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# SHRINKING PROJECTION METHOD OF PROXIMAL-TYPE FOR A GENERALIZED EQUILIBRIUM PROBLEM, A MAXIMAL MONOTONE OPERATOR AND A PAIR OF RELATIVELY NONEXPANSIVE MAPPINGS

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**Abstract.** The purpose of this paper is to introduce and consider a shrinking projection method of proximal-type for finding a common element of the set $EP$ of solutions of a generalized equilibrium problem, the set $F(S) \cap F(\widetilde{S})$ of common fixed points of a pair of relatively nonexpansive mappings $S, \widetilde{S}$ and the set $T^{-1}0$ of zeros of a maximal monotone operator $T$ in a uniformly smooth and uniformly convex Banach space. It is proven that under appropriate conditions, the sequence generated by the shrinking projection method of proximal-type, converges strongly to some point in $EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0$. This new result represents the improvement, generalization and development of the previously known ones in the literature.

## 1. INTRODUCTION

Let $E$ be a real Banach space with the dual $E^*$ and $C$ be a nonempty closed convex subset of $E$. We denote by $J$ the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if $E$ is smooth then $J$ is single-valued and norm-to-weak$^*$ continuous, and that if $E$ is uniformly smooth,

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then $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$. We shall still denote by $J$ the single-valued duality mapping. Let $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to \mathcal{R}$ be a bifunction, where $\mathcal{R}$ denotes the sets of real numbers. In this paper we consider the following generalized equilibrium problem of finding $u \in C$ such that

$$f(u, y) + \langle Au, y - u\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP$, i.e.,

$$EP = \{u \in C : f(u, y) + \langle Au, y - u\rangle \geq 0, \ \forall y \in C\}.$$

Whenever $E = H$ a Hilbert space, problem (1.1) was introduced and studied by Takahashi and Takahashi [14]. We remark that problem (1.1) and related problems have been extensively studied recently. See, e.g., [31-50].

In the case of $A \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that

$$f(u, y) \geq 0, \quad \forall y \in C,$$

which is called the equilibrium problem. The set of its solutions is denoted by $EP(f)$.

In the case of $f \equiv 0$, problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0, \quad \forall y \in C,$$

which is called the variational inequality of Browder type. The set of its solutions is denoted by $VI(C, A)$.

The problem (1.1) is very general in the sense that it includes, as spacial cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, e.g., [18,29]. A mapping $S : C \to E$ is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of $S$, that is, $F(S) = \{x \in C : Sx = x\}$. A mapping $A : C \to E^*$ is called $\alpha$-inverse-strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is easy to see that if $A : C \to E^*$ is an $\alpha$-inverse-strongly monotone mapping, then it is $1/\alpha$-Lipschitzian.

Very recently, motivated by Takahashi and Zembayashi [11], Chang [28] proved the following strong convergence theorem for finding a common element of the set of solutions to the generalized equilibrium problem (1.1) and the set of common fixed points of a pair of relatively nonexpansive mappings in a Banach space.

**Theorem 1.1.** (see [28, Theorem 3.1]). *Let $E$ be a uniformly smooth and uniformly convex Banach space, and $C$ be a nonempty closed convex subset of $E$. Let $A : C \to E^*$ be an $\alpha$-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ be a bifunction satisfying the following conditions (A1)-(A4):*

$(A1)$ *$f(x, x) = 0$ for all $x \in C$,*

$(A2)$ *$f$ is monotone, i.e., $f(x, y) + f(y, x) \le 0$, for all $x, y \in C$,*

$(A3)$ *for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y)$,*

$(A4)$ *for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.*

*Let $S, \widetilde{S} : C \to C$ be two relatively nonexpansive mappings such that $F(S) \cap F(\widetilde{S}) \cap EP \ne \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(1.2)\quad \begin{cases} x_0 \in C,\ C_0 = C; \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S x_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J \widetilde{S} z_n), \\ u_n \in C \text{ such that} \\ \quad f(u_n, y) + \langle A u_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0,\ \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \beta_n \phi(v, x_n) \\ \quad +(1-\beta_n)\phi(v, z_n) \le \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ \forall n \ge 0, \end{cases}$$

*where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, $\forall x, y \in E$, $\Pi_C : E \to C$ is the generalized projection operator, $J : E \to E^*$ is the single-valued normalized duality mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If the following conditions are satisfied:*

$(i)$ $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$,

$(ii)$ $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,

*then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap F(\widetilde{S})\cap EP} x_0$, where $\Pi_{F(S)\cap F(\widetilde{S})\cap EP}$ is the generalized projection of $E$ onto $F(S) \cap F(\widetilde{S}) \cap EP$.*

Let $E$ be a real Banach space with the dual $E^*$. A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \ne \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \ge 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator $T$ is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. Denote by $I$ the identity operator on $E = H$ a Hilbert space. The proximal point algorithm generates, for any initial point $x_0 = x \in H$, a sequence $\{x_n\}$ in $H$, by the iterative scheme

$$x_{n+1} = (I + r_n T)^{-1} x_n, \quad n = 0, 1, 2, ...,$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$. Note that this iteration is equivalent to

$$0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1} - x_n), \quad n = 0, 1, 2, ....$$

This algorithm was first introduced by Martinet [18] and generally studied by Rockafellar [24] in the framework of a Hilbert space. Later many authors studied its convergence in a Hilbert space or a Banach space. See for instance, [7,9,10,13,21,25] and the references therein. On the other hand, Kamimura and Takahashi [12] recently introduced and studied the following proximal-type algorithm for finding an element of $T^{-1}0$ in a uniformly smooth and uniformly convex Banach space $E$, which is an extension of Solodov and Svaiter's proximal-type algorithm [26]:

$$(1.3) \qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \ v_n \in Ty_n, \\ H_n = \{v \in E : \langle v - y_n, v_n \rangle \leq 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \ n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$ and $J$ is the normalized duality mapping on $E$. They derived a strong convergence theorem which extends and improves Solodov and Svaiter's result [26].

Recently, utilizing Nakajo and Takahashi's idea [16], Qin and Su [20] first introduced one iterative algorithm (i.e., modified Ishikawa iteration) for a relatively nonexpansive mapping $S : C \to C$, with $C$ a closed convex subset of a uniformly smooth and uniformly convex Banach space $E$

$$(1.4) \qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen}, \\ z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_n) + (1 - \alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases}$$

They proved that under appropriate conditions the sequence $\{x_n\}$ generated by algorithm (1.4), converges strongly to $\Pi_{F(S)} x_0$.

Let $E$ be a real Banach space with the dual $E^*$. Assume that $T : E \to 2^{E^*}$ is a maximal monotone operator and $S : E \to E$ is a relatively nonexpansive mapping. Very recently, inspired by algorithms (1.3)-(1.4), Ceng, Petruşel and Wu [27] introduced and studied the following hybrid proximal-type algorithm for finding an element of $F(S) \cap T^{-1}0$ in a uniformly smooth and uniformly convex Banach space $E$.

$$(1.5)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ H_n = \{v \in E : \phi(v, y_n) \le \alpha_n\phi(v, \widetilde{x}_n) \\ \quad +(1-\alpha_n)\phi(v, z_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0,\ n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$. The authors proved that under appropriate conditions the sequence $\{x_n\}$ generated by algorithm (1.5), converges strongly to $\Pi_{F(S)\cap T^{-1}0}x_0$.

Let $E$ be a reflexive, strictly convex, and smooth Banach space with the dual $E^*$ and $C$ be a nonempty closed convex subset of $E$. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, and $S, \widetilde{S} : C \to C$ be a pair of relatively nonexpansive mappings. Let $A : C \to X^*$ be an $\alpha$-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4). The purpose of this paper is to introduce and investigate a shrinking projection method of proximal-type for finding an element of $EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0$, i.e., the following iterative algorithm

$$(1.6)\qquad \begin{cases} x_0 \in C_0 \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in C \text{ such that} \\ \ f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \ge 0,\ \forall y \in C, \\ z_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \le \beta_n\phi(v, u_n) \\ \quad + + (1-\beta_n)\phi(v, z_n) \le \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\ D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}} x_0,\ n = 0, 1, 2, ..., \end{cases}$$

where $C_0 = C$, $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

In this paper, we proposed a shrinking projection method of proximal-type in a uniformly smooth and uniformly convex Banach space and established some strong convergence results which represent the improvement, generalization and development of the previously known ones in the literature including Solodov and Svaiter [26], Kamimura and Takahashi [12], Qin and Su [20], Ceng, Petruşel and Wu [27] and Chang [28].

In the rest of this paper the symbol $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

## 2. Preliminaries

A Banach space $E$ is called strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of $E$. Then the Banach space $E$ is called smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. If $E$ is smooth then $J$ is single-valued. We shall still denote the single-valued duality mapping by $J$.

It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$. A Banach space $E$ is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if $E$ is uniformly convex, then $E$ has the Kadec-Klee property; see [8,19] for more details.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $P_C : H \to C$ be the metric projection of $H$ onto C. Then $P_C$ is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. Nevertheless, Alber [2] recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that $E$ is a smooth Banach space. Consider the functional defined as in [1,2] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.1}$$

It is clear that in a Hilbert space $H$, (2.1) reduces to $\phi(x, y) = \|x-y\|^2$, $\forall x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping $J$ (see, e.g., [3]). In a Hilbert space $H$, $\Pi_C = P_C$. From [2], in uniformly smooth and uniformly convex Banach spaces, we have

$$(2.3) \qquad (\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E.$$

Let $C$ be a nonempty closed convex subset of $E$, and let $S$ be a mapping from $C$ into itself. A point $p \in C$ is called an asymptotically fixed point of $S$ [17] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $Sx_n - x_n \to 0$. The set of asymptotical fixed points of $S$ will be denoted by $\widehat{F}(S)$. A mapping $S$ from $C$ into itself is called relatively nonexpansive [4-6] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(S)$.

We remark that if $E$ is a reflexive, strictly convex and smooth Banach space, then for any $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|y\|^2$. From the definition of $J$, we have $Jx = Jy$. Therefore, we have $x = y$; see [8,19] for more details.

We need the following lemmas for the proof of our main results.

**Lemma 2.1.** (see [12]). *Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

**Lemma 2.2.** (see [2,12]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $x \in E$ and let $z \in C$. Then*

$$z = \Pi_C x \quad \Leftrightarrow \quad \langle y - z, Jx - Jz \rangle \le 0, \quad \forall y \in C.$$

**Lemma 2.3.** (see [2,12]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x,y), \quad \forall x \in C \text{ and } y \in E.$$

**Lemma 2.4.** (see [15]). *Let $C$ be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space $E$, and let $S : C \to C$ be a relatively nonexpansive mapping. Then $F(S)$ is closed and convex.*

The following result is due to Blum and Oettli [22].

**Lemma 2.5.** (see [22]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to*

*$\mathcal{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \text{for all } y \in C.$$

Motivated by Combettes and Hirstoaga [23] in a Hilbert space, Takahashi and Zembayashi [11] established the following lemma.

**Lemma 2.6.** (see [11]). *Let $C$ be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$, and let $f$ be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in C\}$$

*for all $x \in E$. Then, the following hold:*

($i$) *$T_r$ is single-valued;*

($ii$) *$T_r$ is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

($iii$) *$F(T_r) = \widehat{F}(T_r) = EP(f)$;*

($iv$) *$EP(f)$ is closed and convex.*

Using Lemma 2.6, one has the following result.

**Lemma 2.7.** (see [11]). *Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)-(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Utilizing Lemmas 2.5, 2.6 and 2.7 as above, Chang [28] derived the following result.

**Proposition 2.1.** (see [28, Lemma 2.5]). *Let $E$ be a smooth, strictly convex and reflexive Banach space and $C$ be a nonempty closed convex subset of $E$. Let $A : C \to E^*$ be an $\alpha$-inverse-strongly monotone mapping, let $f$ be a bifunction from $C \times C$ to $\mathcal{R}$ satisfying (A1)-(A4), and let $r > 0$. Then there hold the following*

($I$) *for $x \in E$, there exists $u \in C$ such that*

$$f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \quad \forall y \in C;$$

$(II)$ *if $E$ is additionally uniformly smooth and $K_r : E \to C$ is defined as*

$$K_r(x) = \{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in E, \tag{2.4}$$

*then the mapping $K_r$ has the following properties:*

$(i)$ *$K_r$ is single-valued,*

$(ii)$ *$K_r$ is a firmly nonexpansive-type mapping, i.e.,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle, \quad \forall x, y \in E,$$

$(iii)$ *$F(K_r) = \widehat{F}(K_r) = EP$,*

$(iv)$ *$EP$ is a closed convex subset of $C$,*

$(v)$ *$\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x),\ \forall p \in F(K_r)$.*

*Proof.* Define a bifunction $F : C \times C \to \mathcal{R}$ as follows:

$$F(x, y) = f(x, y) + \langle Ax, y - x\rangle, \quad \forall x, y \in C.$$

Then it is easy to verify that $F$ satisfies the conditions (A1)-(A4). Therefore, The conclusions (I) and (II) of Proposition 2.1 follow immediately from Lemmas 2.5, 2.6 and 2.7. ■

## 3. Main Results

Throughout this section, unless otherwise stated, we assume that $T : E \to 2^{E^*}$ is a maximal monotone operator, $S, \widetilde{S} : C \to C$ are a pair of relatively nonexpansive mappings, $A : C \to E^*$ is an $\alpha$-inverse-strongly monotone mapping and $f : C \times C \to \mathcal{R}$ is a bifunction satisfying (A1)-(A4), where $C$ is a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space $E$. In this section, we study the following algorithm

$$\begin{cases} x_0 \in C_0 \text{ arbitrarily chosen,} \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in C \text{ such that} \\ \quad f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \geq 0,\ \forall y \in C, \\ z_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1 - \beta_n)J\widetilde{S}z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n \phi(v, u_n) \\ \quad +(1 - \beta_n)\phi(v, z_n) \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}} x_0,\ n = 0, 1, 2, ..., \end{cases} \tag{3.1}$$

where $C_0 = C$, $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

First we investigate the condition under which the algorithm (3.1) is well defined. Rockafellar [30] proved the following result.

**Lemma 3.1.** (Rockafellar [30]). *Let $E$ be a reflexive, strictly convex, and smooth Banach space and let $T : E \to 2^{E^*}$ be a multivalued operator. Then there hold the following*

(*i*) *$T^{-1}0$ is closed and convex if $T$ is maximal monotone such that $T^{-1}0 \neq \emptyset$;*

(*ii*) *$T$ is maximal monotone if and only if $T$ is monotone with $R(J + rT) = E^*$ for all $r > 0$.*

*Utilizing this result, we can show the following lemma.*

**Lemma 3.2.** *Let $E$ be a reflexive, strictly convex, and smooth Banach space. If $EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (3.1) is well defined.*

*Proof.* First it is easy to see that $D_n$ is a closed and convex subset of $C$ for all $n \geq 1$. Second, let us show that $C_n$ is a closed and convex subset of $C$ for all $n \geq 1$. Indeed, observe that

$$\begin{aligned}
&\phi(v, y_n) \leq \beta_n \phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \\
\Leftrightarrow\ & 2\langle v, (1 - \beta_n)Jz_n + \beta_n Ju_n - Jy_n\rangle \leq (1 - \beta_n)\|z_n\|^2 - \|y_n\|^2 + \beta_n\|u_n\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\beta_n\phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \leq \phi(v, \widetilde{x}_n) \\
\Leftrightarrow\ & 2\langle v, J\widetilde{x}_n - (1 - \beta_n)Jz_n - \beta_n Ju_n\rangle \leq \|\widetilde{x}_n\|^2 - (1 - \beta_n)\|z_n\|^2 - \beta_n\|u_n\|^2.
\end{aligned}$$

Hence we have

$$\begin{aligned}
C_{n+1} &= \{v \in C_n : \phi(v, y_n) \leq \beta_n\phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \\
&\leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\} = \{v \in C_n : \phi(v, y_n) \\
&\leq \beta_n\phi(v, u_n) + (1 - \beta_n)\phi(v, z_n)\} \cap \{v \in C_n : \beta_n\phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \\
&\leq \phi(v, \widetilde{x}_n)\} \cap \{v \in C_n : \langle v - \widetilde{x}_n, v_n\rangle \leq 0\} \\
&= \{v \in C_n : 2\langle v, (1 - \beta_n)Jz_n + \beta_n J\widetilde{x}_n - Jy_n\rangle \\
&\leq (1 - \beta_n)\|z_n\|^2 - \|y_n\|^2 + \beta_n\|\widetilde{x}_n\|^2\} \\
&\quad \cap \{v \in C_n : 2\langle v, J\widetilde{x}_n - (1 - \beta_n)Jz_n - \beta_n Ju_n\rangle \\
&\leq \|\widetilde{x}_n\|^2 - (1 - \beta_n)\|z_n\|^2 - \beta_n\|u_n\|^2\} \\
&\quad \cap \{v \in C_n : \langle v, v_n\rangle \leq \langle \widetilde{x}_n, v_n\rangle\}.
\end{aligned}$$

Thus, this implies that $C_n$ is closed and convex for each $n \geq 1$.

On the other hand, let $w \in EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0$ be arbitrarily chosen. Then $w \in EP,\ w \in F(S) \cap F(\widetilde{S})$ and $w \in T^{-1}0$. From (3.1), it follows that

$$\begin{aligned}
&\phi(w, y_n) \\
&= \phi(w, J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n)) \\
&= \|w\|^2 - 2\langle w, \beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n\rangle + \|\beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n\|^2 \\
&\leq \|w\|^2 - 2\beta_n\langle w, Ju_n\rangle - 2(1-\beta_n)\langle w, J\widetilde{S}z_n\rangle + \beta_n\|u_n\|^2 + (1-\beta_n)\|\widetilde{S}z_n\|^2 \\
&= \beta_n\phi(w, u_n) + (1-\beta_n)\phi(w, \widetilde{S}z_n) \\
&\leq \beta_n\phi(w, u_n) + (1-\beta_n)\phi(w, z_n) \\
&= \beta_n\phi(w, u_n) + (1-\beta_n)\phi(w, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSu_n)) \\
&= \beta_n\phi(w, u_n) + (1-\beta_n)[\|w\|^2 - 2\langle w, \alpha_n Ju_n + (1-\alpha_n)JSu_n\rangle \\
&\quad + \|\alpha_n Ju_n + (1-\alpha_n)JSu_n\|^2] \\
&\leq \beta_n\phi(w, u_n) + (1-\beta_n)[\|w\|^2 - 2\alpha_n\langle w, Ju_n\rangle - 2(1-\alpha_n)\langle w, JSu_n\rangle \\
&\quad + \alpha_n\|u_n\|^2 + (1-\alpha_n)\|Su_n\|^2] \\
&= \beta_n\phi(w, u_n) + (1-\beta_n)[\alpha_n\phi(w, u_n) + (1-\alpha_n)\phi(w, Su_n)] \\
&\leq \beta_n\phi(w, u_n) + (1-\beta_n)[\alpha_n\phi(w, u_n) + (1-\alpha_n)\phi(w, u_n)] \\
&= \phi(w, u_n) = \phi(w, K_{r_n}\widetilde{x}_n) \leq \phi(w, \widetilde{x}_n),
\end{aligned}$$

for all $n \geq 0$. Now, from Lemma 3.1 it follows that there exists $(\widetilde{x}_0, v_0) \in E \times E^*$ such that $0 = v_0 + \frac{1}{r_0}(J\widetilde{x}_0 - Jx_0)$ and $v_0 \in T\widetilde{x}_0$. Since $T$ is monotone, it follows that $\langle \widetilde{x}_0 - w, v_0\rangle \geq 0$, which implies that $w \in C_1$. Furthermore, it is clear that $w \in D_1 = C$. Then $w \in C_1 \cap D_1$, and therefore $x_1 = \Pi_{C_1 \cap D_1}x_0$ is well defined. Suppose that $w \in C_n \cap D_n$ and $x_n$ is well defined for some $n \geq 1$. Again by Lemma 3.1, we deduce that $(\widetilde{x}_n, v_n) \in E \times E^*$ such that $0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n)$ and $v_n \in T\widetilde{x}_n$. Then from the monotonicity of $T$ we conclude that $\langle \widetilde{x}_n - w, v_n\rangle \geq 0$, which implies that $w \in C_{n+1}$. It follows from Lemma 2.4 that

$$\langle w - x_n, Jx_0 - Jx_n\rangle = \langle w - \Pi_{C_n \cap D_n}x_0, Jx_0 - J\Pi_{C_n \cap D_n}x_0\rangle \leq 0,$$

which implies that $w \in D_{n+1}$. Consequently, $w \in C_{n+1} \cap D_{n+1}$. This shows that $EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \subset C_n \cap D_n$ for all $n \geq 1$. Therefore $x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}}x_0$ is well defined. Then, by induction, the sequence $\{x_n\}$ generated by (3.1) is well defined for each integer $n \geq 0$. ■

**Remark 3.1.** From the above proof, we obtain that

$$EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \subset C_n \cap D_n$$

for each integer $n \geq 1$.

We are now in a position to prove the main theorem.

**Theorem 3.1.** *Let $E$ be a uniformly smooth and uniformly convex Banach space. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in $(0,\infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0,1]$ such that*

$$(3.2)\quad \liminf_{n\to\infty} r_n > 0, \quad \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \beta_n(1-\beta_n) > 0.$$

*Let $EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \neq \emptyset$. If $S$ is uniformly continuous, then the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to $\Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$.*

*Proof.* We divide the proof into several steps.

**Step 1.** We claim that $\{x_n\}$ is bounded, and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, it follows from the definition of $D_n$ that $x_n = \Pi_{D_{n+1}}x_0$. Since $x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}}x_0 \in D_{n+1}$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0.$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. Also from $x_n = \Pi_{D_{n+1}}x_0$ and Lemma 2.3, we have that

$$\phi(x_n, x_0) = \phi(\Pi_{D_{n+1}}x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \subset D_{n+1}$ and for each $n \geq 0$. Consequently, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2,$$

we conclude that $\{x_n\}$ is bounded. Thus, we have that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From Lemma 2.3, we derive

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{D_{n+1}}x_0)\\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{D_{n+1}}x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that $\phi(x_{n+1}, x_n) \to 0$. So it follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

**Step 2.** We claim that $\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\widetilde{x}_n\| = 0.$

Indeed, since $x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}} x_0 \in C_{n+1}$, from the definition of $C_{n+1}$, we have

$$\begin{aligned} \phi(x_{n+1}, y_n) &\le \beta_n \phi(x_{n+1}, u_n) + (1-\beta_n)\phi(x_{n+1}, z_n) \\ &\le \phi(x_{n+1}, \widetilde{x}_n) \text{ and } \langle x_{n+1} - \widetilde{x}_n, v_n\rangle \le 0. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} \phi(\Pi_{C_{n+1}} x_n, x_n) - \phi(\widetilde{x}_n, x_n) &= \|\Pi_{C_{n+1}} x_n\|^2 - \|\widetilde{x}_n\|^2 + 2\langle \widetilde{x}_n - \Pi_{C_{n+1}} x_n, Jx_n\rangle \\ &\ge 2\langle \Pi_{C_{n+1}} x_n - \widetilde{x}_n, J\widetilde{x}_n\rangle + 2\langle \widetilde{x}_n - \Pi_{C_{n+1}} x_n, Jx_n\rangle \\ &= 2\langle \widetilde{x}_n - \Pi_{C_{n+1}} x_n, Jx_n - J\widetilde{x}_n\rangle. \end{aligned}$$

Since $\Pi_{C_{n+1}} x_n \in C_{n+1}$ and $v_n = \frac{1}{r_n}(Jx_n - J\widetilde{x}_n)$, it follows that

$$\langle \widetilde{x}_n - \Pi_{C_{n+1}} x_n, Jx_n - J\widetilde{x}_n\rangle = r_n \langle \widetilde{x}_n - \Pi_{C_{n+1}} x_n, v_n\rangle \ge 0$$

and hence that $\phi(\Pi_{C_{n+1}} x_n, x_n) \ge \phi(\widetilde{x}_n, x_n)$. Further, from $x_{n+1} \in C_{n+1}$, we have $\phi(x_{n+1}, x_n) \ge \phi(\Pi_{C_{n+1}} x_n, x_n)$, which yields

$$\phi(x_{n+1}, x_n) \ge \phi(\Pi_{C_{n+1}} x_n, x_n) \ge \phi(\widetilde{x}_n, x_n).$$

Then it follows from $\phi(x_{n+1}, x_n) \to 0$ that $\phi(\widetilde{x}_n, x_n) \to 0$. Hence it follows from Lemma 2.1 that $\widetilde{x}_n - x_n \to 0$. Since from (3.3) we derive

$$\begin{aligned} &\phi(x_{n+1}, \widetilde{x}_n) - \phi(\widetilde{x}_n, x_n) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\widetilde{x}_n\rangle + \|\widetilde{x}_n\|^2 - (\|\widetilde{x}_n\|^2 - 2\langle \widetilde{x}_n, Jx_n\rangle + \|x_n\|^2) \\ &= \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1}, J\widetilde{x}_n\rangle + 2\langle \widetilde{x}_n, Jx_n\rangle \\ &= \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1} - \widetilde{x}_n, J\widetilde{x}_n - Jx_n\rangle \\ &\quad -2\langle x_{n+1} - \widetilde{x}_n, Jx_n\rangle + 2\langle \widetilde{x}_n, Jx_n - J\widetilde{x}_n\rangle \\ &= (\|x_{n+1}\| - \|x_n\|)(\|x_{n+1}\| + \|x_n\|) + 2r_n\langle x_{n+1} - \widetilde{x}_n, v_n\rangle \\ &\quad -2\langle x_{n+1} - \widetilde{x}_n, Jx_n\rangle + 2\langle \widetilde{x}_n, Jx_n - J\widetilde{x}_n\rangle \\ &\le \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) + 2\|x_{n+1} - \widetilde{x}_n\|\|x_n\| + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\| \\ &\le \|x_{n+1} - x_n\|(\|x_{n+1}\| \\ &\quad +\|x_n\|) + 2(\|x_{n+1} - x_n\| + \|x_n - \widetilde{x}_n\|)\|x_n\| + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\|, \end{aligned}$$

we have

$$\begin{aligned} \phi(x_{n+1}, \widetilde{x}_n) &\le \phi(\widetilde{x}_n, x_n) + \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) \\ &\quad +2(\|x_{n+1} - x_n\| + \|x_n - \widetilde{x}_n\|)\|x_n\| + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\|. \end{aligned}$$

Thus from $\phi(\widetilde{x}_n, x_n) \to 0$, $x_n - \widetilde{x}_n \to 0$ and $x_{n+1} - x_n \to 0$, we know that $\phi(x_{n+1}, \widetilde{x}_n) \to 0$. Consequently, from (3.3) it follows that

$$\begin{aligned} \lim_{n\to\infty} \phi(x_{n+1}, y_n) &= \lim_{n\to\infty} \phi(x_{n+1}, u_n) \\ &= \lim_{n\to\infty} \phi(x_{n+1}, z_n) = \lim_{n\to\infty} \phi(x_{n+1}, \widetilde{x}_n) = 0. \end{aligned} \tag{3.4}$$

Utilizing Lemma 2.1 we deduce that

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1}-y_n\| &= \lim_{n\to\infty}\|x_{n+1}-u_n\| = \lim_{n\to\infty}\|x_{n+1}-z_n\| \\ &= \lim_{n\to\infty}\|x_{n+1}-\widetilde{x}_n\| = 0.\end{aligned} \tag{3.5}$$

Since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$, from (3.5) we get

$$\begin{aligned}\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\| &= \lim_{n\to\infty}\|Jx_{n+1}-Ju_n\| = \lim_{n\to\infty}\|Jx_{n+1}-Jz_n\| \\ &= \lim_{n\to\infty}\|Jx_{n+1}-J\widetilde{x}_n\| = 0.\end{aligned} \tag{3.6}$$

**Step 3.** We claim that

$$\lim_{n\to\infty}\|u_n-Su_n\| = \lim_{n\to\infty}\|z_n-\widetilde{S}z_n\| = 0. \tag{3.7}$$

Indeed, it follows from (3.6) that

$$\lim_{n\to\infty}\|Jy_n-Jz_n\| = \lim_{n\to\infty}\|Ju_n-Jz_n\| = 0.$$

Also, it follows from (3.1) that

$$Jz_n-Ju_n = (1-\alpha_n)(JSu_n-Ju_n),$$

and

$$Jy_n-Jz_n = \beta_n(Ju_n-Jz_n)+(1-\beta_n)(J\widetilde{S}z_n-Jz_n).$$

Thus, we have

$$(1-\alpha_n)\|JSu_n-Ju_n\| = \|Jz_n-Ju_n\| \to 0,$$

and

$$\begin{aligned}(1-\beta_n)\|J\widetilde{S}z_n-Jz_n\| &= \|Jy_n-Jz_n-\beta_n(Ju_n-Jz_n)\| \\ &\le \|Jy_n-Jz_n\|+\beta_n\|Ju_n-Jz_n\| \to 0.\end{aligned}$$

This implies that $\|JSu_n-Ju_n\|\to 0$ and $\|J\widetilde{S}z_n-Jz_n\|\to 0$. Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded subsets of $E^*$, we conclude that $\|Su_n-u_n\|\to 0$ and $\|\widetilde{S}z_n-z_n\|\to 0$.

**Step 4.** We claim that $\omega_w(\{x_n\})\subset EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0$, where

$$\omega_w(\{x_n\}) := \{\hat{x}\in C : x_{n_k}\rightharpoonup\hat{x} \text{ for some subsequence } \{n_k\}\subset\{n\} \text{ with } n_k\uparrow\infty\}.$$

Indeed, since $\{x_n\}$ is bounded and $E$ is reflexive, we know that $\omega_w(\{x_n\})\neq\emptyset$. Take $\hat{x}\in\omega_w(\{x_n\})$ arbitrarily. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. Hence from $x_{n+1} - x_n \to 0$ and (3.5) it follows that $u_n - x_n \to 0$ and $z_n - x_n \to 0$. So we deduce that $u_{n_k} \rightharpoonup \hat{x}$ and $z_{n_k} \rightharpoonup \hat{x}$. Since $S$ and $\widetilde{S}$ are relatively nonexpansive, from (3.7) we obtain that $\hat{x} \in \widehat{F}(S) = F(S)$ and $\hat{x} \in \widehat{F}(\widetilde{S}) = F(\widetilde{S})$. This implies that $\hat{x} \in F(S) \cap F(\widetilde{S})$.

Now let us show that $\hat{x} \in T^{-1}0$. Since $x_n - \widetilde{x}_n \to 0$, we have that $\widetilde{x}_{n_k} \rightharpoonup \hat{x}$. Moreover, since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$ and $\liminf_{n\to\infty} r_n > 0$, we obtain

$$v_n = \frac{1}{r_n}(Jx_n - J\widetilde{x}_n) \to 0.$$

It follows from $v_n \in T\widetilde{x}_n$ and the monotonicity of $T$ that

$$\langle z - \widetilde{x}_n, z' - v_n \rangle \geq 0$$

for all $z \in D(T)$ and $z' \in Tz$. This implies that

$$\langle z - \hat{x}, z' \rangle \geq 0$$

for all $z \in D(T)$ and $z' \in Tz$. Thus from the maximality of $T$, we infer that $\hat{x} \in T^{-1}0$. Therefore, $\hat{x} \in F(S) \cap F(\widetilde{S}) \cap T^{-1}0$. Further, let us show that $\hat{x} \in EP$. Since $\widetilde{x}_n - x_n \to 0$ (due to Step 2), from $x_{n_k} \rightharpoonup \hat{x}$ we know that $\widetilde{x}_{n_k} \rightharpoonup \hat{x}$.

Since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$, from $u_n - \widetilde{x}_n \to 0$ (due to (3.5)) we derive

$$\lim_{n\to\infty} \|Ju_n - J\widetilde{x}_n\| = 0.$$

From $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \frac{\|Ju_n - J\widetilde{x}_n\|}{r_n} = 0. \tag{3.8}$$

By the definition of $u_n := K_{r_n}\widetilde{x}_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n \rangle \geq 0, \quad \forall y \in C,$$

where

$$F(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n \rangle.$$

Replacing $n$ by $n_k$, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - J\widetilde{x}_{n_k} \rangle \geq -F(u_{n_k}, y) \geq F(y, u_{n_k}), \quad \forall y \in C.$$

Since $y \mapsto f(x, y) + \langle Ax, y - x \rangle$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in the last inequality, from (3.8) and (A4) we have

$$F(y, \hat{x}) \leq 0, \quad \forall y \in C.$$

For $t$, with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, then $y_t \in C$ and hence $F(y_t, \hat{x}) \leq 0$. So, from (A1) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \leq tF(y_t, y).$$

Dividing by $t$, we have

$$F(y_t, y) \geq 0, \quad \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) it follows that

$$F(\hat{x}, y) \geq 0, \quad \forall y \in C.$$

So, $\hat{x} \in EP$. Therefore, we obtain that $\omega_w(\{x_n\}) \subset EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0$ by the arbitrariness of $\hat{x}$.

**Step 5.** We claim that $\omega_w(\{x_n\}) = \{\Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0\}$ and $x_n \to \Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$.

Indeed, put $\overline{x} = \Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$. From $x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}}x_0$ and $\overline{x} \in EP \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \subset C_{n+1} \cap D_{n+1}$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive for each $\hat{x} \in \omega_w(\{x_n\})$

$$\begin{aligned}
\phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \phi(\overline{x}, x_0).
\end{aligned}$$

It follows from the definition of $\Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$ that $\hat{x} = \overline{x}$ and hence

$$\lim_{k\to\infty} \phi(x_{n_k}, x_0) = \phi(\overline{x}, x_0).$$

So we have $\lim_{k\to\infty} \|x_{n_k}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of $E$, we conclude that $\{x_{n_k}\}$ converges strongly to $\Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$. Since $\{x_{n_k}\}$ is an arbitrary weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\Pi_{EP\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$. This completes the proof. ■

The following corollaries can be obtained from Theorem 3.1 immediately.

**Corollary 3.1.** *Let $E$ and $C$ be the same as in Theorem 3.1. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4), and $S, \widetilde{S} : C \to C$ be a pair of relatively nonexpansive mappings such that $EP(f) \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$(3.9)\qquad \begin{cases} x_0 \in C,\ C_0 = C, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in C \text{ such that} \\ \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \geq 0,\ \forall y \in C, \\ z_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n\phi(v, u_n) + (1-\beta_n)\phi(v, z_n) \\ \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}}x_0,\ n = 0, 1, 2, ..., \end{cases}$$

*where* $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *satisfy (3.2). Then* $\{x_n\}$ *converges strongly to* $\Pi_{EP(f)\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$.

*Proof.* Put $A \equiv 0$ in Theorem 3.1. Then $EP = EP(f)$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. ■

**Corollary 3.2.** *Let* $E$ *and* $C$ *be the same as in Theorem 3.1. Let* $T : E \to 2^{E^*}$ *be a maximal monotone operator,* $A : C \to E^*$ *be an* $\alpha$*-inverse-strongly monotone mapping and* $S, \widetilde{S} : C \to C$ *be a pair of relatively nonexpansive mappings such that* $VI(C, A) \cap F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by*

$$(3.10)\qquad \begin{cases} x_0 \in C,\ C_0 = C, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in C \text{ such that} \\ \quad \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \geq 0,\ \forall y \in C, \\ z_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{S}z_n), \\ \quad C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n\phi(v, u_n) + (1-\beta_n)\phi(v, z_n) \\ \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ \quad D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}}x_0,\ n = 0, 1, 2, ..., \end{cases}$$

*where* $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *satisfy (3.2). Then* $\{x_n\}$ *converges strongly to* $\Pi_{VI(C,A)\cap F(S)\cap F(\widetilde{S})\cap T^{-1}0}x_0$.

*Proof.* Put $f \equiv 0$ in Theorem 3.1. Then $EP = VI(C, A)$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. ■

**Corollary 3.3.** *Let $E$ and $C$ be the same as in Theorem 3.1. Let $A : C \to E^*$ be an $\alpha$-inverse-strongly monotone mapping, $f : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4), and $S, \widetilde{S} : C \to C$ be a pair of relatively nonexpansive mappings such that $EP \cap F(S) \cap F(\widetilde{S}) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$
(3.11) \quad \begin{cases} x_0 \in C, \ C_0 = C, \\ u_n \in C \text{ such that} \\ \quad f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jx_n \rangle \geq 0, \ \forall y \in C, \\ z_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n) JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1 - \beta_n) J\widetilde{S} z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n \phi(v, u_n) \\ \quad + (1 - \beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}} x_0, \ n = 0, 1, 2, ..., \end{cases}
$$

*where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{EP \cap F(S) \cap F(\widetilde{S})} x_0$.*

*Proof.* Put $T \equiv 0$ in Theorem 3.1. Then $v_n \equiv 0$ and so $\widetilde{x}_n = x_n, \ \forall n \geq 0$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. ■

**Corollary 3.4.** *Let $E$ and $C$ be the same as in Theorem 3.1. Let $T : E \to 2^{E^*}$ be a maximal monotone operator, and $S, \widetilde{S} : C \to C$ be a pair of relatively nonexpansive mappings such that $F(S) \cap F(\widetilde{S}) \cap T^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$
(3.12) \quad \begin{cases} x_0 \in C, \ C_0 = C, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n), \ v_n \in T\widetilde{x}_n, \\ u_n = \Pi_C \widetilde{x}_n, \\ z_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n) JSu_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1 - \beta_n) J\widetilde{S} z_n), \\ \quad C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n \phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \\ \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n \rangle \leq 0\}, \\ \quad D_{n+1} = \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}} x_0, \ n = 0, 1, 2, ..., \end{cases}
$$

*where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap F(\widetilde{S}) \cap T^{-1}0} x_0$.*

*Proof.* Put $A \equiv 0$ and $f \equiv 0$ in Theorem 3.1. Then $u_n = \Pi_C \widetilde{x}_n, \ \forall n \geq 0$. Hence from Theorem 3.1 we immediately obtain the desired conclusion. ■

## 4. Applications

Let $E$ be a reflexive, strictly convex, and smooth Banach space. Let $U, \widetilde{U} : E \to 2^{E^*}$ be two maximal monotone operators. For $r > 0$, define the resolvent of $U$ and $\widetilde{U}$ by $J_r = (J + rU)^{-1}J$ and $\widetilde{J}_r = (J + r\widetilde{U})^{-1}J$, respectively. Then, $J_r$ (resp. $\widetilde{J}_r$) is a single-valued mapping from $E$ to $D(U)$ (resp. from $E$ to $D(\widetilde{U})$). Also, for $r > 0$,

$$U^{-1}0 = F(J_r) \text{ (resp. } \widetilde{U}^{-1}0 = F(\widetilde{J}_r)), \tag{4.1}$$

where $F(J_r)$ (resp. $F(\widetilde{J}_r)$) is the set of fixed points of $J_r$ (resp. $\widetilde{J}_r$). We can define, for $r > 0$, the Yosida approximation of $U$ (resp. $\widetilde{U}$) by $A_r = (J - JJ_r)/r$ (resp. $\widetilde{A}_r = (J - J\widetilde{J}_r)/r$). For $r > 0$ and $x \in E$, we know that $A_r x \in UJ_r x$ and $\widetilde{A}_r x \in \widetilde{U}\widetilde{J}_r x$.

**Lemma 4.1.** *Let $E$ be a reflexive, strictly convex, and smooth Banach space, and let $U : E \to 2^{E^*}$ be a maximal monotone operator with $U^{-1}0 \neq \emptyset$. Then there hold the following*

($i$) *(see [29]) $\phi(z, J_r x) + \phi(J_r x, x) \leq \phi(z, x)$ for all $r > 0, \ z \in U^{-1}0$ and $x \in E$;*

($ii$) *(see [28]) $J_r : E \to D(U)$ is a relatively nonexpansive mapping.*

We are now in a position to apply Theorem 3.1 to proving the following result.

**Theorem 4.1.** *Let $E$ be a uniformly smooth and uniformly convex Banach space, $r > 0$ be a positive constant, $A : E \to E^*$ be an $\alpha$-inverse-strongly monotone mapping, and $f : E \times E \to \mathcal{R}$ be a bifunction satisfying (A1)-(A4). Let $T, U, \widetilde{U} : E \to 2^{E^*}$ be three maximal monotone operators such that $EP \cap U^{-1}0 \cap \widetilde{U}^{-1}0 \cap T^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in E, \ C_0 = E, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n), \ v_n \in T\widetilde{x}_n, \\ u_n \in E \text{ such that} \\ \quad f(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n \rangle \geq 0, \ \forall y \in E, \\ z_n = J^{-1}(\alpha_n Ju_n + (1 - \alpha_n)JJ_r u_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1 - \beta_n)J\widetilde{J}_r z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n \phi(v, u_n) + (1 - \beta_n)\phi(v, z_n) \\ \quad \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n \rangle \leq 0\}, \\ D_{n+1} = \{v \in E : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1} \cap D_{n+1}} x_0, \ n = 0, 1, 2, ..., \end{cases} \tag{4.2}$$

*where* $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *satisfy (3.2). Then* $\{x_n\}$ *converges strongly to* $\Pi_{EP \cap U^{-1}0 \cap \widetilde{U}^{-1}0 \cap T^{-1}0} x_0$.

*Proof.* From (4.1) and Lemma 4.1 it follows that $J_r : E \to D(U)$ and $\widetilde{J_r} : E \to D(\widetilde{U})$ both are relatively nonexpansive mappings and $U^{-1}0 = F(J_r)$, $\widetilde{U}^{-1}0 = F(\widetilde{J_r})$. Now put $S = J_r$ and $\widetilde{S} = \widetilde{J_r}$ in Theorem 3.1. Then from Theorem 3.1 we immediately obtain the desired conclusion. ■

From Theorem 4.1, we can derive the following corollaries.

**Corollary 4.1** Let $E$ and $r > 0$ be the same as in Theorem 4.1. Let $A : E \to E^*$ be an $\alpha$-inverse-strongly monotone mapping and $T, U, \widetilde{U} : E \to 2^{E^*}$ be three maximal monotone operators such that $VI(E, A) \cap U^{-1}0 \cap \widetilde{U}^{-1}0 \cap T^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by

$$(4.3)\qquad \begin{cases} x_0 \in E,\ C_0 = E, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in E \text{ such that} \\ \quad \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \geq 0,\ \forall y \in E, \\ z_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JJ_r u_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{J}_r z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n\phi(v, u_n) + (1-\beta_n)\phi(v, z_n) \\ \quad \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ D_{n+1} = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}} x_0,\ n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ satisfy (3.2). Then $\{x_n\}$ converges strongly to $\Pi_{VI(E,A)\cap U^{-1}0\cap \widetilde{U}^{-1}0\cap T^{-1}0} x_0$.

*Proof.* Put $f \equiv 0$ in Theorem 4.1. Then from Theorem 4.1 we immediately obtain the desired conclusion. ■

**Corollary 4.2.** *Let* $E$ *and* $r > 0$ *be the same as in Theorem 4.1. Let* $f : E \times E \to \mathcal{R}$ *be a bifunction satisfying (A1)-(A4) and* $T, U, \widetilde{U} : E \to 2^{E^*}$ *be three maximal monotone operators such that* $EP(f) \cap U^{-1}0 \cap \widetilde{U}^{-1}0 \cap T^{-1}0 \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by*

$$(4.4)\qquad \begin{cases} x_0 \in E,\ C_0 = E, \\ 0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n),\ v_n \in T\widetilde{x}_n, \\ u_n \in E \text{ such that} \\ \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - J\widetilde{x}_n\rangle \geq 0,\ \forall y \in E, \\ z_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JJ_r u_n), \\ y_n = J^{-1}(\beta_n Ju_n + (1-\beta_n)J\widetilde{J}_r z_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \beta_n\phi(v, u_n) + (1-\beta_n)\phi(v, z_n) \\ \quad \leq \phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ D_{n+1} = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_{n+1}\cap D_{n+1}} x_0,\ n = 0, 1, 2, ..., \end{cases}$$

*where* $\{r_n\} \subset (0, \infty)$ *and* $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ *satisfy (3.2). Then* $\{x_n\}$ *converges strongly to* $\Pi_{EP(f)\cap U^{-1}0\cap\widetilde{U}^{-1}0\cap T^{-1}0}x_0$.

*Proof.* Put $A \equiv 0$ in Theorem 4.1. Then from Theorem 4.1 we immediately obtain the desired conclusion. ■

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