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DUALITY OF HARDY SPACE WITH BMO ON THE SHILOV BOUNDARY OF THE PRODUCT DOMAIN IN $\mathbb{C}^{2n}$

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Abstract. In this paper, we introduce the BMO space via heat kernels on $\widetilde{M}$, where $\widetilde{M} = M_1 \times \cdots \times M_n$ is the Shilov boundary of the product domain in $\mathbb{C}^{2n}$ defined by Nagel and Stein ([16], see also [17]), each M_i is the boundary of a weakly pseudoconvex domain of finite type in $\mathbb{C}^2$ and the vector fields of M_i are uniformly of finite type ([14]). And we prove that it is the dual space of product Hardy space $H^1(\widetilde{M})$ introduced in [11].

1. Introduction

In [14], Nagel and Stein studied the initial value problem and the regularity properties of the heat operator $\mathcal{H} = \partial_s + \Box_b$ for the Kohn-Laplacian $\Box_b$ on M, where M is the boundary of a weakly pseudoconvex domain Ω of finite type in $\mathbb{C}^2$. And in [16], they obtained the optimal estimates for solution of the Kohn-Laplacian on q-forms, $\Box_b = \Box_b^{(q)}$, which is defined on the boundary $\overline{M} = \partial\Omega$ of a decoupled domain $\Omega \subseteq \mathbb{C}^n$. The method they used is to deduce the results about regularity of $\Box_b$ on $\overline{M}$ from corresponding results on $\widetilde{M} \subset \mathbb{C}^{2n}$ via projection, where $\widetilde{M} = M_1 \times \cdots \times M_n$ is the Cartesian product of boundaries of domains in $\mathbb{C}^2$ mentioned above. Namely, $\widetilde{M}$ is the Shilov boundary of the product domain $\Omega_1 \times \cdots \times \Omega_n$.

In [17], they developed an L^p $(1 < p < \infty)$ theory of product singular integral operators on product space $\widetilde{M} = M_1 \times \cdots \times M_n$ in sufficient generality, which can be used in a number of different situations, particularly for estimates of fundamental solutions of $\Box_b$ mentioned above. They carried this out by first considering the initial value problem of the heat operator $\mathcal{H} = \partial_s + \mathcal{L}$ for each M_i, where $\mathcal{L}$ is the

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sub-Laplacian on M_i in self-adjoint form, then using the heat kernel to introduce a Littlewood-Paley theory for each M_i and finally passing to the corresponding product theory.

In [11], the product Hardy space H^p on $\widetilde{M}$ has been introduced and they obtained the H^p boundedness of the product singular intergal operators studied by Nagel and Stein in [17].

The main purpose of this paper is to introduce the product BMO space on restrictive product space $\widetilde{M}$. More precisely, each factor M_i satisfies the assumption that the vector fields on M_i are uniformly of finite type (Assumption 3.1, Definition 2.2, see also [14]). And we prove that it is the dual of the Hardy space $H^1(\widetilde{M})$. Namely, we will show the following

Theorem 1.1. $\big(H^1(\widetilde{M})\big)' = BMO(\widetilde{M})$.

As a consequence of duality, we obtain that the product singular intergal operators defiend by Nagel and Stein in [17] is bounded on $BMO(\widetilde{M})$ and from $L^\infty(\widetilde{M})$ to $BMO(\widetilde{M})$.

We shall point out that in [11], to establish the Hardy spaces $H^p(\widetilde{M})$, we do not need to impose any additional condition on $\widetilde{M}$, while introducing the BMO space and showing the duality the Assumption 3.1 mentioned above is crucial.

We remark that the duality of Hardy space on $\mathbb{R}^n$ was first obtained in [9] by C. Fefferman and Stein. For the multi-parameter product case, S.Y. Chang and R. Fefferman in [3] proved that the dual of $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$. Recently, in [10], the Carleson measure space $CMO^p(\mathcal{X} \times \mathcal{X})$ was introduced and it is proved to be the dual space of $H^p(\mathcal{X} \times \mathcal{X})$, where $(\mathcal{X}, d, \mu)$ is space of homogeneous type in the sense of Coifman and Weiss ([6]), μ satisfies

$$C_1 r \leq \mu(B(x,r)) \leq C_2 r$$

for all $x \in \mathcal{X}$ and $r > 0$, where $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$ and d satisfies some the Lipschitz condition, see more details in [10].

In this paper, to show Theorem 1.1, we will follow the ideas in [10]. The basic scheme is as follows.

Without lost of generality, we first concentrate on the product space of two factors, namely $\widetilde{M} = M_1 \times M_2$. For the sake of simplicity, we assume that $M_1 = M_2$, dropping the subscript.

To begin with, we impose Assumption 3.1 on M, then for such $\widetilde{M}$, we give the definition of $BMO(\widetilde{M})$ and establish the Plancherel-Pôlya-type inequality by using the discrete Calderón reproducing formula. Next, we introduce the product sequence spaces s^1 and c^1 and prove that the dual of s^1 is c^1 by following the constructive proof of Theorem 4.2 in [10]. Then we prove that $BMO(\widetilde{M})$ can be lifted to c^1 and c^1 can be projected to $BMO(\widetilde{M})$ and the combination of the lifting and projection operators equals the identity on $BMO(\widetilde{M})$. Similar results also hold for $H^1(\widetilde{M})$. From these results, Theorem 1.1 follows.

A brief description of the content of this paper is as follows. In Section 2, we provide some preliminaries introduced by Nagel and Stein ([17], [14], [16]) and the product Hardy space $H^1(\widetilde{M})$ introduced in [11]. The next three sections focus on $\widetilde{M} = M \times M$. In Section 3 we give the precise definition of $BMO(\widetilde{M})$ and establish the Plancherel-Pôlya-type inequality. In Section 4, we develop the product sequence spaces s^1 and c^1 and prove that $(s^1)' = c^1$. Theorem 1.1 will be proved in Section 5. Finally, in Section 6, we describe the results on $\widetilde{M} = M_1 \times \cdots \times M_n$.

2. Preliminaries

2.1. Geometry on $\widetilde{M} = M_1 \times \cdots \times M_n$

We recall the corresponding geometric structure in [16] (See also case (B) of [17]) by concentrating on each factor M_i, which we denote by M, dropping the subscript i.

Here M arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. Let $\Omega = \{(z,w) \in \mathbb{C}^2 : \ \mathrm{Im}(w) > P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m. Then $M = \partial\Omega = \{(z,w) \in \mathbb{C}^2 : \ \mathrm{Im}(w) = P(z)\}$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z,t) : \ z \in \mathbb{C}, t \in \mathbb{R}\}$ so that the point $(z, t + iP(z))$ corresponds to the point (z,t). The basic $(0,1)$ Levi vector field is then $\bar{Z} = \frac{\partial}{\partial \bar{z}} - i\frac{\partial P}{\partial \bar{z}}\frac{\partial}{\partial t}$, and we write $\bar{Z} = \mathbb{X}_1 + i\mathbb{X}_2$. The real vector fields $\{\mathbb{X}_1, \mathbb{X}_1\}$ and their commutators of order $\leq m$ span the tangent space to M at each point.

One variant of the control distance is defined as follows:

For each $x, y \in M$, let $AC(x,y,\delta)$ denote the collection of absolutely continuous mapping $\varphi : \ [0,1] \to M$ with $\varphi(0) = x, \ \varphi(1) = y$, and for almost every $t \in [0,1]$, $\varphi'(t) = \sum\limits_{j=1}^{2} a_j(t)\mathbb{X}_j(\varphi(t))$ with $|a_j(t)| \leq \delta$. The control distance $\rho(x,y)$ from x to y is the infimum of the set of $\delta > 0$ such that $AC(x,y,\delta) \neq \emptyset$. The result we need is that there is a pseudo-metric $d \approx \rho$ [1] equivalent to this control metric which has the optimal smoothness ; i.e. $d(x,y)$ is C^∞ on $\{M \times M - \mathrm{diagonal}\}$, and for $x \neq y$

$$|\partial_X^K \partial_Y^L d(x,y)| \lesssim d(x,y)^{1-K-L}. \tag{2.1}$$

(Here ∂_X^K is a product of K of the real vector fields $\{\mathbb{X}_1, \mathbb{X}_2\}$ acting as derivatives on the x variable, and ∂_Y^L are a corresponding L vector fields acting on the y

[1] Here, and throughout the paper, $A \approx B$ means that the ratio A/B is bounded and bounded away from zero by constants that do not depend on the relevant variables in A and B. $A \lesssim B$ means that the ratio A/B is bounded by a constant independent of the relevant variables. $a \vee b = \max\{a,b\}$ and $a \wedge b = \min\{a,b\}$.

variable). For the existence of such a pseudo-metric, see Theorem 3.3.1 and 4.4.6 in [15], where d is denoted by $\widetilde{\rho}$.

When integrating on M, we use Lebesgue measure on $\mathbb{C} \times \mathbb{R}$. Denote by $|E|$ the measure of E. The corresponding nonisotropic ball is $B(x,\delta) = \{y \in M : d(x,y) < \delta\}$ and $|B(x,\delta)|$ denotes its volume. The volume functions are introduced as follows:

$$V(x,y) = |B(x, d(x,y))|. \tag{2.2}$$

The volume of the ball $B(x,\delta)$ is essentially a polynomial in δ with coefficients that depend on x.

Let $\mathbb{T} = \frac{\partial}{\partial t}$ so that at each point of M the tangent space is spanned by the vectors $\{\mathbb{X}_1, \mathbb{X}_2, \mathbb{T}\}$. Write the commutator $[\mathbb{X}_1, \mathbb{X}_2] = \lambda\mathbb{T} + a_1\mathbb{X}_1 + a_2\mathbb{X}_2$, where $\lambda, a_1, a_2 \in C^\infty(M)$. For $k \geq 2$, set $\Lambda_k(x) = \sum_{\alpha \leq k-2} |\partial^\alpha \lambda(x)|$, where ∂^α is a product of α of the real vector fields $\{\mathbb{X}_1, \mathbb{X}_2\}$. Then the following formula holds for the volume $|B(x,\delta)|$:

$$|B(x,\delta)| \approx \sum_{k=2}^{m} \big(|\Lambda_k(x)\delta^k|\big)\delta^2. \tag{2.3}$$

The balls have the required doubling property

$$|B(x, 2\delta)| \leq C|B(x,\delta)| \quad \text{for all } \delta > 0.$$

We now recall the following construction given by Christ in [1] , which provides an analogue of the grid of Euclidean dyadic cubes on space of homogeneous type.

Lemma 2.1. [1]. *Let $(\mathcal{X}, \rho, \mu)$ be a space of homogeneous type, then, there exists a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and $C_1, C_2 > 0$, such that*

(*i*) $\mu(\mathcal{X} \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \bigcap Q_\beta^k = \Phi$ if $\alpha \neq \beta$;*

(*ii*) *for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \bigcap Q_\alpha^k = \Phi$;*

(*iii*) *for each (k,α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*

(*iv*) $\operatorname{diam}(Q_\alpha^k) \leq C_1 2^{-k}$;

(*v*) *each Q_α^k contains some ball $B(z_\alpha^k, C_2 2^{-k})$, where $z_\alpha^k \in \mathcal{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough 2^{-k} centered at z_α^k. As a result, we consider CQ_α^k to be the cube with the same center as Q_α^k and diameter $C\operatorname{diam}(Q_\alpha^k)$.

Using Lemma 2.1, we can obtain a grid of dyadic cubes on M.

Next we recall Definition 3.3.1 in [14] which characterizes the assumption imposed on M.

Definition 2.2. [14]. Vector fields $\mathbb{X}_1, \mathbb{X}_2, \mathbb{T}$ are uniformly of finite type m on an open set $U \subset \mathbb{R}^3$ if the derivatives of all coefficients of the vector fields are uniformly bounded on U and if the quantity $\sum_{j=2}^{m} \Lambda_j(q)$ is uniformly bounded and uniformly bounded away from zero on U. The vector fields $\mathbb{Y}, \mathbb{X}_1, \mathbb{X}_2, \mathbb{T}$ are uniformly of finite type m on an open set $V \subset \mathbb{R}^4$ if the derivatives of all coefficients of the vector fields are uniformly bounded on U and if the quantity $\sum_{j=2}^{m} \Lambda_j(q)$ is uniformly bounded and uniformly bounded away from zero on V.

2.2. The Heat Equation

In [17], the Littlewood-Paley square function was defined in terms of the heat kernel. More precisely, Nagel and Stein considered the sub-Laplacian $\mathcal{L}$ on M in self-adjoint form, given by

$$\mathcal{L} = \sum_{j=1}^{2} \mathbb{X}_j^* \mathbb{X}_j.$$

Here $(\mathbb{X}_j^* \varphi, \psi) = (\varphi, \mathbb{X}_j \psi)$, where $(\varphi, \psi) = \int\limits_M \varphi(x)\bar{\psi}(x)d\mu(x)$, and $\varphi, \psi \in C_0^\infty(M)$, the space of C^∞ functions on M with compact support. In general, $\mathbb{X}_j^* = -\mathbb{X}_j + a_j$, where $a_j \in C^\infty(M)$. The solution of the following initial value problem for the heat equation,

$$\frac{\partial u}{\partial s}(x,s) + \mathcal{L}_x u(x,s) = 0$$

with $u(x,0) = f(x)$, is given by $u(x,s) = H_s(f)(x)$, where H_s is the operator given via the spectral theorem by $H_s = e^{-s\mathcal{L}}$, and an appropriate self-adjoint extension of the non-negative operator $\mathcal{L}$ initially defined on $C_0^\infty(M)$. And they proved that for $f \in L^2(M)$,

$$H_s(f)(x) = \int_M H(s,x,y)f(y)d\mu(y).$$

Moreover $H(s,x,y)$ has some nice properties(see Proposition 2.3.1 in [17] and Theorem 2.3.1 in [14]). We restate them as follows:

(1) $H(s,x,y) \in C^\infty\big([0,\infty) \times M \times M \backslash \{s = 0 \text{ and } x = y\}\big)$.

(2) For very integer $N \geq 0$,

$$\begin{aligned} &|\partial_s^j \partial_X^L \partial_Y^K H(s,x,y)| \\ &\lesssim \frac{1}{(d(x,y)+\sqrt{s})^{2j+K+L}} \frac{1}{V(x,y)+V_{\sqrt{s}}(x)+V_{\sqrt{s}}(y)} \left(\frac{\sqrt{s}}{d(x,y)+\sqrt{s}}\right)^{\frac{N}{2}}. \end{aligned}$$

(3) For each integer $L \geq 0$ there exists an integer N_L and a constant C_L so that if $\varphi \in C_0^\infty(B(x_0,\delta))$, then for all $s \in (0,\infty)$

$$|\partial_X^L H_s[\varphi](x_0)| \leq C_L \delta^{-L} \sup_x \sum_{|J| \leq N_L} \delta^{|J|} |\partial_X^J \varphi(x)|.$$

(4) For all $(s,x,y) \in (0,\infty) \times M \times M$, $H(s,x,y) = H(s,y,x)$ and $H(s,x,y) \geq 0$.

(5) For all $(s,x) \in (0,\infty) \times M$, $\int H(s,x,y)dy = 1$.

(6) For $1 \leq p \leq \infty$, $\|H_s[f]\|_{L^p(M)} \leq \|f\|_{L^p(M)}$.

(7) For every $\varphi \in C_0^\infty(M)$ and every $t \geq 0$, $\lim\limits_{s \to 0} \|H_s[\varphi] - \varphi\|_t = 0$, where $\|\cdot\|_t$ denotes the Sobolev norm.

To introduce the reproducing identity and the Littlewood-Paley square function, they define a bounded operator $Q_s = 2s\frac{\partial H_s}{\partial s}$, $s > 0$, on $L^2(M)$. Denote by $q_s(x,y)$ the kernel of Q_s. Then from the estimates of $H(s,x,y)$, we have

(a) $q_s(x,y) \in C^\infty\big(M \times M \backslash \{x=y\}\big)$.

(b) For every integer $N \geq 0$,

$$|\partial_X^L \partial_Y^K q_s(x,y)| \lesssim \frac{1}{(d(x,y)+\sqrt{s})^{K+L}} \frac{1}{V(x,y)+V_{\sqrt{s}}(x)+V_{\sqrt{s}}(y)} \left(\frac{\sqrt{s}}{d(x,y)+\sqrt{s}}\right)^{\frac{N}{2}}.$$

(c) $\displaystyle\int q_s(x,y)dy = \int q_s(x,y)dx = 0.$

In [11], to develop the product Hardy space on $\widetilde{M}$, they discretize the operator Q_s by considering the sequence of bounded operators $\{Q_j\}_{j\in\mathbb{Z}}$, where $Q_j = -\frac{1}{2}\int_{2^{-2j}}^{2^{-2j+2}} Q_s \frac{ds}{s}$. From the behavior of operator H_s, it follows that $\sum_j Q_j = Id$ on $L^2(M)$. Denote by $q_j(x,y)$ the kernel of Q_j. From the estimates of $q_s(x,y)$, for each j, $q_j(x,y)$ satisfies that

(a') $q_j(x,y) \in C^\infty\big(M \times M \backslash \{x=y\}\big)$.

(b') For every integer $N \geq 0$,

$$|\partial_X^L \partial_Y^K q_j(x,y)| \lesssim \frac{1}{(d(x,y)+2^{-j})^{K+L}} \frac{1}{V(x,y)+V_{2^{-j}}(x)+V_{2^{-j}}(y)} \left(\frac{2^{-j}}{d(x,y)+2^{-j}}\right)^{\frac{N}{2}}.$$

(c') $\displaystyle\int q_j(x,y)dy = \int q_j(x,y)dx = 0.$

2.3. The Hardy space H^1 on product space $\widetilde{M} = M \times M$

To recall the definition of $H^1(\widetilde{M})$, we need to introduce the the test function space on $\widetilde{M}$

Definition 2.3. ([11]). Let $(x_0, y_0) \in \widetilde{M}$, $\gamma_1, \gamma_2, r_1, r_2 > 0$, $0 < \beta_1, \beta_2 \le 1$. A function on $\widetilde{M}$ is said to be a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ if there exists a constant $C \ge 0$ such that

(i) $|f(x,y)| \le C \dfrac{1}{V_{r_1}(x_0) + V(x_0,x)} \left(\dfrac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \dfrac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\dfrac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$ for all $(x,y) \in \widetilde{M}$;

(ii) $|f(x,y) - f(x',y)| \le C \left(\dfrac{d(x,x')}{r_1 + d(x,x_0)}\right)^{\beta_1} \dfrac{1}{V_{r_1}(x_0) + V(x_0,x)} \left(\dfrac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \times \dfrac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\dfrac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$ for all $x, x' \in \widetilde{M}$ satisfying that $d(x,x') \le (r_1 + d(x,x_0))/2$;

(iii) Property (ii) also holds with x and y interchanged;

(iv) $|f(x,y) - f(x',y) - f(x,y') + f(x',y')| \le C \left(\dfrac{d(x,x')}{r_1 + d(x,x_0)}\right)^{\beta_1} \dfrac{1}{V_{r_1}(x_0) + V(x_0,x)} \times \left(\dfrac{r_1}{r_1 + d(x,x_0)}\right)^{\gamma_1} \left(\dfrac{d(y,y')}{r_2 + d(y,y_0)}\right)^{\beta_2} \dfrac{1}{V_{r_2}(y_0) + V(y_0,y)} \left(\dfrac{r_2}{r_2 + d(y,y_0)}\right)^{\gamma_2}$ for all $x, x', y, y' \in \widetilde{M}$ satisfying that $d(x,x') \le (r_1 + d(x,x_0))/2$ and $d(y,y') \le (r_2 + d(y,y_0))/2$;

(v) $\displaystyle\int_{\widetilde{M}} f(x,y)dx = 0$ for all $y \in \widetilde{M}$;

(vi) $\displaystyle\int_{\widetilde{M}} f(x,y)dy = 0$ for all $x \in \widetilde{M}$.

If f is a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$, we write $f \in G(x_0, y_0; r_1, r_2; \beta_1, \beta_2;\ \gamma_1, \gamma_2)$ and we define the norm of f by

$$\|f\|_{G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \ (i), (ii), (iii) \text{ and } (iv) \ \text{ hold}\}.$$

We denote by $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $G(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$ for any fixed $(x_0, y_0) \in \widetilde{M}$. We can check that $G(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with equivalent norms for all $(x_0, y_0) \in \widetilde{M}$ and $r_1, r_2 > 0$. Furthermore, it is easy to check that $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

Now for $\vartheta_1, \vartheta_2 \in (0,1)$, let $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ be the completion of the space $G(\vartheta_1,\vartheta_2;\vartheta_1,\vartheta_2)$ in $G(\beta_1,\beta_2;\gamma_1,\gamma_2)$ when $0<\beta_i,\gamma_i<\vartheta_i$ with $i=1,2$. We define the dual space $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$ to be the set of all linear functionals L from $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$,

$$|L(f)| \leq C\|f\|_{\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

Next we recall the product square function $\widetilde{S}$ defined via the sequence of operators $\{Q_j\}_{j\in\mathbb{Z}}$ in [11]. If $f(x,y)$ is a function on $\widetilde{M}$ we define $Q_{j_1}\cdot Q_{j_2} = Q_{j_1}\otimes Q_{j_2}$ with Q_{j_1} acting on the first variable and Q_{j_2} on the second. $\widetilde{S}$ is then given by

$$\widetilde{S}(f)(x,y) = \Big\{ \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} \big|Q_{j_1}\cdot Q_{j_2}(f)(x,y)\big|^2 \Big\}^{\frac{1}{2}}. \tag{2.4}$$

And we have $\|\widetilde{S}(f)\|_{L^p(\widetilde{M})} \approx \|f\|_{L^p(\widetilde{M})}$ for $1<p<\infty$ ([11]). Then $H^1(\widetilde{M})$ is defined as follows.

Definition 2.4. ([11]). *Let $0<\vartheta_i<1$ and $0<\beta_i,\gamma_i<\vartheta_i$ for $i=1,2$. The Hardy space $H^1(\widetilde{M})$ is defined to be the set of all $f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ such that $\|\widetilde{S}[f]\|_{L^1(\widetilde{M})} < \infty$, and we define*

$$\|f\|_{H^1(\widetilde{M})} = \|\widetilde{S}[f]\|_{L^1(\widetilde{M})}.$$

Now we recall the discrete Calderón reproducing formula, the Plancherel-Pôlya-type inequality for $H^1(\widetilde{M})$ and the almost orthogonality estimate as follows.

Lemma 2.5. ([11]). *For $\vartheta_i \in (0,1)$, $0<\beta_i,\gamma_i<\vartheta_i$ with $=1,2$ and $f \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$,*

$$f(x,y) = \sum_{k_1,k_2}\sum_{I,J} |I||J| \widetilde{q}_{k_1}\widetilde{q}_{k_2}(x,x_I,y,y_J) Q_{k_1}Q_{k_2}[f](x_I,y_J), \tag{2.5}$$

where $\widetilde{q}_{k_1}\widetilde{q}_{k_2} \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$; $I,J \subset M$ are dyadic cubes with length $2^{-k_1-N_0}$ and $2^{-k_2-N_0}$ for a fixed integer N_0; x_I, y_J are any fixed points in I and J, respectively. The series in (2.5) converges in the norm of $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$. Moreover, for $f \in \big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$, (2.5) holds in the dual space $\big(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}$ $(\beta_1,\beta_2;\gamma_1,\gamma_2)\big)'$.

Lemma 2.6. ([11]). *Suppose* $0 < \vartheta_i < 1$ *and* $0 < \beta_i, \gamma_i < \vartheta_i$ *for* $i = 1, 2$. *Then, for all* $f \in (\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$,

$$\begin{aligned}(2.6)\qquad &\left\| \left\{ \sum_{k_1,k_2} \sum_{I,J} \sup_{u\in I, v\in J} |Q_{k_1} Q_{k_2}[f](u,v)|^2 \, \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}} \right\|_{L^1(\widetilde{M})} \\ &\approx \left\| \left\{ \sum_{k_1,k_2} \sum_{I,J} \inf_{u\in I, v\in J} |Q_{k_1} Q_{k_2}[f](u,v)|^2 \, \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}} \right\|_{L^1(\widetilde{M})},\end{aligned}$$

where I, J *are the same as in Lemma 2.5.*

Let $\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x, x_I, y, y_J)$ be the same as in Lemma 2.5. Note that for any $\beta_1, \beta_2, \gamma_1, \gamma_2 \in (0, 1)$, $\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x, x_I, y, y_J) \in G(\beta_1, \beta_2, \gamma_1, \gamma_2)$. We have that for any $\gamma_1, \gamma_2 \in (0, 1)$ and $\epsilon_1 \in (0, \gamma_1)$, $\epsilon_2 \in (0, \gamma_2)$,

$$\begin{aligned}(2.7)\qquad &|q_{j_1} q_{j_2} \widetilde{q}_{k_1}\widetilde{q}_{k_2}(x, x_I, y, y_J)| \lesssim 2^{-|j_1-k_1|\epsilon_1} 2^{-|j_2-k_2|\epsilon_2} \\ &\frac{1}{V(x, x_I) + V_{2^{-(j_1\wedge k_1)}}(x) + V_{2^{-(j_1\wedge k_1)}}(x_I)} \times \left(\frac{2^{-(j_1\wedge k_1)}}{2^{-(j_1\wedge k_1)} + d(x, x_I)} \right)^{\gamma_1} \\ &\frac{1}{V(y, y_J) + V_{2^{-(j_2\wedge k_2)}}(y) + V_{2^{-(j_2\wedge k_2)}}(y_J)} \left(\frac{2^{-(j_2\wedge k_2)}}{2^{-(j_2\wedge k_2)} + d(y, y_J)} \right)^{\gamma_2}.\end{aligned}$$

3. Product BMO Space and the Plancherel-Pôlya-type Inequality

In this section, to characterize the dual space of $H^1(\widetilde{M})$, we introduce the product BMO space on $\widetilde{M} = M \times M$, which is motivated by ideas of Chang and R. Fefferman([2]), see also [10]. To carry this out, we impose the following assumption on M in all rest sections.

Assumption 3.1. Let M and the real vector fields $\{\mathbb{X}_1, \mathbb{X}_2, \mathbb{T}\}$ be the same as in §2.1. Assume that $\{\mathbb{X}_1, \mathbb{X}_2, \mathbb{T}\}$ are uniformly of finite type m on M (see Definition 2.2).

Now we give the definition of BMO space on $\widetilde{M} = M \times M$ via the sequence of operators $\{Q_j\}_{j\in\mathbb{Z}}$ as follows.

Definition 3.2. Suppose $0 < \vartheta_i < 1$ and $0 < \beta_i, \gamma_i < \vartheta_i$ for $i = 1, 2$. We define the space $BMO(\widetilde{M})$ to be the set of all $f \in \left(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$ such that

$$\begin{aligned}(3.1)\qquad &\|f\|_{BMO(\widetilde{M})} \\ &= \sup_{\Omega} \left\{ \frac{1}{|\Omega|} \int_{\Omega} \sum_{k_1,k_2} \sum_{I\times J\subseteq\Omega} \left| Q_{k_1} Q_{k_2}[f](x,y) \right|^2 \chi_I(x)\chi_J(y) dx dy \right\}^{\frac{1}{2}} < \infty,\end{aligned}$$

where the supermum is taken over all open sets Ω in $\widetilde{M}$ with finite measure and for each k_1 and k_2, I, J range over all the dyadic cubes with length $\ell(I) = 2^{-k_1-N_0}$ and $\ell(J) = 2^{-k_2-N_0}$, respectively.

To see this definition is independent of the choice of Q_j, we first establish the Plancherel-Pôlya-type inequality for $BMO(\widetilde{M})$.

Theorem 3.3. *Let all notation be the same as in Definition 3.2. Then for each* $f \in BMO(\widetilde{M})$,

$$\begin{aligned}&\sup_{\Omega}\left\{\frac{1}{|\Omega|}\int_{\Omega}\sum_{k_1,k_2}\sum_{I\times J\subseteq\Omega}\sup_{u\in I,v\in J}\left|Q_{k_1}Q_{k_2}[f](u,v)\right|^2\chi_I(x)\chi_J(y)dxdy\right\}^{\frac{1}{2}}\\ &\approx \sup_{\Omega}\left\{\frac{1}{|\Omega|}\int_{\Omega}\sum_{k_1,k_2}\sum_{I\times J\subseteq\Omega}\inf_{u\in I,v\in J}\left|Q_{k_1}Q_{k_2}[f](u,v)\right|^2\chi_I(x)\chi_J(y)dxdy\right\}^{\frac{1}{2}}.\end{aligned}\tag{3.2}$$

Proof. Since M satisfies Assumption 3.1, then the quantity $\sum_{j=2}^m \Lambda_j(q)$ is uniformly bounded and uniformly bounded away from zero on M. Thus, from (2.3), we have

$$|B(x,\delta)| \approx |B(y,\delta)| \qquad \text{for all } x, y \in M,\tag{3.3}$$

and

$$|B(x,\delta)| \approx \delta^{m+2} \quad \text{for } \delta \geq 1; \qquad |B(x,\delta)| \approx \delta^4 \quad \text{for } \delta \leq 1.\tag{3.4}$$

The estimates in (3.11) are crucial, namely, if $\mathrm{diam} I \approx \delta, \delta \leq 1$, then $|I| \approx \delta^4$ and if $\mathrm{diam} I \approx \delta, \delta > 1$, then $|I| \approx \delta^{m+2}$. These estimates will be often used in the following proof.

Now for any $f \in BMO(\widetilde{M})$, by using the discrete product reproducing identity (2.5), the Hölder inequality and the almost orthogonality estimate (2.7), we have

$$\begin{aligned}&\sup_{u\in I,v\in J}\left|Q_{k_1}Q_{k_2}[f](u,v)\right|^2\\ &\lesssim \sum_{k_1',k_2'} 2^{-|k_1-k_1'|\epsilon_1}2^{-|k_2-k_2'|\epsilon_2}\sum_{I',J'}|I'||J'|\frac{1}{V(x_I,x_{I'})+V_{2^{-(k_1\wedge k_1')}}(x_I)+V_{2^{-(k_1\wedge k_1')}}(x_{I'})}\\ &\quad\times\Big(\frac{2^{-(k_1\wedge k_1')}}{2^{-(k_1\wedge k_1')}+d(x_I,x_{I'})}\Big)^{\gamma_1}\frac{1}{V(y_J,y_{J'})+V_{2^{-(k_2\wedge k_2')}}(y_J)+V_{2^{-(k_2\wedge k_2')}}(y_{J'})}\\ &\quad\times\Big(\frac{2^{-(k_2\wedge k_2')}}{2^{-(k_2\wedge k_2')}+d(y_J,y_{J'})}\Big)^{\gamma_2}\left|Q_{k_1'}Q_{k_2'}[f](x_{I'},y_{J'})\right|^2,\end{aligned}$$

where ϵ_i is chosen to satisfy $\epsilon_i \in (\vartheta_i, 1)$ for $i = 1, 2$, I' and J' range over all dyadic cubes with length $\ell(I') \approx 2^{-k_1'-N_0}$ and $\ell(J') \approx 2^{-k_2'-N_0}$, respectively. Moreover, x_I, $x_{I'}$ and y_J, $y_{J'}$ can be any fixed points in I, I' and J, J', respectively.

Note that $2^{-|k_1-k_1'|} \approx \frac{\mathrm{diam}(I)}{\mathrm{diam}(I')} \wedge \frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}$, $2^{-(k_1\wedge k_1')} \approx \mathrm{diam}(I) \vee \mathrm{diam}(I')$, $d(x_I, x_{I'}) \geq \mathrm{dist}(I, I')$ and that similar results hold for k_2, k_2' and J, J'. Then

$$\begin{aligned}
&\sup_{u\in I, v\in J} \left|Q_{k_1}Q_{k_2}[f](u,v)\right|^2 \\
&\lesssim \sum_{k_1',k_2'}\sum_{I',J'} |I'||J'| \Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')} \wedge \frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1} \Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')} \wedge \frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2} \\
&\times \frac{1}{V_{\mathrm{dist}(I,I')}(x_I) + |I| \vee |I'|} \Big(\frac{\mathrm{diam}(I) \vee \mathrm{diam}(I')}{\mathrm{diam}(I) \vee \mathrm{diam}(I') + \mathrm{dist}(I,I')}\Big)^{\gamma_1} \\
&\times \frac{1}{V_{\mathrm{dist}(J,J')}(y_J) + |J| \vee |J'|} \Big(\frac{\mathrm{diam}(J) \vee \mathrm{diam}(J')}{\mathrm{diam}(J) \vee \mathrm{diam}(J') + \mathrm{dist}(J,J')}\Big)^{\gamma_2} \\
&\times \left|Q_{k_1'}Q_{k_2'}[f](x_{I'}, y_{J'})\right|^2.
\end{aligned}$$

Combining the above estimate with the facts that $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J' respectively and $ab = (a \vee b)^2\big(\frac{a}{b} \wedge \frac{b}{a}\big)$ for any $a, b > 0$ implies that for any open set $\Omega \in \widetilde{M}$ with finite measure,

$$\begin{aligned}
&\frac{1}{|\Omega|}\sum_{k_1,k_2}\sum_{I\times J\subset\Omega} |I||J| \sup_{u\in I, v\in J} \left|Q_{k_1}Q_{k_2}[f](u,v)\right|^2 \\
&\lesssim \frac{1}{|\Omega|}\sum_{k_1,k_2}\sum_{I\times J\subset\Omega}\sum_{k_1',k_2'}\sum_{I',J'} \Big[\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big]\Big[\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big]\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1} \\
&\times \Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')} \wedge \frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2} \cdot \big(|I| \vee |I'|\big)\big(|J| \vee |J'|\big) \\
&\times \frac{|I| \vee |I'|}{V_{\mathrm{dist}(I,I')}(x_I) + |I| \vee |I'|} \Big(\frac{\mathrm{diam}(I) \vee \mathrm{diam}(I')}{\mathrm{diam}(I) \vee \mathrm{diam}(I') + \mathrm{dist}(I,I')}\Big)^{\gamma_1} \\
&\times \frac{|J| \vee |J'|}{V_{\mathrm{dist}(J,J')}(y_J) + |J| \vee |J'|} \Big(\frac{\mathrm{diam}(J) \vee \mathrm{diam}(J')}{\mathrm{diam}(J) \vee \mathrm{diam}(J') + \mathrm{dist}(J,J')}\Big)^{\gamma_2} \\
&\times \inf_{u\in I', v\in J'} \left|Q_{k_1'}Q_{k_2'}[f](u,v)\right|^2.
\end{aligned} \tag{3.5}$$

For our convenience, let $R = I \times J$ and $R' = I' \times J'$, where I, J, I' and J' range over all dyadic cubes on M. And set

$$\sum_{k_1,k_2}\sum_{I\times J\subset\Omega} = \sum_{R\subset\Omega};\ \sum_{k_1',k_2'}\sum_{I',J'} = \sum_{R'};$$

$$|R| = |I| \times |J|;\ \ |R'| = |I'| \times |J'|;$$

$$r(R,R')=\Big[\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big]\Big[\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big]\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1}\Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\wedge\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2};$$
$$v(R,R')=\Big(|I|\vee|I'|\Big)\Big(|J|\vee|J'|\Big);$$
$$R(R,R')=\frac{|I|\vee|I'|}{V_{\mathrm{dist}(I,I')}(x_I)+|I|\vee|I'|}\Big(\frac{\mathrm{diam}(I)\vee\mathrm{diam}(I')}{\mathrm{diam}(I)\vee\mathrm{diam}(I')+\mathrm{dist}(I,I')}\Big)^{\gamma_1}$$
$$\times\frac{|J|\vee|J'|}{V_{\mathrm{dist}(J,J')}(y_J)+|J|\vee|J'|}\Big(\frac{\mathrm{diam}(J)\vee\mathrm{diam}(J')}{\mathrm{diam}(J)\vee\mathrm{diam}(J')+\mathrm{dist}(J,J')}\Big)^{\gamma_2};$$

$$S_R=\sup_{u\in I,v\in J}\big|Q_{k_1}Q_{k_2}[f](u,v)\big|^2;$$
$$T_{R'}=\inf_{u\in I',v\in J'}\big|Q_{s_1'}Q_{s_2'}[f](u,v)\big|^2.$$

Then, (3.5) can be rewritten as

$$\frac{1}{|\Omega|}\sum_{R\subset\Omega}|R|S_R\lesssim\frac{1}{|\Omega|}\sum_{R\subset\Omega}\sum_{R'}r(R,R')v(R,R')P(R,R')T_{R'}. \tag{3.6}$$

To complete the proof, we need to prove that the right-hand side of (3.6) can be controlled by

$$\sup_{\overline{\Omega}}\frac{1}{|\overline{\Omega}|}\sum_{R'\subset\overline{\Omega}}|R'|T_{R'},$$

where $\overline{\Omega}$ ranges over all open sets in $\widetilde{M}$ with finite measure.

Let $\Omega^{i,\ell}=\bigcup_{R=I\times J\subset\Omega}3(2^iI\times 2^\ell J)$ for $i,\ell\geq 0$ and

$$B_{0,0}=\{R'=I'\times J':3R'\bigcap\Omega^{0,0}\neq\emptyset\};$$
$$B_{i,0}=\{R'=I'\times J':3(2^iI'\times J')\bigcap\Omega^{i,0}\neq\emptyset,3(2^{i-1}I'\times J')\bigcap\Omega^{i-1,0}=\emptyset\};$$
$$B_{0,\ell}=\{R'=I'\times J':3(I'\times 2^\ell J')\bigcap\Omega^{0,\ell}\neq\emptyset,3(I'\times 2^{\ell-1}J')\bigcap\Omega^{0,l-1}=\emptyset\};$$
$$B_{i,\ell}=\{R'=I'\times J':3(2^iI'\times 2^\ell J')\bigcap\Omega^{i,\ell}\neq\emptyset,3(2^{i-1}I'\times 2^{\ell-1}J')\bigcap\Omega^{i-1,l-1}=\emptyset\},$$

where $i,\ell\geq 1$.

First, it is obvious that $\bigcup_{i,\ell\geq 0}B_{i,\ell}\subset\{R'=I'\times J',I',J'$ are dyadic cubes$\}$. Moreover, since $\lim_{i,\ell\to\infty}\Omega^{i,\ell}=\widetilde{M}$, we can see that for any dyadic rectangle R', it must belong to some $B_{i,\ell}$. Thus, $\{R'=I'\times J',I',J'$ are dyadic cubes$\}\subset\bigcup_{i,\ell\geq 0}B_{i,\ell}$. Hence we have

$$\{R'=I'\times J',I',J'\text{ are dyadic cubes}\}=\bigcup_{i,\ell\geq 0}B_{i,\ell}.$$

As a consequence, the right-hand side of (3.6) can be controlled by

$$\frac{1}{|\Omega|}\sum_{R\subset\Omega}\Big(\sum_{R'\in B_{0,0}}+\sum_{i\geq 1}\sum_{R'\in B_{i,0}}+\sum_{\ell\geq 1}\sum_{R'\in B_{0,\ell}}+\sum_{i,\ell\geq 1}\sum_{R'\in B_{i,\ell}}\Big)$$
$$r(R,R')v(R,R')P(R,R')T_{R'}$$
$$=: \mathbb{I}+\mathbb{II}+\mathbb{III}+\mathbb{IV}.$$

We first estimate $\mathbb{I}$. Note that when $R'\in B_{0,0}$, $3R'\bigcap\Omega^{0,0}\neq\emptyset$, so let $\mathcal{F}_h^{0,0}=\{R':|3R'\bigcap\Omega^{0,0}|\geq\frac{1}{2^h}|3R'|\}$, $\mathcal{D}_h^{0,0}=\mathcal{F}_h^{0,0}\backslash\mathcal{F}_{h-1}^{0,0}$, $\mathcal{F}_{-1}^{0,0}=\emptyset$ and $\Omega_h^{0,0}=\bigcup_{R'\in\mathcal{D}_h^{0,0}}R'$, where $h\geq 0$. Since $B_{0,0}=\bigcup_{h\geq 0}\mathcal{D}_h^{0,0}$, we have

$$\mathbb{I}\leq\frac{1}{|\Omega|}\sum_{h\geq 0}\sum_{R'\in\mathcal{D}_h^{0,0}}\sum_{R\subset\Omega}r(R,R')v(R,R')P(R,R')T_{R'}. \tag{3.7}$$

To estimate (3.7), for each $R'\in\mathcal{D}_h^{0,0}$, we decompose $\{R:R\subset\Omega\}$ by

$$A_{0,0}(R')=\Big\{R\subseteq\Omega:\mathrm{dist}(I,I')\leq\mathrm{diam(I)}\vee\mathrm{diam(I')},\ \mathrm{dist}(J,J')\leq\mathrm{diam(J)}\vee\mathrm{diam(J')}\Big\};$$
$$A_{j,0}(R')=\Big\{R\subseteq\Omega:2^{j-1}\big(\mathrm{diam(I)}\vee\mathrm{diam(I')}\big)<\mathrm{dist}(I,I')\leq 2^j\big(\mathrm{diam(I)}\vee\mathrm{diam(I')}\big),$$
$$\mathrm{dist}(J,J')\leq\mathrm{diam(J)}\vee\mathrm{diam(J')}\Big\};$$
$$A_{0,k}(R')=\Big\{R\subseteq\Omega:\mathrm{dist}(I,I')\leq\mathrm{diam(I)}\vee\mathrm{diam(I')},$$
$$2^{k-1}\big(\mathrm{diam(J)}\vee\mathrm{diam(J')}\big)<\mathrm{dist}(J,J')\leq 2^k\big(\mathrm{diam(J)}\vee\mathrm{diam(J')}\big)\Big\};$$
$$A_{j,k}(R')=\Big\{R\subseteq\Omega:2^{j-1}\big(\mathrm{diam(I)}\vee\mathrm{diam(I')}\big)<\mathrm{dist}(I,I')\leq 2^j\big(\mathrm{diam(I)}\vee\mathrm{diam(I')}\big),$$
$$2^{k-1}\big(\mathrm{diam(J)}\vee\mathrm{diam(J')}\big)<\mathrm{dist}(J,J')\leq 2^k\big(\mathrm{diam(J)}\vee\mathrm{diam(J')}\big)\Big\},$$

where $j,k\geq 1$. Then we split the right-hand side of (3.7) into

$$\frac{1}{|\Omega|}\sum_{h\geq 0}\sum_{R'\in\mathcal{D}_h^{0,0}}\Big(\sum_{R\in A_{0,0}(R')}+\sum_{j\geq 1}\sum_{R\in A_{j,0}(R')}+\sum_{k\geq 1}\sum_{R\in A_{0,k}(R')}+\sum_{j,k\geq 1}\sum_{R\in A_{j,k}(R')}\Big)v(R,R')$$
$$\times r(R,R')P(R,R')T_{R'}=:\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4.$$

Now we first estimate $\mathbb{I}_1$. To do this, we only need to consider

$$\sum_{R\in A_{0,0}(R')}r(R,R')v(R,R') \tag{3.8}$$

for any $R' \in \mathcal{D}_h^{0,0}$ and $h \geq 0$, since $P(R,R^{'}) \leq 1$ in this case. In what follows, we use the geometrical argument as we deal with the homogeneous space, which is a generalization of Chang and R. Fefferman's idea, see more details in [10] and [2]. Note that when $R \in A_{0,0}(R^{'})$, $3R\bigcap 3R^{'} \neq \emptyset$. So we can split (3.8) into four cases:

Case 1. $|I'| \geq |I|$, $|J'| \leq |J|$.

We first consider the comparison of the diameters of I, I' and J, J'. Note that $\mathrm{diam}(I) \approx 2^{-k_1}$ and $\mathrm{diam}(I^{'}) \approx 2^{-k_1^{'}}$. As we remarked above, the following geometric arguments follow from Assumption 3.1.

If $2^{-k_1}, 2^{-k_1^{'}} \geq 1$, then $2^{-k_1^{'}(m+2)} \approx |I'| \geq |I| \approx 2^{-k_1(m+2)}$. This yields $2^{-k_1^{'}} \gtrsim 2^{-k_1}$.

If $2^{-k_1}, 2^{-k_1^{'}} \leq 1$, then $2^{-k_1^{'}\cdot 4} \approx |I'| \geq |I| \approx 2^{-k_1\cdot 4}$. This also implies $2^{-k_1^{'}} \gtrsim 2^{-k_1}$.

If $2^{-k_1^{'}} \geq 1 \geq 2^{-k_1}$, then obviously $2^{-k_1^{'}} \geq 2^{-k_1}$.

If $2^{-k_1^{'}} \leq 1 \leq 2^{-k_1}$, we can see that this is impossible since in Case1, $|I'| \geq |I|$.

Combining the above four results, we can see that $\mathrm{diam}(I^{'}) \gtrsim \mathrm{diam}(I)$. Similarly, we can obtain that $\mathrm{diam}(J^{'}) \lesssim \mathrm{diam}(J)$.

From this, we have

$$\frac{|I|}{|3I'|}|3R'| \lesssim |3R\bigcap 3R'| \lesssim |3R'\bigcap \Omega^{0,0}| \lesssim \frac{1}{2^{h-1}}|3R'|,$$

then $2^{h-1}|I| \leq |3I'| \lesssim |I'|$. Thus $|I'| \approx 2^{h-1+n_1}|I|$, for some $n_1 \geq 0$. For each fixed n_1, the number of such I's must be $\lesssim 2^{n_1}$. As for J, $|J| \approx 2^{n_2}|J'|$ for some $n_2 \geq 0$. For each fixed n_2, the number of such $J^{'}$'s is less than a constant independent of n_2, since $3J\bigcap 3J^{'} \neq \emptyset$ and $|J| \geq |J'|$.

Again, by Assumption 3.1, if $2^{-k_1}, 2^{-k_1^{'}} \geq 1$, then $2^{-k_1^{'}(m+2)} \approx |I'| \approx 2^{h-1+n_1}|I| \approx 2^{h-1+n_1}2^{-k_1(m+2)}$. This yields that $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})} \approx 2^{-\frac{h-1+n_1}{m+2}}$.

Similarly, if $2^{-k_1^{'}} \geq 1 \geq 2^{-k_1}$, then $2^{-k_1^{'}(m+2)} \approx |I'| \approx 2^{h-1+n_1}|I| \approx 2^{h-1+n_1}2^{-k_1\cdot 4}$. This implies that $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})} \lesssim 2^{-\frac{h-1+n_1}{m+2}}$.

Finally, if $2^{-k_1}, 2^{-k_1^{'}} \leq 1$, then $2^{-k_1^{'}\cdot 4} \approx |I'| \approx 2^{h-1+n_1}|I| \approx 2^{-k_1\cdot 4}$. Hence, $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})} \approx 2^{-\frac{h-1+n_1}{4}}$.

Combining the above cases, we have $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})} \lesssim 2^{-\frac{h-1+n_1}{m+2}}$. Similarly, $\frac{\mathrm{diam}(J^{'})}{\mathrm{diam}(J)} \lesssim 2^{-\frac{n_2}{m+2}}$.

Thus

$$\begin{aligned}
&\sum_{R\in case1} r(R,R^{'})v(R,R^{'})\\
&= \sum_{R\in case1}\left(\frac{|I|}{|I'|}\right)\left(\frac{|J'|}{|J|}\right)\left(\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})}\right)^{\epsilon_1}\left(\frac{\mathrm{diam}(J^{'})}{\mathrm{diam}(J)}\right)^{\epsilon_2}|I'||J|\\
&\lesssim \sum_{n_1,n_2\geq 0} 2^{-(h-1+n_1)(1+\frac{\epsilon_1}{m+2})}2^{-n_2(1+\frac{\epsilon_2}{m+2})}2^{n_1}|I'|2^{n_2}|J'|\\
&\lesssim 2^{-h(1+\frac{\epsilon_1}{m+2})}|R'|.
\end{aligned}$$

Case 2. $|I'|\leq|I|$, $|J'|\geq|J|$.
This can be handled similarly as Case 1. We have

$$\sum_{R\in case2} r(R,R^{'})v(R,R^{'}) \quad\lesssim\quad 2^{-h(1+\frac{\epsilon_2}{m+2})}|R'|.$$

Case 3. $|I'|\geq|I|$, $|J'|\geq|J|$.

Similar to Case1, by comparing 2^{-k_i} and $2^{-k_i^{'}}$ with 1 respectively, we can obtain that $\mathrm{diam}(I^{'})\gtrsim\mathrm{diam}(I)$ and $\mathrm{diam}(J^{'})\gtrsim\mathrm{diam}(J)$. Thus we have

$$|R|\lesssim|3R^{'}\bigcap 3R|\leq|3R^{'}\bigcap\Omega_{0,0}|\leq\frac{1}{2^{h-1}}|3R'|.$$

thus $2^{h-1}|R|\lesssim|R'|$. Hence $|R'|\approx 2^{h-1+n}|R|$ for some $n\geq 0$. For each fixed n, the number of such R's is $\lesssim 2^n$.

Now we further consider the diameter of the cubes I,I',J,J'.

If $2^{-k_1},2^{-k_1^{'}}\geq 1$ and $2^{-k_2},2^{-k_2^{'}}\geq 1$, then $2^{-k_1^{'}(m+2)}2^{-k_2^{'}(m+2)}\approx 2^{h-1+n}$ $2^{-k_1(m+2)}2^{-k_2(m+2)}$. Hence $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})}\frac{\mathrm{diam}(J)}{\mathrm{diam}(J^{'})}\lesssim 2^{-\frac{h-1+n}{m+2}}$.

Similarly, by continuing comparing 2^{-k_i} and $2^{-k_i^{'}}$ with 1, respectively, we have $\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})}\frac{\mathrm{diam}(J)}{\mathrm{diam}(J^{'})}\lesssim 2^{-\frac{h-1+n}{m+2}}$. As a consequence, we have

$$\begin{aligned}
\sum_{R\in case3} r(R,R^{'})v(R,R^{'}) &= \sum_{R\in case3}\frac{|R|}{|R'|}\left(\frac{\mathrm{diam}(I)}{\mathrm{diam}(I^{'})}\right)^{\epsilon_1}\left(\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\right)^{\epsilon_2}|R'|\\
&\lesssim \sum_{n\geq 0}2^{-(h-1+n)(1+\frac{\epsilon_3}{m+2})}|R'|\\
&\lesssim 2^{-h(1+\frac{\epsilon_3}{m+2})}|R'|,
\end{aligned}$$

where $\epsilon_3=\epsilon_1\wedge\epsilon_2$.

Case 4. $|I'| \le |I|$, $|J'| \le |J|$.

Similar to Case 3, we have $\mathrm{diam}(I^{'}) \lesssim \mathrm{diam}(I)$ and $\mathrm{diam}(J^{'}) \lesssim \mathrm{diam}(J)$, which implies that

$$|R'| \lesssim |3R^{'} \bigcap 3R| \le |3R^{'} \bigcap \Omega_{0,0}| \le \frac{1}{2^{h-1}}|3R^{'}|.$$

Hence there exists a constant $h_0 > 0$ independent of R and R' such that $0 \le h \le h_0$. We obtain that $|R| \approx 2^{h-1+n}|R^{'}|$ for some $n \ge 0$ and that for each fixed n, the number of such $R^{'}$'s is less then a constant independent of n. Also, by using the same skills as in Case3, we have $\frac{\mathrm{diam}(I^{'})}{\mathrm{diam}(I)}\frac{\mathrm{diam}(J^{'})}{\mathrm{diam}(J)} \lesssim 2^{-\frac{h-1+n}{m+2}}$. Therefore

$$\begin{aligned}&\sum_{R\in case4} r(R,R^{'})v(R,R^{'})\\ &= \sum_{R\in case4} \frac{|R'|}{|R|}\left(\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\right)^{\epsilon_1}\left(\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\right)^{\epsilon_2}|R| \lesssim 2^{-h\frac{\epsilon_3}{m+2}}|R'|,\end{aligned}$$

where ϵ_3 is the same as in Case3.

Now let us turn to $\mathbb{I}_1$.

$$\begin{aligned}\mathbb{I}_1 &= \frac{1}{|\Omega|}\sum_h \sum_{R'\in\mathcal{D}_h^{0,0}}\left(\sum_{R\in\ case1} + \sum_{R\in\ case2} + \sum_{R\in\ case3} + \sum_{R\in\ case4}\right) r(R,R^{'})v(R,R^{'})T_{R'}\\ &=: \mathbb{I}_{11} + \mathbb{I}_{12} + \mathbb{I}_{13} + \mathbb{I}_{14}.\end{aligned}$$

Obviously, combining the fact that $|\Omega_h^{0,0}| \lesssim h2^h|\Omega|$ for $h \ge 1$, $|\Omega_0^{0,0}| \lesssim |\Omega|$, $\epsilon_i \in (\vartheta_i, 1)$ for $i = 1, 2$, we have

$$\begin{aligned}\mathbb{I}_{11}, \mathbb{I}_{12}, \mathbb{I}_{13} &\lesssim \frac{1}{|\Omega|}\sum_h \sum_{R'\in\mathcal{D}_h^{0,0}} 2^{-h(1+\frac{\epsilon_3}{m+2})}|R'|T_{R'}\\ &\lesssim \sum_h 2^{-h(1+\frac{\epsilon_3}{m+2})}\frac{|\Omega_h^{0,0}|}{|\Omega|}\frac{1}{|\Omega_h^{0,0}|}\sum_{R'\subset\ \Omega_h^{0,0}} |R'|T_{R'}\\ &\lesssim \sum_h 2^{-h(1+\frac{\epsilon_3}{m+2})}h2^h \sup_{\bar\Omega}\frac{1}{|\bar\Omega|}\sum_{R'\subset\bar\Omega}\mu(R^{'})T_{R'}\\ &\lesssim \sup_{\bar\Omega}\frac{1}{|\bar\Omega|}\sum_{R'\subset\bar\Omega}\mu(R^{'})T_{R'}.\end{aligned}$$

As for $\mathbb{I}_{14}$, noting that $0 \le h \le h_0$ then we can get the same estimate as above.

Then, following the same routine and skills as in the proof of Theorem 3.2 in [10], we can obtain the estimates of other three terms in $\mathbb{I}$ and similarly we can deal with $\mathbb{II}$, $\mathbb{III}$ and $\mathbb{IV}$ with only minor differences that we need to compare the diameter of the dyadic cube with 1 according to the volume of the cube.

This completes the proof of Theorem 3.3. ■

4. Product Sequence Spaces and Duality

In this section, we introduce the product sequence space c^1 and prove that c^1 is the dual space of s^1. Let $\widetilde{M} = M \times M$, where M is mentioned in Section 2.1. We first recall the definition of s^1 introduced in [11].

Definition 4.1. [11]. Set $\tilde{\chi}_R(x_1, x_2) = |R|^{-1/2}\chi_R(x_1, x_2)$ for any dyadic rectangle R in $\widetilde{M}$. The product sequence space s^1 is defined as the collection of all complex-value sequences $s = \{s_R\}_R$ such that

$$\|s\|_{s^1} = \big\|\big\{\sum_R (|s_R|\tilde{\chi}_R(x_1, x_2))^2\big\}^{1/2}\big\|_{L^1(\widetilde{M})}. \tag{4.1}$$

Definition 4.2. The product sequence space c^1 is defined as the collection of all complex-value sequences $t = \{t_R\}_R$ such that

$$\|t\|_{c^1} = \sup_{\Omega}\big\{\frac{1}{|\Omega|}\sum_{R\subseteq\Omega}|t_R|^2\big\}^{1/2}, \tag{4.2}$$

where the sup is taken over all open sets $\Omega \in \widetilde{M}$ with finite measure and R ranges over all the dyadic rectangles in $\widetilde{M}$.

The main result in this section is the following duality theorem.

Theorem 4.3. $\big(s^1\big)' = c^1$.

Proof. First, we prove that for all $t \in c^1$, let

$$L(s) = \sum_R s_R \cdot \bar{t}_R, \quad \forall s \in s^1, \tag{4.3}$$

then $|L(s)| \lesssim \|s\|_{s^1}\|t\|_{c^1}$.

To see this, let

$$\begin{aligned}
\Omega_k &= \big\{(x_1, x_2) \in \widetilde{M} \ : \big\{\sum_R \big(|s_R|\tilde{\chi}_R(x_1, x_2)\big)^2\big\}^{1/2} > 2^k\big\};\\
B_k &= \big\{R : |\Omega_k \bigcap R| > \frac{1}{2}|R|,\ |\Omega_{k+1} \bigcap R| \le \frac{1}{2}|R|\big\};\\
\tilde{\Omega}_k &= \big\{(x_1, x_2) \in \widetilde{M} \ : \mathcal{M}_s(\chi_{\Omega_k}) > \frac{1}{2}\big\},
\end{aligned}$$

where M_s is the strong maximal function on $\widetilde{M}$. By (4.3) and the Hőlder inequality,

$$
\begin{aligned}
|L(s)| &\le \sum_k \Big(\sum_{R\in B_k} |s_R|^2\Big)^{\frac12} \Big(\sum_{R\in B_k} |t_R|^2\Big)^{\frac12} \\
&\le \sum_k |\tilde{\Omega}_k|^{\frac12} \Big(\sum_{R\in B_k} |s_R|^2\Big)^{\frac12} \Big(\frac{1}{|\tilde{\Omega}_k|} \sum_{R\subset\tilde{\Omega}_k} |t_R|^2\Big)^{\frac12} \qquad (4.4)\\
&\le \sum_k |\tilde{\Omega}_k|^{\frac12} \Big(\sum_{R\in B_k} |s_R|^2\Big)^{\frac12} \|t\|_{c^1}.
\end{aligned}
$$

Combining the facts that $\int\limits_{\tilde{\Omega}_k\backslash\Omega_{k+1}} \sum_{R\in B_k} \big(|s_R|\tilde{\chi}_R(x)\big)^2 dx \le 2^{2(k+1)}|\tilde{\Omega}_k\backslash\Omega_{k+1}| \le C2^{2k}|\Omega_k|$ and that

$$
\begin{aligned}
\int_{\tilde{\Omega}_k\backslash\Omega_{k+1}} \sum_{R\in B_k} \big(|s_R|\tilde{\chi}_R(x)\big)^2 dx &\ge \sum_{R\in B_k} |s_R|^2|R|^{-1}|\tilde{\Omega}_k\backslash\Omega_{k+1}\bigcap R| \\
&\qquad \textit{since } R\in B_k \textit{ then } R \textit{ is contained in } \tilde{\Omega}_k \\
&\ge \sum_{R\in B_k} |s_R|^2|R|^{-1}\frac12|R| \\
&\ge \frac12 \sum_{R\in B_k} |s_R|^2,
\end{aligned}
$$

we have $\Bigg(\sum\limits_{R\in B_k} |s_R|^2\Bigg)^{\frac12} \lesssim 2^k|\Omega_k|^{\frac12}$. Substituting this back into the last term of (4.4) yields that $|L(s)| \lesssim \|s\|_{s^1}\|t\|_{c^1}$.

Conversely, we need to verify that for any $L \in \big(s^1\big)^{'}$, there exists $t \in c^1$ with $\|t\|_{c^1} \le \|L\|$ such that for all $s \in s^1$, $L(s) = \sum\limits_R s_R\overline{t}_R$. Here we adapt a similar idea given by Frazier and Jawerth in [8] in one-parameter case to our multi-parameter situation.

We define $s^i_R = 1$ when $R = R_i$ and $s^i_R = 0$ for all other R. Then it is easy to see that $\|S^i_R\|_{s^1} = 1$. Now for all $s \in s^1$, $s = \{s_R\} = \sum\limits_i s_{R_i}s^i_{R_i}$, the limit holds in the norm of s^1, where $\{R_i\}_{i\in\mathbb{Z}}$ are denoted by all dyadic rectangles in $\widetilde{M}$. For any $L \in \big(s^1\big)^{'}$, let $\overline{t}_{R_i} = L(s^i)$, then $L(s) = L(\sum\limits_i s_{R_i}s^i) = \sum\limits_i s_{R_i}\overline{t}_{R_i} = \sum\limits_R s_R\overline{t}_R$. Let $t = \{t_R\}$. Then we only need to check that $\|t\|_{c^1} \le \|L\|$.

For any open set $\Omega \subset \widetilde{M}$ with finite measure, let $\bar{\mu}$ be a new measure such that $\bar{\mu}(R) = \dfrac{|R|}{|\Omega|}$ when $R \subset \Omega$, $\bar{\mu}(R) = 0$ when $R \not\subseteq \Omega$. And let $l^2(\bar{\mu})$ be a sequence space such that when $s \in l{-}^2\,(\bar{\mu})$, $\big(\sum\limits_{R\subset\Omega} |s_R|^2\frac{|R|}{|\Omega|}\big)^{1/2} < \infty$. It is easy to see that

$(l^2(\bar{\mu}))' = l^2(\bar{\mu})$. Then,

$$\begin{aligned}\Big\{\frac{1}{|\Omega|}\sum_{R\subset\Omega}|t_R|^2\Big\}^{1/2} &= \big\||R|^{-1/2}|t_R|\big\|_{l^2(\bar{\mu})}\\ &= \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\Big|\sum_{R\subseteq\Omega}(t_R|R|^{-1/2})\cdot\overline{s}_R\cdot\frac{|R|}{|\Omega|}\Big|\\ &\le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\Big|L\Big(\chi_{\{R\subseteq\Omega\}}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|}\Big)\Big|\end{aligned}$$

$$\le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\|L\|\cdot\Big\|\chi_{\{R\subseteq\Omega\}}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|}\Big\|_{s^1}.$$

By (4.1) and the Hölder inequality, we have

$$\Big\|\chi_{\{R\subseteq\Omega\}}(R)\frac{|R|^{1/2}|s_R|}{|\Omega|}\Big\|_{s^1} \le \Big(\sum_{R\subseteq\Omega}|s_R|^2\frac{|R|}{|\Omega|}\Big)^{1/2}.$$

Hence,

$$\|t\|_{c^1} \le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\|L\|\cdot\|s\|_{l^2(\bar{\mu})} \le \|L\|.$$

This completes the proof of Theorem 4.3. ■

5. Duality of $H^1(\widetilde{M})$ with $BMO(\widetilde{M})$

In this section, we prove Theorem 1.1. Let $\widetilde{M} = M\times M$, where M satisfies Assumption 3.1. First, we define the lifting and projection operators as follows.

Definition 5.1. Suppose $\vartheta_i\in(0,1)$ and $0<\beta_i,\gamma_i<\vartheta_i$ for $i=1,2$. For any $f\in(\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$, define the lifting operator S_Q by

$$S_Q(f) = \Big\{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}Q_{k_1}Q_{k_2}[f](x_I,y_J)\Big\}_{k_1,k_2,I,J}, \tag{5.1}$$

where $k_1,k_2\in\mathbb{Z}$, I,J are the same as in Lemma 2.5 and $R=I\times J$, x_I and y_J are the centers of I and J, respectively.

Definition 5.2. For any complex-value sequence $\lambda=\{\lambda_{k_1,k_2,I,J}\}_{k_1,k_2,I,J}$, define the projection operator $T_{\widetilde{Q}}$ by

$$T_{\widetilde{Q}}(\lambda)(x,y)=\sum_{j,k}\sum_{I,J}|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x,x_I,y,y_J)\cdot\lambda_{j,k,I,J}, \tag{5.2}$$

where $\widetilde{q}_{s_1}\widetilde{q}_{s_2}(x,x_I,y,y_J)$ are the same as in Lemma 2.5, and k_1,k_2; I,J; x_I,y_J are the same as in the above definition. Moreover,

$$T_{\widetilde{Q}}\big(S_Q(f)\big)(x,y)=\sum_{k_1,k_2}\sum_{I,J}|I||J|\widetilde{q}_{k_1}\widetilde{q}_{k_2}(x,x_I,y,y_J)Q_{k_1}Q_{k_2}[f](x_I,y_J).$$

For the above lifting and projection operators, we first recall the following result on $H^1(\widetilde{M})$ showed in [11].

Lemma 5.3. ([11]). *For any $f\in H^1(\widetilde{M})$, we have*

$$\|S_Q(f)\|_{s^1}\lesssim\|f\|_{H^1(\widetilde{M})}. \tag{5.3}$$

Conversely, for any $s\in s^1$,

$$\|T_{\widetilde{Q}}(s)\|_{H^1(\widetilde{M})}\lesssim\|s\|_{s^1}. \tag{5.4}$$

Moreover, $T_{\widetilde{Q}}S_Q$ equals the identity on $H^1(\widetilde{M})$.

We now establish a similar result on $BMO(\widetilde{M})$ as follows.

Lemma 5.4. *For any $f\in BMO(\widetilde{M})$, we have*

$$\|S_Q(f)\|_{c^1}\lesssim\|f\|_{BMO(\widetilde{M})}. \tag{5.5}$$

Conversely, for any $t\in c^1$,

$$\|T_Q(t)\|_{BMO(\widetilde{M})}\lesssim\|t\|_{c^1}. \tag{5.6}$$

Moreover, $T_{\widetilde{Q}}S_Q$ equals the identity on $BMO(\widetilde{M})$.

Proof. According to Definition 4.2, 5.1 and 3.2, (5.5) follows directly from the Plancherel-Pôlya-type inequality for $BMO(\widetilde{M})$ (Theorem 3.3).

Now let us prove (5.6). For any $t \in c^1$, by Definition 3.2 and 5.2 and using the same skills as in the estimate of (3.5), we obtain that

$$\begin{aligned}
&\frac{1}{|\Omega|}\sum_{k_1,k_2}\sum_{I\times J\subset\Omega}|I||J|\sup_{u\in I,v\in J}\big|Q_{k_1}Q_{k_2}[T_{\widetilde{Q}}(t)](u,v)\big|^2\\
&\lesssim \frac{1}{|\Omega|}\sum_{k_1,k_2}\sum_{I\times J\subset\Omega}\sum_{k_1',k_2'}\sum_{I',J'}\Big[\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\Big]\Big[\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\Big]\Big[\frac{\mathrm{diam}(I)}{\mathrm{diam}(I')}\wedge\frac{\mathrm{diam}(I')}{\mathrm{diam}(I)}\Big]^{\epsilon_1}\\
&\times\Big[\frac{\mathrm{diam}(J)}{\mathrm{diam}(J')}\wedge\frac{\mathrm{diam}(J')}{\mathrm{diam}(J)}\Big]^{\epsilon_2}\cdot\big(|I|\vee|I'|\big)\big(|J|\vee|J'|\big)\\
&\times\frac{|I|\vee|I'|}{V_{\mathrm{dist}(I,I')}(x_I)+|I|\vee|I'|}\Big(\frac{\mathrm{diam}(I)\vee\mathrm{diam}(I')}{\mathrm{diam}(I)\vee\mathrm{diam}(I')+\mathrm{dist}(I,I')}\Big)^{\gamma_1}\\
&\times\frac{|J|\vee|J'|}{V_{\mathrm{dist}(J,J')}(y_J)+|J|\vee|J'|}\Big(\frac{\mathrm{diam}(J)\vee\mathrm{diam}(J')}{\mathrm{diam}(J)\vee\mathrm{diam}(J')+\mathrm{dist}(J,J')}\Big)^{\gamma_2}\\
&\times\big|t_{k_1',k_2',I',J'}\ \ |I'|^{-\frac12}|J'|^{-\frac12}\big|^2.
\end{aligned}$$

In fact, we now deal with the same estimates as (3.5) with only minor modification that $\inf_{u\in I',v\in J'}\big|Q_{k_1'}Q_{k_2'}[f](u,v)\big|^2$ is replaced by $\big|t_{k_1',k_2',I',J'}\ \ |I'|^{-\frac12}|J'|^{-\frac12}\big|^2$. Thus, following the proof of Theorem 3.3, we can obtain that

$$\|T_{\widetilde{Q}}(t)\|_{BMO(\widetilde{M})}\lesssim\Big(\sup_{\Omega}\frac{1}{|\Omega|}\sum_{k_1,k_2}\sum_{I\times J\subset\Omega}|I||J|\big|t_{k_1,k_2,I,J}\ \ |I|^{-\frac12}|J|^{-\frac12}\big|^2\Big)^{\frac12}\lesssim\|t\|_{c^1}.$$

Finally, we can easily get that from the Calderón reproducing formula $T_{\widetilde{Q}}S_Q$ is the identity operator on $BMO(\widetilde{M})$. The proof of Lemma 5.4 is completed. ■

We now prove the main result, Theorem 1.1.

Proof of Theorem 1.1. First, for any $g\in\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $0<\beta_i,\gamma_i<\vartheta_i$ for $i=1,2$ and $f\in BMO(\widetilde{M})$, from Lemma 2.5, we have

$$<f,g>=\sum_{k_1,k_2}\sum_{I,J}|I||J|\widetilde{Q}_{k_1}\widetilde{Q}_{k_2}[f](x_I,y_J)Q_{k_1}Q_{k_2}[f](x_I,y_J).$$

Here we use $\widetilde{Q}_{k_i}$ to denote the operator whose kernel is $\widetilde{q}_{k_i}(x,y)$. Following the idea of (4.4), we have $|<f,g>|\le C\|S_{\widetilde{Q}}(g)\|_{s^1}\|S_Q(f)\|_{c^1}$, where $S_{\widetilde{Q}}(g)=\big\{|I|^{\frac12}|J|^{\frac12}\widetilde{Q}_{k_1}\widetilde{Q}_{k_2}[g]\ (x_I,y_J)\big\}_{k_1,k_2,I,J}$.

From the Definition 4.1, the Calderón reproducing formula and the Plancherel-Pôlya-type inequality (6.2), we can get that $\|S_{\widetilde{Q}}(g)\|_{s^1}\lesssim\|g\|_{H^1(\widetilde{M})}$. And from

Lemma 5.4, we have $\|S_Q(f)\|_{c^1} \le C\|f\|_{BMO(\widetilde{M})}$. Thus,

$$|<f,g>| \le C\|f\|_{BMO(\widetilde{M})}\|g\|_{H^1(\widetilde{M})}.$$

Since $\overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ is dense in $H^1(\widetilde{M})$, it follows from a standard density argument that $BMO(\widetilde{M}) \subseteq (H^1(\widetilde{M}))'$.

Conversely, suppose $L \in (H^1(\widetilde{M}))'$. Then $L_1 = L \circ T_{\widetilde{Q}} \in (s^1)'$ by Lemma 5.3. So by Theorem 4.3, there exists $t \in c^1$ such that $L_1(s) =< t, s >$ for all $s \in s^1$ and that $\|t\|_{c^1} \approx \|L_1\| \lesssim \|L\|$ since $T_{\widetilde{Q}}$ is bounded. Hence for any $g \in \overset{\circ}{G}_{\vartheta_1,\vartheta_2}(\beta_1,\beta_2;\gamma_1,\gamma_2)$, $L(g) = L(T_{\widetilde{Q}}S_Q(g)) =< t, S_Q(g) >$. From Definition 4.2, we have

$$\begin{aligned}
< t, S_Q(g) > &= \sum_{k_1,k_2}\sum_{I,J}|I|^{\frac12}|J|^{\frac12}Q_{k_1}Q_{k_2}[g](x_I,y_J)\ \cdot t_{k_1,k_2,I,J}\\
&= \int_{\widetilde{M}} \sum_{k_1,k_2}\sum_{I,J}|I|^{\frac12}|J|^{\frac12}q_{k_1}q_{k_2}(x,x_I,y,y_J)t_{k_1,k_2,I,J}\ \cdot\ g(x,y)dxdy\\
&= < T_Q(t), g > .
\end{aligned}$$

By using the Plancherel-Pôlya-type inequality in Theorem 3.3, we can get that $\|T_Q(t)\|_{BMO(\widetilde{M})} \le C\|t\|_{c^1} \le C\|L\|$. By the density argument, we have that for any $g \in H^1(\widetilde{M})$,

$$L(g) =< T_Q(t), g >,$$

which shows that $(H^1(\widetilde{M}))' \subseteq BMO(\widetilde{M})$. ■

6. Product Case of n Factors

In this section, we describe the results on $\widetilde{M} = M_1 \times \cdots \times M_n$, where each M_i satisfies Assumption 3.1, since the method we used on $\widetilde{M} = M \times M$ can be applied for the product case of n factors.

To begin with, we state some necessary results in [11]. Denote by $\overset{\circ}{G}_{\vartheta_1,\cdots,\vartheta_n}(\beta_1,\gamma_1;\cdots;\beta_n,\gamma_n)$ and $(\overset{\circ}{G}_{\vartheta_1,\cdots,\vartheta_n}(\beta_1,\gamma_1;\cdots;\beta_n,\gamma_n))'$ the test function space and its dual space, where $\vartheta_i \in (0,1)$ and $0 < \beta_i,\gamma_i < \vartheta_i$ for $i = 1,2,\cdots,n$. The Littlewood-Paley square function associated to the sequence of operators $\{Q_{k_i}\}_{k_i\in\mathbb{Z}}$ on each M_i is defined by

$$\widetilde{S}(f)(x_1,\cdots,x_n) = \left\{\sum_{k_1}\cdots\sum_{k_n}|Q_{k_1}\cdots Q_{k_n}(f)(x_1,\cdots,x_n)|^2\right\}^{\frac12}.$$

In [11] we can see that $\|\widetilde{S}(f)\|_{L^p(\widetilde{M})} \approx \|f\|_{L^p(\widetilde{M})}$ for $1 < p < \infty$. And the Hardy space $H^1(\widetilde{M})$ is defined as follows.

Definition 6.1. ([11]). Let $0 < \vartheta_i < 1$ and $0 < \beta_i, \gamma_i < \vartheta_i$ for $i = 1, \cdots, n$. The Hardy space $H^1(\widetilde{M})$ is defined to be the set of all $f \in (\overset{\circ}{G}_{\vartheta_1,\cdots,\vartheta_n}(\beta_1, \gamma_1; \cdots; \beta_n, \gamma_n))'$ such that $\|\widetilde{S}[f]\|_{L^1(\widetilde{M})} < \infty$, and we define

$$\|f\|_{H^1(\widetilde{M})} = \|\widetilde{S}[f]\|_{L^1(\widetilde{M})}.$$

Now we give the definition of $BMO(\widetilde{M})$ via the sequence of operators $\{Q_{k_i}\}_{k_i \in \mathbb{Z}}$ on each M_i as follows.

Definition 6.2. Let $0 < \vartheta_i < 1$ and $0 < \beta_i, \gamma_i < \vartheta_i$ for $i = 1, \cdots, n$. We define the space $BMO(\widetilde{M})$ to be the set of all $f \in (\overset{\circ}{G}_{\vartheta_1,\cdots,\vartheta_n}(\beta_1, \gamma_1; \cdots; \beta_n, \gamma_n))'$ such that

$$\begin{aligned} &\|f\|_{BMO(\widetilde{M})} \\ (6.1) \quad &= \sup_{\Omega} \left\{ \frac{1}{|\Omega|} \int_{\Omega} \sum_{k_1,\cdots,k_n} \sum_{I_1 \times \cdots \times I_n \subseteq \Omega} \left| Q_{k_1} \cdots Q_{k_n}[f](x_1, \cdots, x_n) \right|^2 \right. \\ &\quad \left. \times \chi_{I_1}(x_1) \cdots \chi_{I_n}(x_n) dx_1 \cdots dx_n \right\}^{\frac{1}{2}} < \infty, \end{aligned}$$

where the sup is taken over all open sets Ω in $\widetilde{M}$ with finite measure and for each k_i, I_i ranges over all the dyadic cubes in M_i with length $\ell(I_i) = 2^{-k_i - N_0}$ for $i = 1, 2, \cdots, n$.

Following the same routine as in the product case of two factors, we can establish the Plancherel-Pôlya-type inequality for $BMO(\widetilde{M})$.

Theorem 6.3. *Let all the notation be the same as in Definition 6.2. Then for all $f \in BMO(\widetilde{M})$,*

$$\begin{aligned} &\sup_{\Omega} \left\{ \frac{1}{|\Omega|} \int_{\Omega} \sum_{k_1,\cdots,k_n} \sum_{I_1 \times \cdots \times I_n \subseteq \Omega} \sup_{u_1 \in I_1, \cdots, u_n \in I_n} \left| Q_{k_1} \cdots Q_{k_n}[f](u_1, \cdots, u_n) \right|^2 \right. \\ &\quad \left. \times \chi_{I_1}(x_1) \cdots \chi_{I_n}(x_n) dx_1 \cdots dx_n \right\}^{\frac{1}{2}} \\ \approx &\sup_{\Omega} \left\{ \frac{1}{|\Omega|} \int_{\Omega} \sum_{k_1,\cdots,k_n} \sum_{I_1 \times \cdots \times I_n \subseteq \Omega} \inf_{u_1 \in I_1, \cdots, u_n \in I_n} \left| Q_{k_1} \cdots Q_{k_n}[f](u_1, \cdots, u_n) \right|^2 \right. \\ &\quad \left. \times \chi_{I_1}(x_1) \cdots \chi_{I_n}(x_n) dx_1 \cdots dx_n \right\}^{\frac{1}{2}}. \end{aligned}$$

Next, we can extend the result of sequence spaces on product space of 2-factors, namely, Theorem 4.3, Lemma 5.3 and 5.4, to product spaces of n-factors. Then, by working on the level of sequence spaces, we can obtain Theorem 1.1 on product case of n factors. For the detail, we omit it here.

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