TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 1, pp. 59-80, February 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

GENERALIZED PROJECTION AND ITERATIVE METHODS FOR APPROXIMATING FIXED POINTS OF ASYMPTOTICALLY WEAKLY SUPPRESSIVE OPERATORS

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Abstract. Let C be a nonempty closed convex proper subset of a real uniformly convex and uniformly smooth Banach space E, let $S : C \to C$ be a relatively nonexpansive mapping, and let $T : C \to E$ be an asymptotically weakly suppressive operator. Using the notion of generalized projection, iterative methods for approximating common fixed points of the mappings S and T are studied. In terms of the modified Ishikawa iteration and modified Halpern one for relatively nonexpansive mappings, we propose two modified versions of Chidume, Khumalo and Zegeye's iterative algorithms [C.E. Chidume, M. Khumalo and H. Zegeye, Generalized projection and approximation of fixed points of nonself maps, J. Appro. Theory, 120 (2003) 242-252] for finding approximate common fixed points of the mappings S and T. Moreover, it is proved that these two iterative algorithms converge strongly to the same common fixed point of the mappings S and T.

1. Introduction

Let E be a real Banach space with the dual E^*. As usual, $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as follows

Received January 1, 2008, accepted June 30, 2008.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 47H09, 47H10, 47H17.
Key words and phrases: Relatively nonexpansive mapping, Asymptotically weakly suppressive operator, Generalized projection, Iterative methods, Uniformly convex and uniformly smooth Banach space, Strong convergence.
[1]This research was partially supported by the National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), and Science and Technology Commission of Shanghai Municipality grant (075105118).
[2]This research was partially supported by a grant from the National Science Council.
[3]This research was partially supported by the grant no. 187 from the National Science Council of Romania.
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$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Recall that if E is smooth, then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote the single-valued duality mapping by J.

Let C be a subset of a Banach space E. A map $T : C \to C$ is called a strict contraction if there exists $k \in [0, 1)$ such that $\|Tx - Ty\| \le k\|x - y\|$ for all $x, y \in C$, and is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The map T is called asymptotically nonexpansive if $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in C$, where $\{k_n\}$ is a sequence of real numbers such that $\lim_{n\to\infty} k_n = 1$. It is clear that for asymptotically nonexpansive mappings it may be assumed that $k_n \ge 1$ and that $k_{i+1} \le k_i,\ i = 1, 2, ...$

It is well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a real smooth Banach space. Consider the functional $\phi : E \times E \to R^+ = [0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

It is clear that in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\phi(\overline{x}, x) = \inf_{y\in C} \phi(y, x)$. Existence and uniqueness of the operator Π_C follow from the properties of the functional ϕ and strict monotonicity of the mapping J (see, e.g., [3]). In Hilbert space H, $\Pi_C = P_C$.

Recently, Chidume, Khumalo and Zegeye [9] introduced and studied several new classes of maps in a real Banach space E.

Definition 1.1. ([9, Definition 3.1]). Let C be a nonempty subset of a real Banach space E. A map $T : C \to E$ is called asymptotically weakly suppressive of class $C_{\psi(t)}$ if there exists a continuous and nondecreasing function $\psi(t)$ defined on R^+ such that ψ is positive on $R^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = +\infty$ and $\forall x, y \in C$ there exists $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\phi(T(\Pi_C T)^{n-1}x, T(\Pi_C T)^{n-1}y) \le k_n\phi(x, y) - \psi(\phi(x, y)), \quad \forall n \ge 1.$$

Let $F(T) := \{x \in C : Tx = x\}$. Then T is called asymptotically weakly hemi-suppressive if $F(T) \ne \emptyset$ and the last inequality holds for every $x \in F(T)$ and $y \in C$.

The map $T : C \to E$ is called asymptotically nonextensive if, for all $x, y \in C$, there exists $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\phi(T(\Pi_C T)^{n-1}x, T(\Pi_C T)^{n-1}y) \leq k_n \phi(x, y), \quad \forall n \geq 1.$$

and it is called asymptotically quasi-nonextensive, if $F(T) \neq \emptyset$ and the last inequality holds for every $x \in F(T)$ and $y \in C$.

Very recently, Zeng, Tanaka and Yao [10] introduced and studied asymptotically Q_C-weakly contractive operators.

Definition 1.2. ([10, Definition 1.5]). Let C be a nonempty closed convex subset of a real Banach space E such that a nonexpansive retraction $Q_C : E \to C$ exists. A mapping $T : C \to E$ is said to be asymptotically Q_C-weakly contractive of class $C_{\psi(t)}$ if there exist a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, and a continuous and increasing function $\psi(t)$ defined on R^+ which is positive on $R^+ \setminus \{0\}$ with $\psi(0) = 0$ and $\lim_{t\to+\infty} \psi(t) = +\infty$ such that

$$\|T(Q_C T)^{n-1}x - T(Q_C T)^{n-1}y\| \leq k_n\|x - y\| - \psi(\|x - y\|)$$

for all $x, y \in C$ and each integer $n \geq 1$.

In [9], Chidume, Khumalo and Zegeye established some results on the successive approximations of fixed points for two classes of nonself maps in the above Definitions.

Theorem 1.1. ([9, Theorem 3.3]). Let C be a closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : C \to E$ be an asymptotically weakly suppressive operator of class $C_{\psi(t)}$ with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose $F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$ let the sequence $\{x_n\}$ be defined by

$$x_{n+1} := (\Pi_C T)^n x_n, \quad n \geq 1. \tag{1.1}$$

Then, $\{x_n\}$ converges strongly to some $x^* \in F(T)$.

Theorem 1.2. ([9, Theorem 3.4]). L et C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let $T : C \to E$ be an asymptotically nonextensive operator with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Suppose $F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$ let the sequence $\{x_n\}$ be defined by

$$x_{n+1} := (\Pi_C T)^n x_n, \quad n \geq 1.$$

(i) If the operator $A := I - T$ is demi-closed and $\|x_{n+1} - x_n\| \to 0$, then $\lim_{n\to\infty} Ax_n = 0$ and all weak accumulation points of $\{x_n\}$ belong to the fixed point set $F(T)$ of T.

(ii) In addition, if either $F(T)$ is a singleton, or the duality mapping J is weakly sequentially continuous (on some bounded set containing $\{x_n\}$), then $\{x_n\}$ converges weakly to a point $x^* \in F(T)$.

In [10], Zeng, Tanaka and Yao also derived some results on the modified retraction descent-like approximation of fixed points for asymptotically Q_C-weakly contractive operator in Definition 1.2.

Theorem 1.3. ([10, Theorem 3.1]). Let $\{\omega_n\}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} \omega_n = \infty$. Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E such that a nonexpansive retraction $Q_C : E \to C$ exists. Let $T : C \to E$ be an asymptotically Q_C-weakly contractive mapping of the class $C_{\psi(t)}$. Suppose that the mapping T has a (unique) fixed point $x^* \in C$. Then:

(i) the iterative sequence $\{x_n\}$ generated from any initial $x_0 \in C$ by

$$x_{n+1} = Q_C[(1-\omega_n)x_n + \omega_n T(Q_C T)^n x_n], \quad n \geq 0, \tag{1.2}$$

converges in norm to x^* as $n \to \infty$;

(ii) there exists a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$, $l = 1, 2, \ldots$, such that

$$\begin{cases} \|x_{n_l} - x^*\| \leq \psi^{-1}\left(\dfrac{1}{\sum\limits_{m=0}^{n_l} \omega_m} + (k_{n_l+1} - 1)\mathrm{diam}(G)\right), \\ \|x_{n_l+1} - x^*\| \leq \psi^{-1}\left(\dfrac{1}{\sum\limits_{m=0}^{n_l} \omega_m} + (k_{n_l+1} - 1)\mathrm{diam}(G)\right) \\ \qquad +\omega_{n_l}(k_{n_l+1} - 1)\mathrm{diam}(G), \\ \|x_n - x^*\| \leq \|x_{n_l+1} - x^*\| - \sum\limits_{m=n_l+1}^{n-1} \dfrac{\omega_m}{\vartheta_m}, \quad n_l+1 < n < n_{l+1},\ \vartheta_m = \sum\limits_{i=0}^{m} \omega_i, \\ \|x_{n+1} - x^*\| \leq \|x_0 - x^*\| - \sum\limits_{m=0}^{n} \dfrac{\omega_m}{\vartheta_m} \leq \|x_0 - x^*\|, \quad 1 \leq n \leq n_l - 1, \\ 1 \leq n_l \leq s_{\max} = \max\left\{s : \sum\limits_{m=0}^{s} \dfrac{\omega_m}{\vartheta_m} \leq \|x_0 - x^*\| v\right\}, \end{cases}$$

where $\mathrm{diam}(G)$ is the diameter of the set G.

Remark 1.1. If the mapping T in (1.2) is a self-mapping of C, then the iterative scheme (1.2) can be rewritten as

$$x_{n+1} = (1-\omega_n)x_n + \omega_n T^{n+1}x_n, \quad n = 0, 1, 2, \ldots.$$

In this case, T is also an asymptotically nonexpansive mapping. Moreover, the iterative scheme (1.2) essentially reduces to the Mann iterative process considered and studied by many authors; for instance, [7, 11, 13, 18].

On the other hand, let C be a nonempty closed convex subset of a real Banach space E. Whenever E is a Hilbert space H, Nakajo and Takahashi [16] proposed the following iterative algorithm for a single nonexpansive mapping $S : C \to C$

$$(1.3) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_n = \{v \in C : \|y_n - v\| \le \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ \qquad x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H and proved that the sequence $\{x_n\}$ converges strongly to $P_{F(S)}x_0$, where $F(S)$ is the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$.

In 2006, Martinez-Yanes and Xu [14] introduced one iterative algorithm for a nonexpansive mapping $S : C \to C$, with C a bounded closed convex subset of a real Hilbert space H

$$(1.4) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)Sx_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sz_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + (1-\alpha_n)(\|z_n\|^2 \\ \qquad -\|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ \qquad x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

and also defined another iterative algorithm

$$(1.5) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Sx_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ \qquad x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in the interval $[0,1]$. They proved that both the sequence $\{x_n\}$ generated by algorithm (1.4) and the sequence $\{x_n\}$ generated by algorithm (1.5) converge strongly to the same point $P_{F(S)}x_0$.

Very recently, utilizing Nakajo and Takahashi's idea [16], Qin and Su [20] modified algorithms (1.4) and (1.5) for relatively nonexpansive mappings in a Banach space E. They first introduced one iterative algorithm (i.e., modified Ishikawa iteration) for a relatively nonexpansive mapping $S : C \to C$, with C a closed convex subset of a uniformly convex and uniformly smooth Banach space E

$$(1.6) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n)JSx_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ \quad x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where J is the single-valued normalized duality mapping on E, $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and $\Pi_C : E \to C$ is the generalized projection. Second, they also defined another iterative algorithm (i.e., modified Halpern iteration)

$$(1.7) \qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n)JSx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases}$$

They proved that under appropriate conditions both the sequence $\{x_n\}$ generated by algorithm (1.6) and the sequence $\{x_n\}$ generated by algorithm (1.7), converge strongly to the same point $\Pi_{F(S)}x_0$.

Let C be a nonempty closed convex subset of a real Banach space E with the dual E^*. Assume that $T : C \to E$ is an asymptotically weakly suppressive operator on C and $S : C \to C$ is a relatively nonexpansive mapping such that $F(S) \neq \emptyset$. The purpose of this paper is to introduce and study new iterative algorithms (1.8) and (1.9) in a uniformly convex and uniformly smooth Banach space E, which combine (1.1) with (1.6) and (1.1) with (1.7), respectively.

Algorithm I.

$$(1.8)\qquad \begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ \widetilde{x}_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n), \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, \widetilde{x}_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \le 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

Algorithm II.

$$(1.9)\qquad \begin{cases} x_0 \in E \text{ chosen arbitrarily,} \\ \widetilde{x}_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n), \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)\}, \\ Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n \rangle \le 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$.

In this paper, strong convergence results on these two iterative algorithms are established; that is, under appropriate conditions, both the sequence $\{x_n\}$ generated by algorithm (1.8) and the sequence $\{x_n\}$ generated by algorithm (1.9), converge strongly to the same point $\Pi_{F(S)}x_0$, which is an element of the $F(T)$. Our results represent the improvement, generalization and development of the previously known results in the literature including Chidume, Khumalo and Zegeye [9], Zeng, Tanaka and Yao [10], and Qin and Su [20].

Notation. $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

2. Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is smooth then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote the single-valued duality mapping by J.

Recall that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [1,2] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E. \tag{2.1}$$

It is clear that in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x-y\|^2, \ \forall x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [3]). In a Hilbert space, $\Pi_C = P_C$. From [2], in uniformly convex and uniformly smooth Banach spaces, we have

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \text{for all } x, y \in E. \tag{2.3}$$

Let C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is called an asymptotically fixed point of S [17] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n - x_n \to 0$. The set of asymptotical fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive [4-6] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$.

A Banach space E is called strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. Then the Banach space E is called smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [8,19] for more details.

Remark 2.1. ([20]). If E is a reflexive, strictly convex and smooth Banach space, then for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [8,19] for more details.

We need the following lemmas, which will be used for the proof of our main results in the sequel.

Lemma 2.1. (Kamimura and Takahashi [12]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2. (Alber [2]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx \rangle \geq 0 \quad \text{for all } z \in C.$$

Lemma 2.3. (Alber [2]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \text{for all } y \in C.$$

Lemma 2.4. (Matsushita and Takahashi [15]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself. Then $F(S)$ is closed and convex.*

3. Main Results

Now we are in a position to prove the main theorems of this paper.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \neq \emptyset$ and let $T : C \to E$ be an asymptotically weakly suppressive operator of class $C_{\psi(t)}$ with sequence $\{k_n\} \subseteq [1, \infty)$ such that*

$\lim_{n\to\infty} k_n = 1$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be sequences in* $[0,1]$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. *Suppose* $F(T) \neq \emptyset$ *and let the sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *be defined by*

$$(3.1)\qquad \begin{cases} x_0 \; inC \text{ chosen arbitrarily}, \\ \widetilde{x}_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n), \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, \widetilde{x}_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x_0, \quad n = 0, 1, 2, \ldots. \end{cases}$$

Assume that S *is uniformly continuous. If* $x_n - (\Pi_C T)^n x_n \to 0$ $(n \to \infty)$, *then* $\{x_n\}$ *converges strongly to* $\Pi_{F(S)}x_0$, *which is an element of* $F(T)$; *conversely, if* $\{x_n\}$ *converges strongly to an element of* $F(T)$, *then* $x_n - (\Pi_C T)^n x_n \to 0$ $(n \to \infty)$.

Proof. First of all, let us show that C_n and Q_n are closed and convex for each $n \ge 0$. Indeed, from the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We claim that C_n is convex. For any $v_1, v_2 \in C_n$ and any $t \in (0,1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Note that the inequality

$$(3.2)\qquad \phi(v, y_n) \le \alpha_n \phi(v, \widetilde{x}_n) + (1-\alpha_n)\phi(v, z_n)$$

is equivalent to the one

$$(3.3)\quad 2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \le \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2.$$

Observe that there hold the following

$$\phi(v, y_n) = \|v\|^2 - 2\langle v, Jy_n\rangle + \|y_n\|^2, \quad \phi(v, \widetilde{x}_n) = \|v\|^2 - 2\langle v, J\widetilde{x}_n\rangle + \|\widetilde{x}_n\|^2$$

and $\phi(v, z_n) = \|v\|^2 - 2\langle v, Jz_n\rangle + \|z_n\|^2$. Thus we have

$$\begin{aligned} & 2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \\ &= 2\alpha_n\langle tv_1 + (1-t)v_2, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle tv_1 + (1-t)v_2, Jz_n\rangle \\ &\quad -2\langle tv_1 + (1-t)v_2, Jy_n\rangle \\ &= 2t\alpha_n\langle v_1, J\widetilde{x}_n\rangle + 2(1-t)\alpha_n\langle v_2, J\widetilde{x}_n\rangle + 2(1-\alpha_n)t\langle v_1, Jz_n\rangle \\ &\quad +2(1-\alpha_n)(1-t)\langle v_2, Jz_n\rangle - 2t\langle v_1, Jy_n\rangle - 2(1-t)\langle v_2, Jy_n\rangle \\ &\le \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2. \end{aligned}$$

This implies that $v \in C_n$. So, C_n is convex. Next let us show that $F(S) \subset C_n$ for all n. Indeed, we have, for all $w \in F(S)$

$$\begin{aligned}\phi(w, y_n) &= \phi(w, J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n)) \\ &= \|w\|^2 - 2\langle w, \alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n\rangle + \|\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n\|^2 \\ &\leq \|w\|^2 - 2\alpha_n\langle w, J\widetilde{x}_n\rangle - 2(1-\alpha_n)\langle w, JSz_n\rangle \\ &\quad +\alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|Sz_n\|^2 \\ &\leq \alpha_n\phi(w, \widetilde{x}_n) + (1-\alpha_n)\phi(w, Sz_n) \\ &\leq \alpha_n\phi(w, \widetilde{x}_n) + (1-\alpha_n)\phi(w, z_n).\end{aligned}$$

So $w \in C_n$ for all $n \geq 0$. Next let us show that

$$F(S) \subset Q_n \quad \text{for all } n \geq 0. \tag{3.4}$$

We prove this by induction. For $n = 0$, we have $F(S) \subset C = Q_0$. Assume that $F(S) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.2, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $F(S) \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F(S)$. This together with the definition of Q_{n+1} implies that $F(S) \subset Q_{n+1}$. Hence (3.4) holds for all $n \geq 0$. This implies that $\{x_n\}$ is well defined.

On the other hand, it follows from the definition of Q_n that $x_n = \Pi_{Q_n}x_0$. Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \text{for all } n \geq 0.$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. And also from $x_n = \Pi_{Q_n}x_0$ and Lemma 2.3 that

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n}x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in F(S) \subset Q_n$ for each $n \geq 0$. Consequently, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2,$$

we conclude that $\{x_n\}$ is bounded. So, we know that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. From Lemma 2.3, we derive

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n}x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n}x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that $\phi(x_{n+1}, x_n) \to 0$. So it follows from Lemma 2.1 that $x_{n+1} - x_n \to 0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, \widetilde{x}_n) + (1-\alpha_n)\phi(x_{n+1}, z_n). \tag{3.5}$$

Observe that

$$\begin{aligned}
&\phi(x_{n+1}, z_n) \\
&= \phi(x_{n+1}, J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n)) \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n\rangle \\
&\quad + \|\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n\|^2 \\
&\leq \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, J\widetilde{x}_n\rangle - 2(1-\beta_n)\langle x_{n+1}, JS\widetilde{x}_n\rangle \\
&\quad + \beta_n\|\widetilde{x}_n\|^2 + (1-\beta_n)\|S\widetilde{x}_n\|^2 \\
&= \beta_n \phi(x_{n+1}, \widetilde{x}_n) + (1-\beta_n)\phi(x_{n+1}, S\widetilde{x}_n).
\end{aligned} \tag{3.6a}$$

At the same time, observe that

$$\begin{aligned}
&\phi(x_{n+1}, \widetilde{x}_n) \\
&= \phi(x_{n+1}, J^{-1}(\gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n)) \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n\rangle \\
&\quad + \|\gamma_n Jx_n + (1-\gamma_n)J(\Pi_C T)^n x_n\|^2 \\
&\leq \|x_{n+1}\|^2 - 2\gamma_n\langle x_{n+1}, Jx_n\rangle - 2(1-\gamma_n)\langle x_{n+1}, J(\Pi_C T)^n x_n\rangle \\
&\quad + \gamma_n\|x_n\|^2 + (1-\gamma_n)\|(\Pi_C T)^n x_n\|^2 \\
&= \gamma_n \phi(x_{n+1}, x_n) + (1-\gamma_n)\phi(x_{n+1}, (\Pi_C T)^n x_n),
\end{aligned} \tag{3.6b}$$

and

$$\begin{aligned}
&\phi(x_{n+1}, (\Pi_C T)^n x_n) \\
&\leq \phi(x_{n+1}, T(\Pi_C T)^{n-1}x_n) - \phi((\Pi_C T)^n x_n, T(\Pi_C T)^{n-1}x_n) \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, JT(\Pi_C T)^{n-1}x_n\rangle + \|T(\Pi_C T)^{n-1}x_n\|^2 \\
&\quad - [\|(\Pi_C T)^n x_n\|^2 - 2\langle(\Pi_C T)^n x_n, JT(\Pi_C T)^{n-1}x_n\rangle \\
&\quad + \|T(\Pi_C T)^{n-1}x_n\|^2] \\
&= \|x_{n+1}\|^2 - \|(\Pi_C T)^n x_n\|^2 - 2\langle x_{n+1} - (\Pi_C T)^n x_n, JT(\Pi_C T)^{n-1}x_n\rangle \\
&= (\|x_{n+1}\| - \|(\Pi_C T)^n x_n\|)(\|x_{n+1}\| + \|(\Pi_C T)^n x_n\|) \\
&\quad - 2\langle x_{n+1} - (\Pi_C T)^n x_n, JT(\Pi_C T)^{n-1}x_n\rangle \\
&\leq \|x_{n+1} - (\Pi_C T)^n x_n\|(\|x_{n+1}\| + \|(\Pi_C T)^n x_n\|) \\
&\quad + 2\|x_{n+1} - (\Pi_C T)^n x_n\|\|T(\Pi_C T)^{n-1}x_n\|.
\end{aligned} \tag{3.6c}$$

Also, observe that

$$\|x_{n+1} - (\Pi_C T)^n x_n\| \le \|x_{n+1} - x_n\| + \|x_n - (\Pi_C T)^n x_n\|.$$

From $x_{n+1} - x_n \to 0$ and $x_n - (\Pi_C T)^n x_n \to 0$ it follows that

$$x_{n+1} - (\Pi_C T)^n x_n \to 0.$$

Therefore, we get

$$\begin{aligned} \|Jx_{n+1} - J\widetilde{x}_n\| &= \|\gamma_n(Jx_{n+1} - Jx_n) + (1-\gamma_n)(Jx_{n+1} - J(\Pi_C T)^n x_n)\| \\ &\le \gamma_n \|Jx_{n+1} - Jx_n\| + (1-\gamma_n)\|Jx_{n+1} - J(\Pi_C T)^n x_n\|. \end{aligned}$$

Utilizing the uniform norm-to-norm continuity of J on bounded subsets of E, we deduce that $Jx_{n+1} - Jx_n \to 0$ and $Jx_{n+1} - J(\Pi_C T)^n x_n \to 0$ and hence $Jx_{n+1} - J\widetilde{x}_n \to 0$. Since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain that $x_{n+1} - \widetilde{x}_n \to 0$ and hence $\{\widetilde{x}_n\}$ is bounded. Thus $\{S\widetilde{x}_n\}$ is also bounded. Note that

$$\|(\Pi_C T)^n x_n\| \le \|(\Pi_C T)^n x_n - x_n\| + \|x_n\|.$$

So we know that $\{(\Pi_C T)^n x_n\}$ is bounded.

Let $x^* \in F(T)$. Then, by the definition of asymptotically weakly suppressive operator, we have

$$\begin{aligned} \phi(x^*, T(\Pi_C T)^{n-1} x_n) &= \phi(T(\Pi_C T)^{n-1} x^*, T(\Pi_C T)^{n-1} x_n) \\ &\le k_n \phi(x^*, x_n) - \psi(\phi(x^*, x_n)) \\ &\le k_n \phi(x^*, x_n), \end{aligned}$$

which together with the boundedness of $\{k_n\}$ and $\{\phi(x^*, x_n)\}$, implies that $\{\phi(x^*, T(\Pi_C T)^{n-1} x_n)\}$ is bounded. Thus $\{T(\Pi_C T)^{n-1} x_n)\}$ is bounded. Consequently, utilizing the boundedness of $\{x_n\}$, $\{(\Pi_C T)^n x_n\}$ and $\{T(\Pi_C T)^{n-1} x_n\}$, from (3.6c) and $x_{n+1} - (\Pi_C T)^n x_n \to 0$ we have $\phi(x_{n+1}, (\Pi_C T)^n x_n) \to 0$. Again from (3.6$b$) and $\phi(x_{n+1}, x_n) \to 0$ we obtain $\phi(x_{n+1}, \widetilde{x}_n) \to 0$. Consequently from (3.6$a$), $\phi(x_{n+1}, \widetilde{x}_n) \to 0$ and $\beta_n \to 1$ it follows that

$$\phi(x_{n+1}, z_n) \to 0. \tag{3.7}$$

Further it follows from (3.5), $\phi(x_{n+1}, \widetilde{x}_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$ that

$$\phi(x_{n+1}, y_n) \to 0. \tag{3.8}$$

Utilizing Lemma 2.1 we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - \widetilde{x}_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E we have

$$(3.10a) \qquad \lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\widetilde{x}_n\| = 0.$$

Furthermore, we have

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|.$$

It follows from $x_{n+1} - x_n \to 0$ and $x_{n+1} - z_n \to 0$ that

$$(3.10b) \qquad \lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Noticing that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n)\| \\ &= \|\alpha_n(Jx_{n+1} - J\widetilde{x}_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JSz_n) - \alpha_n(J\widetilde{x}_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n\|J\widetilde{x}_n - Jx_{n+1}\|, \end{aligned}$$

we have

$$\|Jx_{n+1} - JSz_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|J\widetilde{x}_n - Jx_{n+1}\|).$$

From (3.10a) and $\limsup_{n\to\infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JSz_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$(3.10c) \qquad \lim_{n\to\infty} \|x_{n+1} - Sz_n\| = 0.$$

Observe that

$$\|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\|.$$

Since S is uniformly continuous, it follows from (3.10b), (3.10c) and $x_{n+1} - x_n \to 0$ that $x_n - Sx_n \to 0$.

Next, let us show that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x_0$, which is an element of $F(T)$. Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widetilde{x} \in E$. Then $\widetilde{x} \in F(S)$. Next let us show that $\widetilde{x} = \Pi_{F(S)}x_0$ and convergence is strong. Put $\overline{x} = \Pi_{F(S)}x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n}x_0$ and $\overline{x} \in F(S) \subset C_n \cap Q_n$, we have

$\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive

$$\begin{aligned}\phi(\widetilde{x}, x_0) &= \|\widetilde{x}\|^2 - 2\langle \widetilde{x}, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\leq \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\leq \phi(\overline{x}, x_0).\end{aligned}$$

It follows from the definition of $\Pi_{F(S)}x_0$ that $\widetilde{x} = \overline{x}$ and hence

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0).$$

So we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\Pi_{F(S)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(S)}x_0$. Now, by the definition of asymptotically weakly suppressive operator and property of Π_C, we have for $x^* \in F(T)$

$$\begin{aligned}&\phi(x^*, (\Pi_C T)^n x_n) \\ &\leq \phi(x^*, T(\Pi_C T)^{n-1}x_n) - \phi((\Pi_C T)^n x_n, T(\Pi_C T)^{n-1}x_n) \\ &\leq \phi(x^*, T(\Pi_C T)^{n-1}x_n) \\ &= \phi(T(\Pi_C T)^{n-1}x^*, T(\Pi_C T)^{n-1}x_n) \\ &\leq k_n\phi(x^*, x_n) - \psi(\phi(x^*, x_n)),\end{aligned}$$

and hence

$$\begin{aligned}&\psi(\phi(x^*, x_n)) \\ &\leq k_n\phi(x^*, x_n) - \phi(x^*, (\Pi_C T)^n x_n) \\ &= k_n(\|x^*\|^2 - 2\langle x^*, Jx_n\rangle + \|x_n\|^2) - (\|x^*\|^2 - 2\langle x^*, J(\Pi_C T)^n x_n\rangle + \|(\Pi_C T)^n x_n\|^2) \\ &= (k_n - 1)\|x^*\|^2 - 2(k_n - 1)\langle x^*, Jx_n\rangle + 2\langle x^*, J(\Pi_C T)^n x_n - Jx_n\rangle \\ &\quad +(k_n - 1)\|x_n\|^2 + \|x_n\|^2 - \|(\Pi_C T)^n x_n\|^2 \\ &\leq (k_n - 1)\|x^*\|^2 + 2(k_n - 1)\|x^*\|\|x_n\| + 2\|x^*\|\|J(\Pi_C T)^n x_n - Jx_n\| \\ &\quad +(k_n - 1)\|x_n\|^2 + (\|x_n\| - \|(\Pi_C T)^n x_n\|)(\|x_n\| + \|(\Pi_C T)^n x_n\|) \\ &\leq (k_n - 1)\|x^*\|^2 + 2(k_n - 1)\|x^*\|\|x_n\| + 2\|x^*\|\|J(\Pi_C T)^n x_n - Jx_n\| \\ &\quad +(k_n - 1)\|x_n\|^2 + \|x_n - (\Pi_C T)^n x_n\|(\|x_n\| + \|(\Pi_C T)^n x_n\|).\end{aligned}$$

Since $k_n \to 1$, $(\Pi_C T)^n x_n - x_n \to 0$ and $\{x_n\}$ and $\{(\Pi_C T)^n x_n\}$ are bounded, by the uniform norm-to-norm continuity of J on bounded subsets of E we obtain $\psi(\phi(x^*, x_n)) \to 0$. From the property of the function ψ it follows that $\phi(x^*, x_n) \to 0$. Utilizing Lemma 2.1 we derive $x_n \to x^*$. On account of the uniqueness of the limit of $\{x_n\}$, we know that $x^* = \Pi_{F(S)} x_0$.

Conversely, let $x_n \to x^* \in F(T)$. Then $\{x_n\}$ is bounded. Since

$$\begin{aligned}
\phi(x^*, x_n) &= \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + \|x_n\|^2 \\
&= \langle x^*, Jx^* - Jx_n\rangle + \langle x_n - x^*, Jx_n\rangle \\
&\le \|x^*\|\|Jx^* - Jx_n\| + \|x_n - x^*\|\|x_n\|,
\end{aligned}$$

from the uniform norm-to-norm continuity of J on bounded subsets of E, we obtain $\phi(x^*, x_n) \to 0$. Now, by the definition of asymptotically weakly suppressive operator and property of Π_C, we get

$$\begin{aligned}
\phi(x^*, (\Pi_C T)^n x_n) &\le \phi(x^*, T(\Pi_C T)^{n-1} x_n) - \phi((\Pi_C T)^n x_n, T(\Pi_C T)^{n-1} x_n) \\
&\le \phi(x^*, T(\Pi_C T)^{n-1} x_n) \\
&= \phi(T(\Pi_C T)^{n-1} x^*, T(\Pi_C T)^{n-1} x_n) \\
&\le k_n \phi(x^*, x_n) - \psi(\phi(x^*, x_n)) \\
&\le k_n \phi(x^*, x_n).
\end{aligned}$$

From $\phi(x^*, x_n) \to 0$ it follows that $\phi(x^*, (\Pi_C T)^n x_n) \to 0$. Thus from Lemma 2.1 we have $(\Pi_C T)^n x_n \to x^*$, which together with $x_n \to x^*$, yields

$$(\Pi_C T)^n x_n - x_n \to 0.$$

This completes the proof. ■

In Theorem 3.1, put $\gamma_n = 1$ for all $n \ge 0$. Then we have

$$\begin{aligned}
\widetilde{x}_n &= J^{-1}(\gamma_n J x_n + (1 - \gamma_n) J(\Pi_C T)^n x_n) \\
&= J^{-1}(J x_n + (1 - 1) J(\Pi_C T)^n x_n) \\
&= x_n,
\end{aligned}$$

for all n. Thus algorithm (3.1) reduces to algorithm (3.11). Meantime, observe that in the proof of Theorem 3.1, the condition $(\Pi_C T)^n x_n - x_n \to 0$ is applied to the verification of $Jx_{n+1} - J\widetilde{x}_n \to 0$. In the case when $\gamma_n = 1$, there is no doubt that the condition $(\Pi_C T)^n x_n - x_n \to 0$ can be deleted because $x_n = \widetilde{x}_n$. By the careful analysis of the proof of Theorem 3.1, we conclude that Theorem 3.1 covers [20, Theorem 2.1] as a special case.

Theorem 3.2. Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \neq \emptyset$ and let $T : C \to E$ be an asymptotically weakly suppressive operator of class $C_{\psi(t)}$ with sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{\gamma_n\}_{n=0}^{\infty} \subseteq [0, 1]$ and $\{\alpha_n\}_{n=0}^{\infty} \subseteq (0, 1)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$. Suppose $F(T) \neq \emptyset$ and let the sequence $\{x_n\}_{n=0}^{\infty}$ in C be defined by

$$
(3.12)\qquad \begin{cases}
x_0 \in C \text{ chosen arbitrarily},\\
\widetilde{x}_n = J^{-1}(\gamma_n J x_n + (1-\gamma_n) J(\Pi_C T)^n x_n),\\
y_n = J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J S\widetilde{x}_n),\\
C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)\},\\
Q_n = \{v \in C : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\},\\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots.
\end{cases}
$$

Assume that S is uniformly continuous. If $x_n - (\Pi_C T)^n x_n \to 0$ $(n \to \infty)$, then $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$, which is an element of $F(T)$; conversely, if $\{x_n\}$ converges strongly to an element of $F(T)$, then $x_n - (\Pi_C T)^n x_n \to 0$ $(n \to \infty)$.

Proof. First, Let us show that C_n is closed and convex for each $n \ge 0$. From the definition of C_n, it is obvious that C_n is closed for each $n \ge 0$. We prove that C_n is convex. Similarly to the proof of Theorem 3.1, since

$$\phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)$$

is equivalent to

$$2\alpha_n \langle v, Jx_0\rangle + 2(1-\alpha_n)\langle v, J\widetilde{x}_n\rangle - 2\langle v, Jy_n\rangle \le \alpha_n \|x_0\|^2 + (1-\alpha_n)\|\widetilde{x}_n\|^2 - \|y_n\|^2,$$

we know that C_n is convex. Next, let us show that $F(S) \subset C_n$ for each $n \ge 0$. Indeed, we have, for each $w \in F(S)$

$$
\begin{aligned}
\phi(w, y_n) &= \phi(w, J^{-1}(\alpha_n J x_0 + (1-\alpha_n) JS\widetilde{x}_n))\\
&= \|w\|^2 - 2\langle w, \alpha_n J x_0 + (1-\alpha_n) JS\widetilde{x}_n\rangle + \|\alpha_n J x_0 + (1-\alpha_n) JS\widetilde{x}_n\|^2\\
&\le \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2(1-\alpha_n)\langle w, JS\widetilde{x}_n\rangle + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|S\widetilde{x}_n\|^2\\
&\le \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, S\widetilde{x}_n)\\
&\le \alpha_n \phi(w, x_0) + (1-\alpha_n)\phi(w, \widetilde{x}_n).
\end{aligned}
$$

So $w \in C_n$ for all $n \ge 0$ and $F(S) \subset C_n$. Similarly to the proof of Theorem 3.1, we also obtain $F(S) \subset Q_n$ for all $n \ge 0$. Consequently, $F(S) \subset C_n \cap Q_n$ for all $n \ge 0$.

Therefore, the sequence $\{x_n\}$ generated by (3.12) is well defined. As in the proof of Theorem 3.1, we can obtain $\phi(x_{n+1}, x_n) \to 0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n we also have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n)\phi(x_{n+1}, \widetilde{x}_n).$$

As in the proof of Theorem 3.1, we can deduce from $x_{n+1} - x_n \to 0$ and $x_n - (\Pi_C T)^n x_n \to 0$ that

$$x_{n+1} - (\Pi_C T)^n x_n \to 0$$

and hence

$$\lim_{n\to\infty} \phi(x_{n+1}, \widetilde{x}_n) = 0. \tag{3.13}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we also have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n)\phi(x_{n+1}, \widetilde{x}_n).$$

It follows from (3.13) and $\alpha_n \to 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.14}$$

Utilizing Lemma 2.1 we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - \widetilde{x}_n\| = 0. \tag{3.15}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\widetilde{x}_n\| = 0. \tag{3.16}$$

Note that

$$\begin{aligned} \|JS\widetilde{x}_n - Jy_n\| &= \|JS\widetilde{x}_n - (\alpha_n Jx_0 + (1 - \alpha_n)JS\widetilde{x}_n)\| \\ &= \alpha_n \|Jx_0 - JS\widetilde{x}_n\|. \end{aligned}$$

Therefore, from $\alpha_n \to 0$ we have

$$\lim_{n\to\infty} \|JS\widetilde{x}_n - Jy_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|S\widetilde{x}_n - y_n\| = 0. \tag{3.17}$$

It follows that

$$(3.18)\quad \|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S\widetilde{x}_n\| + \|S\widetilde{x}_n - Sx_n\|.$$

Since S is uniformly continuous, it follows from (3.15) and (3.17) that $x_n - Sx_n \to 0$.

Next, let us show that $\{x_n\}$ converges strongly to $\Pi_{F(S)}x_0$, which is an element of $F(T)$. Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widetilde{x} \in E$. Then $\widetilde{x} \in F(S)$. Next let us show that $\widetilde{x} = \Pi_{F(S)}x_0$ and convergence is strong. Put $\overline{x} = \Pi_{F(S)}x_0$. From $x_{n+1} = \Pi_{C_n \cap Q_n}x_0$ and $\overline{x} \in F(S) \subset C_n \cap Q_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive

$$\begin{aligned} \phi(\widetilde{x}, x_0) &= \|\widetilde{x}\|^2 - 2\langle \widetilde{x}, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\leq \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\leq \phi(\overline{x}, x_0). \end{aligned}$$

It follows from the definition of $\Pi_{F(S)}x_0$ that $\widetilde{x} = \overline{x}$ and hence

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0).$$

So we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\Pi_{F(S)}x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent subsequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(S)}x_0$. Now, by the definition of asymptotically weakly suppressive operator and property of Π_C, we have for $x^* \in F(T)$

$$\begin{aligned} &\phi(x^*, (\Pi_C T)^n x_n) \\ &\leq \phi(x^*, T(\Pi_C T)^{n-1}x_n) - \phi((\Pi_C T)^n x_n, T(\Pi_C T)^{n-1}x_n) \\ &\leq \phi(x^*, T(\Pi_C T)^{n-1}x_n) \\ &= \phi(T(\Pi_C T)^{n-1}x^*, T(\Pi_C T)^{n-1}x_n) \\ &\leq k_n\phi(x^*, x_n) - \psi(\phi(x^*, x_n)), \end{aligned}$$

and hence

$$\begin{aligned} &\psi(\phi(x^*, x_n)) \\ &\leq k_n\phi(x^*, x_n) - \phi(x^*, (\Pi_C T)^n x_n) \\ &= k_n(\|x^*\|^2 - 2\langle x^*, Jx_n\rangle + \|x_n\|^2) - (\|x^*\|^2 \\ &\quad -2\langle x^*, J(\Pi_C T)^n x_n\rangle + \|(\Pi_C T)^n x_n\|^2) \\ &= (k_n - 1)\|x^*\|^2 - 2(k_n - 1)\langle x^*, Jx_n\rangle + 2\langle x^*, J(\Pi_C T)^n x_n - Jx_n\rangle \end{aligned}$$

$$
\begin{aligned}
&+(k_n-1)\|x_n\|^2+\|x_n\|^2-\|(\Pi_C T)^n x_n\|^2\\
&\le (k_n-1)\|x^*\|^2+2(k_n-1)\|x^*\|\|x_n\|+2\|x^*\|\|J(\Pi_C T)^n x_n-Jx_n\|\\
&\quad+(k_n-1)\|x_n\|^2+(\|x_n\|-\|(\Pi_C T)^n x_n\|)(\|x_n\|+\|(\Pi_C T)^n x_n\|)\\
&\le (k_n-1)\|x^*\|^2+2(k_n-1)\|x^*\|\|x_n\|+2\|x^*\|\|J(\Pi_C T)^n x_n-Jx_n\|\\
&\quad+(k_n-1)\|x_n\|^2+\|x_n-(\Pi_C T)^n x_n\|(\|x_n\|+\|(\Pi_C T)^n x_n\|).
\end{aligned}
$$

Since $k_n \to 1$, $(\Pi_C T)^n x_n - x_n \to 0$ and $\{x_n\}$ and $\{(\Pi_C T)^n x_n\}$ are bounded, by the uniform norm-to-norm continuity of J on bounded subsets of E we obtain $\psi(\phi(x^*, x_n)) \to 0$. From the property of the function ψ it follows that $\phi(x^*, x_n) \to 0$. Utilizing Lemma 2.1 we derive $x_n \to x^*$. On account of the uniqueness of the limit of $\{x_n\}$, we know that $x^* = \Pi_{F(S)} x_0$.

Conversely, let $x_n \to x^* \in F(T)$. Then $\{x_n\}$ is bounded. Since

$$
\begin{aligned}
\phi(x^*, x_n) &= \|x^*\|^2 - 2\langle x^*, Jx_n\rangle + \|x_n\|^2\\
&= \langle x^*, Jx^* - Jx_n\rangle + \langle x_n - x^*, Jx_n\rangle\\
&\le \|x^*\|\|Jx^* - Jx_n\| + \|x_n - x^*\|\|x_n\|,
\end{aligned}
$$

from the uniform norm-to-norm continuity of J on bounded subsets of E, we obtain $\phi(x^*, x_n) \to 0$. Now, by the definition of asymptotically weakly suppressive operator and property of Π_C, we get

$$
\begin{aligned}
\phi(x^*, (\Pi_C T)^n x_n) &\le \phi(x^*, T(\Pi_C T)^{n-1} x_n) - \phi((\Pi_C T)^n x_n, T(\Pi_C T)^{n-1} x_n)\\
&\le \phi(x^*, T(\Pi_C T)^{n-1} x_n)\\
&= \phi(T(\Pi_C T)^{n-1} x^*, T(\Pi_C T)^{n-1} x_n)\\
&\le k_n \phi(x^*, x_n) - \psi(\phi(x^*, x_n))\\
&\le k_n \phi(x^*, x_n).
\end{aligned}
$$

From $\phi(x^*, x_n) \to 0$ it follows that $\phi(x^*, (\Pi_C T)^n x_n) \to 0$. Thus from Lemma 2.1 we have $(\Pi_C T)^n x_n \to x^*$, which together with $x_n \to x^*$, yields

$$(\Pi_C T)^n x_n - x_n \to 0.$$

This completes the proof. ■

In Theorem 3.2, put $\gamma_n = 1$ for all $n \ge 0$. Then we have

$$
\begin{aligned}
\widetilde{x}_n &= J^{-1}(\gamma_n Jx_n + (1-\gamma_n) J(\Pi_C T)^n x_n)\\
&= J^{-1}(Jx_n + (1-1) J(\Pi_C T)^n x_n)\\
&= x_n,
\end{aligned}
$$

for all n. Thus under the lack of the uniform continuity of S it follows from (3.18) that $x_n - Sx_n \to 0$. By the careful analysis of the proof of Theorem 3.2, we see that Theorem 3.2 covers [20, Theorem 2.2] as a special case.

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