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FINITENESS RESULT FOR GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a Noetherian ring, let M and N be finitely generated modules and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. Let s be an integer such that $\mathfrak{b}_\mathfrak{p} \subseteq \sqrt{\operatorname{Ann} \mathrm{H}^i_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})}$ for all $i \leq s$ and all prime ideal $\mathfrak{p}$ of R. Then we show the following statements hold:

(1) If $\mathrm{H}^i_\mathfrak{b}(N) = 0$ for all $i < s$, then $\mathrm{H}^i_\mathfrak{a}(M, N)$ is finitely generated for all $i \leq s$.
(2) $\mathfrak{b} \subseteq \sqrt{\operatorname{Ann} \mathrm{H}^2_\mathfrak{a}(M, N)}$.

These statements generalize the corresponding results which are shown in [6] and [1] for standard local cohomology module.

1. INTRODUCTION

Throughout this note the ring R is commutative Noetherian with non–zero identity. Let M and N be R–modules and let $\mathfrak{a}$ be an ideal of R. Then the generalized local cohomology module

$$\mathrm{H}^i_\mathfrak{a}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

was introduced by Herzog in [5] and studied further by Suzuki in [7] and Yassemi in [8]. Note that $\mathrm{H}^i_\mathfrak{a}(R, N) = \mathrm{H}^i_\mathfrak{a}(N)$.

Let M be a finitely generated R–module and let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of R. In [6] Raghavan proved the following statement:

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If $\mathfrak{b}_\mathfrak{p} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^1_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})}$ for every prime ideal $\mathfrak{p}$ of R, and $\mathfrak{b} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^0_\mathfrak{a}(M)}$, then $\mathfrak{b} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^1_\mathfrak{a}(M)}$. If, in addition, $\mathrm{H}^0_\mathfrak{b}(M) = \mathrm{H}^0_\mathfrak{a}(M) = 0$, then $\mathrm{H}^1_\mathfrak{a}(M)$ is finitely generated.

On the other hand in [1] Brodmann, Rotthaus, and Sharp proved the following statement:
If $\mathfrak{b}_\mathfrak{p} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^i_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p})}$ for all $i \leq 2$ and all prime ideal $\mathfrak{p}$ of R then $\mathfrak{b} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^i_\mathfrak{a}(M)}$ for all $i \leq 2$.

In this note we generalize the above results for the generalized local cohomology modules, see Theorems 1 and 2.

2. Main Results

First we recall some known results which we will use in this paper.

Lemma A. ([1, Lemma 2.1]). *Let L be an R–module such that the set of associated primes of L is finite and $\mathfrak{b}_\mathfrak{p} \subseteq \sqrt{Ann_{R_\mathfrak{p}} L_\mathfrak{p}}$ for all prime ideal $\mathfrak{p}$ of R. Then $\mathfrak{b} \subseteq \sqrt{Ann_R L}$.*

Theorem B. ([9, Theorem 2.1]). *Let M and N be finitely generated R-modules. Let $s \in \mathbb{N}_0$ be such that $H^i_\mathfrak{a}(M, N)$ is finitely generated for all $i < s$. Then the set of associated primes of the module $H^s_\mathfrak{a}(M, N)$ is finite.*

In the first step we prove an extension of Faltings' lemma cf. [3, Lemma 3] and [4, pp. 48-49], (see also [2, Prop. 9.1.2] and [2, Theorem 9.6.1]), in the context of generalized local cohomology modules.

Theorem 1. *Let M and N be finitely generated R–modules and let $s \in \mathbb{N}$. Then the following statements are equivalent:*

(1) *$H^i_\mathfrak{a}(M, N)$ is finitely generated for all $i < s$;*

(2) *$H^i_{\mathfrak{a}_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is finitely generated for all $i < s$ and all prime ideal $\mathfrak{p}$ of R;*

(3) *$\mathfrak{a} \subseteq \sqrt{Ann\, H^i_\mathfrak{a}(M, N)}$ for all $i < s$.*

Proof. We show that (1)⇔(2) and (1)⇔(3).

(1)⇒(2) is clear.

(1)⇒(3): It follows from the fact that $\mathrm{H}^i_\mathfrak{a}(M, N)$ is finitely generated and $\mathfrak{a}$–torsion.

(3)⇒(1): We use induction on s. For $s = 1$ the assertion follows from the fact $\mathrm{H}^0_\mathfrak{a}(M, N) = \Gamma_\mathfrak{a}(\mathrm{Hom}\,(M, N)$. Let $s > 1$. By induction hypothesis we know that

$\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finitely generated for $0 \le i \le s-2$. Hence it remains to prove that $\mathrm{H}^{s-1}_{\mathfrak{a}}(M,N)$ is finitely generated. First assume that N is $\mathfrak{a}$–torsion free module. Let x be an element of $\mathfrak{a}$ which is regular on N. Then $x^k\mathrm{H}^{s-1}_{\mathfrak{a}}(M,N) = 0$ for all $k \gg 0$. It follows from [2, Lemma 9.1.1] and the long exact sequence on generalized local cohomology induced by $0 \longrightarrow N \xrightarrow{x^k\cdot} N \longrightarrow N/x^kN \longrightarrow 0$ that $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^i_{\mathfrak{a}}(M, N/x^kN)}$ for all $i < s-1$. Hence by induction hypothesis $\mathrm{H}^i_{\mathfrak{a}}(M, N/x^kN)$ is finitely generated for all $i < s-1$. Thus $\mathrm{H}^{s-1}_{\mathfrak{a}}(M,N)$ is finitely generated (since it is homomorphic image of $\mathrm{H}^{s-2}_{\mathfrak{a}}(M, N/x^kN)$).

Now let N be an arbitrary finitely generated R–module, and set $L = \Gamma_{\mathfrak{a}}(N)$, $K = N/L$. The exact sequence $0 \to L \to N \to K \to 0$ induces the exact sequence $\mathrm{H}^i_{\mathfrak{a}}(M,L) \to \mathrm{H}^i_{\mathfrak{a}}(M,N) \to \mathrm{H}^i_{\mathfrak{a}}(M,K) \to \mathrm{H}^{i+1}_{\mathfrak{a}}(M,L)$. By [9, Lemma 1.1] we know that $\mathrm{H}^i_{\mathfrak{a}}(M,L) = \mathrm{Ext}^i(M,L)$ for all i and so $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^{i+1}_{\mathfrak{a}}(M,L)}$. Thus $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^i_{\mathfrak{a}}(M,K))}$ for all $i < s$. Now since K is $\mathfrak{a}$–torsion free we have that $\mathrm{H}^i_{\mathfrak{a}}(M,K)$ is finitely generated for $i < s$ and hence $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finitely generated for $i < s$.

(2)⇒(1): For $s = 1$ it is clear. Let $s > 1$. By the induction hypothesis we have $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finitely generated for $0 \le i \le s-2$ and hence $|\mathrm{Ass}\,\mathrm{H}^i_{\mathfrak{a}}(M,N)| < \infty$ for all $i \le s-1$ by Theorem B. By Lemma A we have $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^i_{\mathfrak{a}}(M,N)}$ for all $i < s-1$. Now the assertion follows from the equivalence between (1) and (3).

The following theorem is a generalization of the main result in [6].

Theorem 2. *Let M and N be finitely generated modules and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. If s is an integer such that $H^i_{\mathfrak{b}}(N) = 0$ for $i < s$, and $\mathfrak{b}_{\mathfrak{p}} \subseteq \sqrt{Ann\,H^i_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})}$ for all $i \le s$ and all prime ideal $\mathfrak{p}$ of R then $H^i_{\mathfrak{a}}(M,N)$ is finitely generated for all $i \le s$.*

Proof. We use induction on s. Assume that $s = 1$. Since $\mathrm{H}^0_{\mathfrak{p}}(M,N) = \Gamma_{\mathfrak{a}}(\mathrm{Hom}\,(M,N)$ is finitely generated, the set of associated primes of $\mathrm{H}^1_{\mathfrak{a}}(M,N)$ is finite by Theorem B, and hence by Lemma A we have that $\mathfrak{b} \subseteq \sqrt{\mathrm{Ann}\,\mathrm{H}^1_{\mathfrak{a}}(M,N)}$. Therefore, there exists an integer k such that $\mathfrak{b}^k\mathrm{H}^1_{\mathfrak{a}}(M,N) = 0$. Let x be an element of $\mathfrak{b}$ that is regular on N. Since $x^k\mathrm{H}^1_{\mathfrak{a}}(M,N) = 0$, it follows from the long exact sequence on generalized local cohomology induced by $0 \longrightarrow N \xrightarrow{x^k\cdot} N \longrightarrow N/x^kN \longrightarrow 0$ that $\mathrm{H}^1_{\mathfrak{a}}(M,N)$ is a homomorphic image of $\mathrm{H}^0_{\mathfrak{a}}(M, N/x^kN)$ and hence is finitely generated.

Now let $s > 1$. By induction hypothesis we have $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ is finitely generated for all $i < s$ and hence $\mathrm{Ass}\,\mathrm{H}^s_{\mathfrak{a}}(M,N)$ is a finite set by Theorem B. Let x be an element in $\mathfrak{b}$ that is regular on N. By Lemma A we know that $x^k\mathrm{H}^s_{\mathfrak{a}}(M,N) = 0$ for all $k \gg 0$. It follows from the long exact sequence on local cohomology induced by $0 \longrightarrow N \xrightarrow{x^k\cdot} N \longrightarrow N/x^kN \longrightarrow 0$ that $H^i_{\mathfrak{b}}(N/x^kN) = 0$ for all $i < s-1$. Now it follows from the long exact sequence

$$\mathrm{H}^i_{\mathfrak{a}}(M,N)\longrightarrow \mathrm{H}^i_{\mathfrak{a}}(M,N/x^kN)\longrightarrow \mathrm{H}^{i+1}_{\mathfrak{a}}(M,N)\xrightarrow{x^k\cdot}\mathrm{H}^{i+1}_{\mathfrak{a}}(M,N)$$

and using [2, Lemma 9.1.1], we have $\mathfrak{b}_{\mathfrak{p}}\subseteq\sqrt{\operatorname{Ann}\mathrm{H}^i_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}},(N/x^kN)_{\mathfrak{p}})}$ for all $i<s$. Therefore, by induction hypothesis $\mathrm{H}^i_{\mathfrak{a}}(M,N/x^kN)$ is finitely generated for all $i<s$ and hence $\mathrm{H}^s_{\mathfrak{a}}(M,N)$ is finitely generated (since it is a homomorphic image of $\mathrm{H}^{s-1}_{\mathfrak{a}}(M,N/x^kN)$).

Corollary 3. *Let M and N be finitely generated modules and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R. If $\mathfrak{b}_{\mathfrak{p}}\subseteq\sqrt{Ann\,H^i_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})}$ for all $i\leq 2$ and all prime ideal $\mathfrak{p}$ of R then $\mathfrak{b}\subseteq\sqrt{Ann\,H^i_{\mathfrak{a}}(M,N)}$ for all $i\leq 2$.*

Proof. Set $N'=N/\Gamma_{\mathfrak{b}}(N)$. By using the same technique as [1, Remark 1.3(ii)], we have that $\mathfrak{b}_{\mathfrak{p}}\subseteq\sqrt{\operatorname{Ann}\mathrm{H}^i_{\mathfrak{a}_{\mathfrak{p}}}(M_{\mathfrak{p}},N'_{\mathfrak{p}})}$ for all $i\leq 2$ and all prime ideal $\mathfrak{p}$ of R. Since $\mathrm{H}^i_{\mathfrak{b}}(N')=0$ for all $i<1$ we have $\mathrm{H}^i_{\mathfrak{a}}(M,N')$ is finitely generated for all $i\leq 1$ and hence $\operatorname{Ass}\mathrm{H}^2_{\mathfrak{a}}(M,N')$ is a finite set. Therefore, by Lemma A, $\mathfrak{b}\subseteq\sqrt{\operatorname{Ann}\mathrm{H}^i_{\mathfrak{a}}(M,N)}$ for all $i\leq 2$.

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