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BOUNDEDNESS OF OPERATORS ON HARDY SPACES

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Abstract. In [1], the author provided an example which shows that there is a linear functional bounded uniformly on all atoms in $H^1(\mathbb{R}^n)$, and it can not be extended to a bounded functional on $H^1(\mathbb{R}^n)$. In this note, we first give a new atomic decomposition, where the decomposition converges in $L^2(\mathbb{R}^n)$ rather than only in the distribution sense. Then using this decomposition, we prove that for $0 < p \leq 1, T$ is a linear operator which is bounded on $L^2(\mathbb{R}^n)$, then T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if T is bounded uniformly on all $(p, 2)$-atoms in $L^p(\mathbb{R}^n)$. A similar result from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ is also obtained. These results still hold for the product Hardy space and Hardy space on spaces of homogeneous type.

1. Introduction

It is important to verify boundedness for many important classes of operators defined on H^p spaces. And the atomic decompositions of Hardy spaces play an important role in the boundedness of operators on Hardy spaces. The best known example of a class with this property are Calderón-Zygmund operators. As we know, usually, it is indeed sufficient to check that atoms are mapped into bounded elements of quasi-Banach spaces. Recently, in [1], M.Bownik gave an example of a linear functional defined on a dense subspace of Hardy space $H^1(\mathbb{R}^n)$, which maps all atoms into bounded scalars, but it can not be extended to a bounded functional on the whole space $H^1(\mathbb{R}^n)$. As a consequence of his example, it implies that to prove the boundedness of an operator from Hardy space $H^p(\mathbb{R}^n), 0 < p \leq 1$, to some other quasi-Banach space, in general it does not suffice to just verify that this operator maps atoms into bounded elements of this quasi-Banach space. Therefore,

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it should be very carefully to do this. Maybe this problem is based on the atomic decomposition of Hardy spaces. Since Calderón-Zygmund operators are bounded on $L^2(\mathbb{R}^n)$ spaces, the atomic decompositions are converged in the distribution sense (not converged in $L^2(\mathbb{R}^n)$). So, the operators should not be put into each one atom in the series.

In this paper, using the Calderón reproducing formula, we give a new atomic decomposition of a dense subspace $H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ of the Hardy spaces $H^p(\mathbb{R}^n)$, where the decomposition converges also in $L^2(\mathbb{R}^n)$ rather than only in the distribution sense. Then, using this atomic decomposition, we can prove the boundedness of linear operators on Hardy spaces by T is bounded uniformly on all atoms.

The main result of this note is to prove the following Theorem.

Theorem 1.1. *Fix $0 < p \leq 1$. Let T be a linear operator which is bounded on $L^2(\mathbb{R}^n)$. (i) T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $\|Ta\|_p \leq C$ for all $(p, 2)$-atoms, where the constant C is independent of a; (ii) T can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ if and only if $\|Ta\|_{H^p} \leq C$ for all $(p, 2)$-atoms, where the constant C is also independent of a.*

This theorem is achieved by the following new atomic decomposition.

Theorem 1.2. *Let $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$. Then there is a sequence of $(p, 2)$-atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p}$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in $L^2(\mathbb{R}^n)$.*

2. Proof of Theorems

We recall some basic definitions and results.

Let $\psi(x)$ be a radial Schwartz function supported in the unit ball and satisfying the conditions $\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $\int_{\mathbb{R}^n} \psi(x) x^\alpha dx = 0$ for all nonnegative multi-indexes α with $|\alpha| \leq [n(\frac{1}{p} - 1)]$.

Definition 2.1. Suppose that $f \in \mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. Let ψ be a function as above. The Lusin function of $f, S(f)$, is defined by

$$S(f)(x) = \left\{ \int_0^\infty \int_{|y-x|<t} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1}$$

where $\psi_t(x) = t^{-n}\psi(\frac{x}{t})$.

Definition 2.2. The Hardy space $H^p(\mathbb{R}^n), 0 < p \le 1$, is defined by

$$H^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : S(f) \in L^p(\mathbb{R}^n)\}. \tag{2}$$

If $f \in H^p(\mathbb{R}^n)$, the norm of f is defined by $\|S(f)\|_p$. It was known that the definition 2.2 is independent of the choice of the function ψ.

The usual atomic decomposition of $H^p(\mathbb{R}^n)$ is as follows (cf. [2, 4, 6] etc.).

Theorem 2.3. *Let $f \in H^p(\mathbb{R}^n)$. Then there is a sequence of $(p, 2)$-atoms $\{a_j\}$ and a sequence of scalars $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \le C\|f\|_{H^p}^p$ such that $f = \sum_j \lambda_j a_j$, where the series converges to f in the sense of tempered distributions. Conversely, if f is a tempered distribution such that $f = \sum_j \lambda_j a_j$ in the sense of tempered distributions with $\sum_j |\lambda_j|^p < \infty$, and the a_j's being $(p, 2)$-atoms, then $f \in H^p(\mathbb{R}^n)$ and $\|f\|_{H^p}^p \le C \sum_j |\lambda_j|^p$.*

Here a function $a(x)$ is said to be an $(p, 2)$-atom of $H^p(\mathbb{R}^n), 0 < p \le 1$, if $a(x)$ is supported in a cube Q; $\|a\|_2 \le |Q|^{\frac{1}{2}-\frac{1}{p}}$; and finally, $\int a(x)x^\alpha dx = 0$ for all nonnegative multi-indexes α with $|\alpha| \le [n(\frac{1}{p} - 1)]$.

We first prove Theorem 1.2.

Proof of Theorem 1.2. Let ψ be a function mentioned above. Then the following Calderón reproducing formula holds

$$f(x) = \int_0^\infty \psi_t * \psi_t * f(x)\frac{dt}{t}, \tag{3}$$

where the integral converges in $L^2(\mathbb{R}^n)$.

Now, suppose $f \in L^2 \cap H^p$. Let $\Omega_k = \{x \in \mathbb{R}^n : S(f)(x) > 2^k\}$ and $B_k = \{Q :$ dyadic cubes such that $|Q \cap \Omega_k| > \frac{1}{2}|Q|$ and $|Q \cap \Omega_{k+1}| \le \frac{1}{2}|Q|\}$. For each dyadic cube Q, denote $\widehat{Q} = \{(y, t) : y \in Q$ and $\sqrt{n}\ell(Q) \le t < 2\sqrt{n}\ell(Q)\}$, where $\ell(Q)$ is the side length of Q. We claim that

$$f(x) = \sum_k \sum_{\widetilde{Q} \in B_k} \sum_{Q \subseteq \widetilde{Q} \cap B_k} \int_{\widehat{Q}} \psi_t(x - y)\psi_t * f(y)\frac{dydt}{t}, \tag{4}$$

where $\widetilde{Q} \in B_k$ are maximal dyadic cubes in B_k, and the series converges in $L^2(\mathbb{R}^n)$.

To prove the claim, it suffices to show that for any positive integer N,

$$\left\|\sum_{k>N}\sum_{Q\in B_k}\int_{\widehat{Q}}\psi_t(x-y)\psi_t*f(y)\frac{dydt}{t}\right\|_2$$

tends to zero as N goes to infinity.

First let $\widetilde{\Omega}_k=\{x\in\mathbb{R}^n: M(\chi_{\Omega_k})(x)>\frac{1}{2}\}$, where M is the Hardy-Littlewood maximal function. Then $\Omega_k\subseteq\widetilde{\Omega}_k$, and by the maximal theorem, $|\widetilde{\Omega}_k|\le C|\Omega_k|$. Let $\chi(x,y,t)$ be the characterization of $\{(x,y,t): x\in\widetilde{\Omega}_k\backslash\Omega_{k+1}, |x-y|<t\}$. For any $x\in Q\in B_k$, since $|Q\cap\Omega_k|\ge\frac{1}{2}|Q|$(by the definition of B_k), one has $x\in\widetilde{\Omega}_k$, thus if $(y,t)\in\widehat{Q}$, then

$$\begin{aligned}&\int_{\mathbb{R}^n}\chi(x,y,t)dx\ge|Q\cap(\widetilde{\Omega}_k\backslash\Omega_{k+1})|\\&=|Q\cap\widetilde{\Omega}_k|-|Q\cap\Omega_{k+1}|\ge|Q|-\frac{|Q|}{2}=C't^n.\end{aligned}$$

Therefore

$$\begin{aligned}C2^{2k}|\Omega_k|\ge 2^{2k}|\widetilde{\Omega}_k|\ge&\int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}}(Sf)^2(x)dx\\&=\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n}|\psi_t*f(y)|^2\chi(x,y,t)\frac{dydtdx}{t^{n+1}}\\&\ge\sum_{Q\in B_k}\int_{\widehat{Q}}\int_{\mathbb{R}^n}|\psi_t*f(y)|^2\chi(x,y,t)\frac{dydtdx}{t^{n+1}}\\&\ge C'\sum_{Q\in B_k}\int_{\widehat{Q}}|\psi_t*f(y)|^2\frac{dydt}{t}.\end{aligned}\tag{5}$$

Now by duality argument and Hölder's inequality, we have

$$\begin{aligned}&\left\|\sum_{k>N}\sum_{Q\in B_k}\int_{\widehat{Q}}\psi_t(x-y)\psi_t*f(y)\frac{dydt}{t}\right\|_2\\&=\sup_{\|g\|_2\le1}\left|<\sum_{k>N}\sum_{Q\in B_k}\int_{\widehat{Q}}\psi_t(x-y)\psi_t*f(y)\frac{dydt}{t},g>\right|\\&\le\sup_{\|g\|_2\le1}\sum_{k>N}\sum_{Q\in B_k}\int_{\widehat{Q}}|\psi_t*g(y)\psi_t*f(y)|\frac{dydt}{t}\end{aligned}\tag{6}$$

$$\leq \sup_{\|g\|_2 \leq 1} \left\{ \sum_{k>N} \sum_{Q \in B_k} \int_{\widehat{Q}} |\psi_t * g(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}}$$

$$\left\{ \sum_{k>N} \sum_{Q \in B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}}$$

$$\leq \sup_{\|g\|_2 \leq 1} \left\{ \int_{\mathbb{R}^{n+1}_+} |\psi_t * g(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}} \left\{ \sum_{k>N} \sum_{Q \in B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}}$$

$$\leq C \left\{ \sum_{k>N} \sum_{Q \in B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}},$$

where the last inequality follows from the L^2 estimates of the Littlewood-Paley square function

$$\left\{ \int_{\mathbb{R}^{n+1}_+} |\psi_t * g(y)|^2 \frac{dydt}{t} \right\}^{\frac{1}{2}} \leq C\|g\|_{L^2(\mathbb{R}^n)}.$$

Then the estimate in (5) implies that

$$\| \sum_{k>N} \sum_{Q \in B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y) \frac{dydt}{t} \|_2 \leq C(\sum_{k>N} 2^{2k}|\Omega_k|)^{\frac{1}{2}}.$$

The last term tends to zero as N goes to infinity is because of

$$\sum_k 2^{2k}|\Omega_k| \leq C\|S(f)\|_2^2 \leq C\|f\|_2^2 < \infty.$$

Thus (4) hold, and the series converges in $L^2(\mathbb{R}^n)$.

Moreover, (4) gives an atomic decomposition of $H^p(\mathbb{R}^n)$. To see this, we denote

$$b_{\widetilde{Q}}(x) = \sum_{Q \subseteq \widetilde{Q} \cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y) \frac{dydt}{t},$$

then it is easy to see that $b_{\widetilde{Q}}(x)$ is supported in $5\widetilde{Q}$(the same center and 5 times side length of $\widetilde{Q}$). By Hölder's inequality,

$$\begin{aligned}\|b_{\widetilde{Q}}(x)\|_p^p &= \left\| \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\frac{dydt}{t} \right\|_p^p \\ &\leq |5\widetilde{Q}|^{(1-\frac{p}{2})} \left\| \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\frac{dydt}{t} \right\|_2^p .\end{aligned} \tag{7}$$

Using duality argument again, we obtain

$$\begin{aligned}&\left\| \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\frac{dydt}{t} \right\|_2 \\ &= \sup_{\|g\|_2\leq 1} \left| < \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\frac{dydt}{t}, g > \right| \\ &\leq \sup_{\|g\|_2\leq 1} \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * g(y)\psi_t * f(y)|\frac{dydt}{t} \\ &\leq \sup_{\|g\|_2\leq 1} \left\{ \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * g(y)|^2\frac{dydt}{t} \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2\frac{dydt}{t} \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2\frac{dydt}{t} \right\}^{\frac{1}{2}},\end{aligned}$$

where the last inequality also follows from the L^2 estimate of the Littlewood-Paley square function as the same as in (6) used.

Hence, together with the cancellation conditions of ψ, it is easy to see that if we set

$$\begin{aligned}a_{\widetilde{Q}}(x) = C|5\widetilde{Q}|^{(\frac{1}{2}-\frac{1}{p})}\{ \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2\frac{dydt}{t}\}^{-\frac{1}{2}} \\ \times \sum_{Q\subseteq\widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(x-y)\psi_t * f(y)\frac{dydt}{t}\end{aligned}$$

for a suitable constant C, then $a_{\widetilde{Q}}(x)$ is an $(p,2)$-atom.
Finally, by (5), we obtain

$$\sum_k \sum_{\widetilde{Q}\in B_k} |\lambda_{\widetilde{Q}}|^p = \sum_k \sum_{\widetilde{Q}\in B_k} |5\widetilde{Q}|^{(1-\frac{p}{2})} \left\{ \sum_{Q\subseteq \widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t} \right\}^{\frac{p}{2}}$$
$$\leq C \sum_k |\Omega_k|^{(1-\frac{p}{2})} 2^{kp} |\Omega_k|^{\frac{p}{2}} \leq C\|S(f)\|_p^p = C\|f\|_{H^p}^p.$$

Therefore, we have the new atomic decomposition of $H^p(\mathbb{R}^n)$

$$f(x) = \sum_k \sum_{\widetilde{Q}\in B_k} \lambda_{\widetilde{Q}} a_{\widetilde{Q}}(x) \tag{8}$$

which converges in $L^2(\mathbb{R}^n)$.

This ends the proof of Theorem 1.2.

Now, by Theorem 1.2, we can prove Theorem 1.1.

Proof of Theorem 1.1. We only need to prove the "if" parts of the theorem. Suppose that a linear operator T is bounded on $L^2(\mathbb{R}^n)$ and $\|T(a)\|_p \leq C$ uniformly on all $(p,2)-$atoms. By Theorem 1.2, for any $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), 0 < p \leq 1$, we obtain

$$\|Tf\|_p^p = \left\| \sum_k \sum_{\widetilde{Q}\in B_k} T\left(\sum_{Q\subseteq \widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(\cdot - y)\psi_t * f(y) \frac{dydt}{t} \right) \right\|_p^p$$
$$\leq C \sum_k \sum_{\widetilde{Q}\in B_k} |5\widetilde{Q}|^{(1-\frac{p}{2})} \left\{ \sum_{Q\subseteq \widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t} \right\}^{\frac{p}{2}}$$
$$\leq C \sum_k |\Omega_k|^{(1-\frac{p}{2})} 2^{kp} |\Omega_k|^{\frac{p}{2}} \leq C\|S(f)\|_p^p = C\|f\|_{H^p}^p,$$

where the equality follows from the fact that the L^2 convergence of the series implies the convergence for almost everywhere, and the first inequality then follows from the uniform boundedness of T on all $(p,2)$-atoms in $L^p(\mathbb{R}^n)$ and the same estimate as (7).

Similarly, since the decomposition in (4) (or in (8)) converges in $L^2(\mathbb{R}^n)$, as a consequence, it also converges in $\mathcal{S}'$. Applying Lusin function and taking the

pth-power of L^p norm to both sides in (4) yield

$$\|Tf\|_{H^p}^p \leq \sum_k \sum_{\widetilde{Q}\in B_k} \left\| T\left(\sum_{Q\subseteq \widetilde{Q}\cap B_k} \int_{\widehat{Q}} \psi_t(\cdot - y)\psi_t * f(y)\frac{dydt}{t}\right)\right\|_{H^p}^p .$$

Using the fact that T is bounded uniformly on all $(p,2)$-atoms in H^p and repeating the same estimate above give

$$\begin{aligned}\|Tf\|_{H^p}^p &\leq C\sum_k \sum_{\widetilde{Q}\in B_k} |5\widetilde{Q}|^{(1-\frac{p}{2})} \left\{ \sum_{Q\subseteq \widetilde{Q}\cap B_k} \int_{\widehat{Q}} |\psi_t * f(y)|^2 \frac{dydt}{t}\right\}^{\frac{p}{2}} \\ &\leq C\sum_k |\Omega_k|^{(1-\frac{p}{2})} 2^{kp}|\Omega_k|^{\frac{p}{2}} \leq C\|S(f)\|_p^p = C\|f\|_{H^p}^p.\end{aligned}$$

Since $L^2 \cap H^p$ is dense in $H^p(\mathbb{R}^n)$, the "if" parts of Theorem 1.1 are proved, and hence the proof of Theorem 1.1 is complete.

Remark. The proof of the theorems above depends only on the Calderón reproducing formula on L^2 and the characterization of H^p space by the Lusin function. This formula and characterization still hold for product domains and spaces of homogeneous type (cf. [3] and [5]). Therefore, the theorems still hold for these setting.

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