorem 2.** *Let* $f_i(\sigma_i,\tau_i)$,$F_i(s_i,t_i),\alpha_i$ *and* $\beta_i$ *be as in Theorem 1. Let* $p_i(\sigma_i,\tau_i)$ *be n positive functions defined for* $\sigma_i\in(0,x_i),\tau_i\in(0,y_i)(i=1,2,\cdots,n)$ *and define* $P_i(s_i,t_i)=\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)d\sigma_id\tau_i$, *where* $x_i,y_i$ *are positive real numbers and Let* $\phi_i(i=1,2,\cdots,n)$ *be n real-valued nonnegative, concave, and supermultiplicative functions defined on* $R_+$ *Then*

$$\begin{aligned}&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n\phi_i(F_i(s_i,t_i))}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}s_it_i\right)^{1/\alpha}}ds_1dt_1\cdots ds_ndt_n\\ &\geq L(x_1,\ldots,x_n)\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\right.\\ &\left.\left(p_i(s_i,t_i)\phi_i(\frac{f_i(s_i,t_i)}{p_i(s_i,t_i)})\right)^{\beta_i}ds_idt_i\right)^{1/\beta_i},\end{aligned}\tag{6}$$

*where*

$$L(x_1,\ldots,x_n)=\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\right)^{\alpha_i}ds_idt_i\right)^{1/\alpha_i}.$$

*Proof.* By using Jensen integral inquality and inverse Hӧlder integral inequality and noting that $\phi_i(i=1,2,\cdots,n)$ are n real-valued supermultiplicative functions, it is easy to observe that

$$\begin{aligned}&\phi_i\left(F_i(s_i,t_i)\right)\\ &=\phi_i\left(\frac{P_i(s_i,t_i)\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}d\sigma_id\tau_i}{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}\right)\\ &\geq\phi_i(P_i(s_i,t_i))\phi_i\left(\frac{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}d\sigma_id\tau_i}{\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)d\sigma_id\tau_i}\right)\\ &\geq\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)d\sigma_id\tau_i\\ &\geq\left(\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\right)(s_it_i)^{1/\alpha_i}\left(\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i\tau_i)}{p_i(\sigma_i\tau_i)}\right)\right)^{\beta_i}d\sigma_id\tau_i\right)^{1/\beta_i}.\end{aligned}\tag{7}$$

Integrating both sides of (7) over $s_i, t_i$ from $0$ to $x_i, y_i (i = 1, 2, \ldots, n)$ and in view of Hölder integral inequality, the means inequality and Fubini′s theorem, we observe that

$$\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n\phi_i(F_i(s_i,t_i))}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}s_it_i\right)^{1/\alpha}}ds_1dt_1\cdots ds_ndt_n$$
$$\geq\prod_{i=1}^n\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\right)$$
$$\left(\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right)^{\beta_i}d\sigma_id\tau_i\right)^{1/\beta_i}ds_idt_i$$
$$\geq\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\right)^{\alpha_i}ds_idt_i\right)^{1/\alpha_i}$$
$$\times\left(\int_0^{x_i}\int_0^{y_i}\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\phi_i\left(\frac{f_i(\sigma_i,\tau_i)}{p_i(\sigma_i,\tau_i)}\right)\right)^{\beta_i}d\sigma_id\tau_ids_idt_i\right)^{1/\beta_i}$$
$$=L(x_1,\ldots,x_n)\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}\left(x_i-s_i\right)\left(y_i-t_i\right)\right.$$
$$\left.\left(p_i(s_i,t_i)\phi_i(\frac{f_i(s_i,t_i)}{p_i(s_i,t_i)})\right)^{\beta_i}ds_idt_i\right)^{1/\beta_i},$$

where

$$L(x_1,\ldots,x_n)=\prod_{i=1}^n\left(\int_0^{x_i}\int_0^{y_i}\left(\frac{\phi_i\left(P_i(s_i,t_i)\right)}{P_i(s_i,t_i)}\right)^{\alpha_i}ds_idt_i\right)^{1/\alpha_i}.$$

This completes the proof of Theorem 2.

**Remark 3.** Let $f_i(\sigma_i,\tau_i)$, $p_i(\sigma_i,\tau_i)$, $P(s_i,t_i)$and $F_i(s_i,t_i)$ change to $f_i(\sigma_i)$, $p_i(\sigma_i)$, $P_i(s_i)$ and $F_i(s_i)$, respectively and with suitable changes, we have

**Corollary 2.** *Let $f_i(\sigma_i)$,$F_i(s_i)$, $\alpha_i$ and $\beta_i$ be as in Corollary 1. Let $p_i(\sigma_i)$ be n positive functions defined for $\sigma_i \in (0, x_i)(i = 1, 2, \cdots, n)$ and define $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i)d\sigma_i$, where $x_i$ are positive real numbers and Let $\phi_i(i = 1, 2, \cdots, n)$ be n real-valued nonnegative, concave, and supermultiplicative functions defined on $R_+$. Then*

(8)
$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^n \frac{1}{\alpha_i}s_i\right)^{1/\alpha}}ds_1\cdots ds_n \geq L'(x_1,\cdots,x_n)\prod_{i=1}^n\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\phi_i(\frac{f_i(s_i)}{p_i(s_i)})\right)^{\beta_i}ds_i\right)^{1/\beta_i},$$

*where*

$$L'(x_1,\cdots,x_n)=\prod_{i=1}^n\left(\int_0^{x_i}\left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)^{\alpha_i}ds_i\right)^{1/\alpha_i}.$$

**Remark 4.** Taking $n=2, \beta_i=\frac{1}{2}$ to (8), (8) changes to

$$\int_0^{x_1}\int_0^{x_2}\frac{\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1+s_2)^{-2}}ds_1ds_2 \geq L'(x_1,x_2)\left(\int_0^{x_1}(x_1-s_1)\left(p_1(s_1)\phi_1\Big(\frac{f_1(s_1)}{p_1(s_1)}\Big)\right)^{1/2}ds_1\right)^2 \times\left(\int_0^{x_2}(x_2-s_2)\left(p_2(s_2)\phi_2\Big(\frac{f_2(s_2)}{p_2(s_2)}\Big)\right)^{1/2}ds_2\right)^2,$$

where

$$L'(x_1,x_2)=4\left(\int_0^{x_1}\left(\frac{\phi_1\Big(P_1(s_1)\Big)}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_0^{x_2}\left(\frac{\phi_2\Big(P_2(s_2)\Big)}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}.$$

This is an inverse of the following inequality in Theorem B which was established by Pachpatte [1].

$$\int_0^x\int_0^y\frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s+t}dsdt\leq L(x,y)\left(\int_0^x(x-s)\left(p(s)\phi\Big(\frac{f(s)}{p(s)}\Big)\right)^2ds\right)^{1/2} \times\left(\int_0^y(y-t)\left(q(t)\psi\Big(\frac{g(t)}{q(t)}\Big)\right)^2dt\right)^{1/2},$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\phi\Big(P(s)\Big)}{P(s)}\right)^2ds\right)^{1/2}\left(\int_0^y\left(\frac{\psi\Big(Q(t)\Big)}{Q(t)}\right)^2dt\right)^{1/2}.$$

On the other hand, for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$, (8) changes to

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(F_i(s_i))}{(s_1 + \cdots + s_n)^{-n/(n-1)}} ds_1 \cdots ds_n$$
$$\geq \bar{L}(x_1, \cdots, x_n) \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) \left( p_i(s_i)\phi_i(\frac{f_i(s_i)}{p_i(s_i)}) \right)^{(n-1)/n} ds_i \right)^{n/(n-1)},$$

where

$$\bar{L}(x_1, \cdots, x_n) = n^{n/(n-1)} \prod_{i=1}^n \left( \int_0^{x_i} \left( \frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{-(n-1)} ds_i \right)^{-1/(n-1)}.$$

In [1], Pachpatte also established the following Hilbert type integral inequality.

**Theorem C.** *Let $f, g, p, q, P, Q$ be as in Theorem B, and define $F(s) = \frac{1}{P(s)} \int_0^s p(\sigma)d\sigma$ and $G(t) = \frac{1}{Q(t)} \int_0^t q(\tau)d\tau$, for $s \in (0, x), t \in (0, y)$, where $x, y$ are positive real numbers. Let $\phi$ and $\psi$ be two real-valued, nonnegative and convex functions defined on $R+ = [0, \infty)$. Then*

$$\int_0^x \int_0^y \frac{P(s)Q(t)\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s+t} dsdt$$
(9)
$$\leq \frac{1}{2}(xy)^{1/2} \left( \int_0^x \Big(x - s\Big) \Big(p(s)\phi\Big(f(s)\Big)\Big)^2 ds \right)^{1/2} \left( \int_0^y \Big(y - t\Big) \Big(q(t)\psi\Big(g(t)\Big)\Big)^2 dt \right)^{1/2}.$$

In this section, we establish a general inverse of inequality (9) as follows.

**Theorem 3.** *Let $f_i(\sigma_i, \tau_i), p_i(\sigma_i, \tau_i), P_i(\sigma_i, \tau_i), \alpha_i$ and $\beta_i$ be as Theorem 2 and define $F_i(s_i, t_i) = \frac{1}{P_i(s_i,t_i)} \int_0^{s_i} \int_0^{t_i} p_i(\sigma_i, \tau_i)f_i(\sigma_i, \tau_i)d\sigma_i d\tau_i$* for $\sigma_i, s_i \in (0, x_i), \tau_i, t_i \in (0, y_i)$ *where $x_i, y_i$ are positive real numbers. Let $\phi_i(i = 1, 2, \ldots, n)$ be n real-valued, nonnegative, and concave functions on $R_+$. Then*

$$\int_0^{x_1} \int_0^{y_1} \cdots \int_0^{x_n} \int_0^{y_n} \frac{\prod_{i=1}^n P_i(s_i, t_i) \cdot \phi_i(F_i(s_i, t_i))}{\left( \alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i t_i \right)^{1/\alpha}} ds_1 dt_1 \cdots ds_n dt_n$$
(10)
$$\geq \prod_{i=1}^n (x_i y_i)^{1/\alpha_i} \left( \int_0^{x_i} \int_0^{y_i} (x_i - s_i)(y_i - t_i)(p_i(s_i, t_i) \cdot \phi_i(f_i(s_i, t_i)))^{\beta_i} ds_i dt_i \right)^{1/\beta_i}.$$

*Proof.* From the hypotheses and by using Jensen integral inequality and the inverse Hölder integral inequlity, we have

$$\begin{aligned}
&\phi_i\left(F_i(s_i,t_i)\right)\\
&=\phi_i\left(\frac{1}{P_i(s_i,t_i)}\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)f_i(\sigma_i,\tau_i)d\sigma_i d\tau_i\right)\\
&\geq\frac{1}{P_i(s_i,t_i)}\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)\cdot\phi_i\left(f_i(\sigma_i,\tau_i)\right)d\sigma_i d\tau_i \qquad (11)\\
&\geq\frac{1}{P_i(s_i,t_i)}(s_it_i)^{1/\alpha_i}\left(\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\cdot\phi_i\left(f_i(\sigma_i,\tau_i)\right)\right)^{\beta_i}d\sigma_i d\tau_i\right)^{1/\beta_i}.
\end{aligned}$$

Integrating both sides of (11) over $s_i, t_i$ from $0$ to $x_i, y_i (i = 1, 2, \ldots, n)$ and in view of Hölder′s integral inequality, the means inequality and Fubini′s theorem, we observe that

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\cdots\int_0^{x_n}\int_0^{y_n}\frac{\prod_{i=1}^n P_i(s_i,t_i)\phi_i(F_i(s_i,t_i))}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}s_it_i\right)^{1/\alpha}}ds_1dt_1\cdots ds_ndt_n\\
&\geq\prod_{i=1}^n\int_0^{x_i}\int_0^{y_i}\left(\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\phi_i(f_i(\sigma_i,\tau_i))\right)^{\beta_i}d\sigma_i d\tau_i\right)^{1/\beta_i}ds_idt_i\\
&\geq\prod_{i=1}^n(x_iy_i)^{1/\alpha_i}\left(\int_0^{x_i}\int_0^{y_i}\int_0^{s_i}\int_0^{t_i}\left(p_i(\sigma_i,\tau_i)\phi_i(f_i(\sigma_i,\tau_i))\right)^{\beta_i}d\sigma_i d\tau_i ds_idt_i\right)^{1/\beta_i}\\
&=\prod_{i=1}^n(x_iy_i)^{1/\alpha_i}\left(\int_0^{x_i}\int_0^{y_i}(x_i-s_i)(y_i-t_i)\left(p_i(s_i,t_i)\phi_i(f_i(s_i,t_i))\right)^{\beta_i}ds_idt_i\right)^{1/\beta_i}.
\end{aligned}$$

The proof is complete.

**Remark 5.** Let $f_i(\sigma_i,\tau_i)$, $p_i(\sigma_i,\tau_i)$, $P(s_i,t_i)$ and

$$F_i(s_i,t_i)=\frac{1}{P_i(s_i,t_i)}\int_0^{s_i}\int_0^{t_i}p_i(\sigma_i,\tau_i)f_i(\sigma_i,\tau_i)d\sigma_i d\tau_i$$

change to $f_i(\sigma_i)$, $p_i(\sigma_i)$, $P_i(s_i)$ and

$$F_i(s_i)=\frac{1}{P_i(s_i)}\int_0^{s_i}p_i(\sigma_i)f_i(\sigma_i)d\sigma_i,$$

respectively and with suitable changes, we have

**Corollary 3.** *Let $f_i(\sigma_i), p_i(\sigma_i), P_i(\sigma_i), \alpha_i$ and $\beta_i$ be as Corollary 2 and define $F_i(s_i)=\frac{1}{P_i(s_i)}\int_0^{s_i}p_i(\sigma_i)f_i(\sigma_i)d\sigma_i$ for $\sigma_i, s_i \in (0, x_i)$, where $x_i$ are positive real* *numbers. Let $\phi_i (i = 1, 2, \ldots, n)$ be $n$ real-valued, nonnegative, and concave functions on $R_+$. Then*

$$\begin{aligned}
&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n P_i(s_i)\phi_i(F_i(s_i))}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n \\
&\geq \prod_{i=1}^n x_i^{1/\alpha_i} \left( \int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(f_i(s_i))\right)^{\beta_i} ds_i \right)^{1/\beta_i} .
\end{aligned} \tag{12}$$

**Remark 6.** Taking $n = 2, \beta_i = \frac{1}{2}$ to (12), (12) changes to

$$\begin{aligned}
&\int_0^{x_1} \int_0^{x_2} \frac{P_1(s_1)P_2(s_2)\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1 + s_2)^{-2}} ds_1 ds_2 \\
&\geq 4(x_1x_2)^{-1} \left( \int_0^{x_1} \Big(x_1 - s_1\Big)\left(p_1(s_1)\phi_1\Big(f_1(s_1)\Big)\right)^{1/2} ds_1 \right)^2 \\
&\times \left( \int_0^{x_2} \Big(x_2 - s_2\Big)\left(p_2(s_2)\phi_2\Big(f_2(s_2)\Big)\right)^{1/2} ds_2 \right)^2 .
\end{aligned} \tag{13}$$

This is an inverse inequality of the following inequality in Theorem C which was proved by Pachpatte [1].

$$\begin{aligned}
&\int_0^{x} \int_0^{y} \frac{P(s)Q(t)\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s + t} dsdt \\
&\leq \frac{1}{2}(xy)^{1/2} \left( \int_0^{x} \Big(x - s\Big)\left(p(s)\phi\Big(f(s)\Big)\right)^2 ds \right)^{1/2} \\
&\times \left( \int_0^{y} \Big(y - t\Big)\left(q(t)\psi\Big(g(t)\Big)\right)^2 dt \right)^{1/2}
\end{aligned}$$

Moreover, in (13), let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore (13) changes to

$$\begin{aligned}
&\int_0^{x_1} \int_0^{x_2} \frac{\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1t_1)^{-1}(s_1 + s_2)^{-2}} ds_1 ds_2 \\
&\geq 4(x_1x_2)^{-1} \left( \int_0^{x_1} \Big(x_1 - s_1\Big)\left(\phi_1\Big(f_1(s_1)\Big)\right)^{1/2} ds_1 \right)^2 \\
&\times \left( \int_0^{x_2} \Big(x_2 - s_2\Big)\left(\phi_2\Big(f_2(s_2)\Big)\right)^{1/2} ds_2 \right)^2 .
\end{aligned}$$

This is an inverse inequality of the following inequality which was proved by Pachpatte [1].

$$\int_0^x \int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{(st)^{-1}(s+t)} dsdt$$
$$\leq \frac{1}{2}(xy)^{1/2}\left(\int_0^x \Big(x-s\Big)\Big(\phi\Big(f(s)\Big)\Big)^2 ds\right)^{1/2}\left(\int_0^y \Big(y-t\Big)\Big(\psi\Big(g(t)\Big)\Big)^2 dt\right)^{1/2}.$$

On the other hand, for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$, (12) changes to

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n P_i(s_i)\phi_i(F_i(s_i))}{(s_1 + \cdots + s_n)^{-n/(n-1)}} ds_1 \cdots ds_n$$
$$\geq n^{n/(n-1)} \prod_{i=1}^n x_i^{-1/(n-1)} \left(\int_0^{x_i} (x_i - s_i)\,(p_i(s_i)\phi_i(f_i(s_i)))^{(n-1)/n}\, ds_i\right)^{n/(n-1)}.$$

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