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MINIMIZERS AND GAMMA-CONVERGENCE OF ENERGY FUNCTIONALS DERIVED FROM p-LAPLACIAN EQUATION

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Abstract. This paper presents the existence of minimizers and Γ-convergence for the energey functionals

$$E_\epsilon(u) = \int_\Omega \{W(u(x)) + \epsilon|\nabla u(x)|^p\}\, dx, \text{ for all } \epsilon > 0, \quad p > 1$$

with Neumann boundary condition and the constraint

$$\int_\Omega u(x)dx = m|\Omega|, \text{ where } 0 < m < 1.$$

The energy functionals discussed in this paper are associated with the Euler-Lagrange p-Laplacian equation. We employ the direct method in the calculus of variations to show the existence of minimizers. The Γ-convergence is achieved with the help of coarea formula and Young's inequality.

1. INTRODUCTION

In this paper, we study the existence of minimizers and Γ-convergence of the energy functionals derived from the p-Laplacian equation

$$-\Delta_p u + g(x,u) = 0, \quad \text{in} \quad \Omega, \tag{1}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \tag{2}$$

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and the conserved density constraint,

$$\int_\Omega u(x)dx = m|\Omega|, \quad \text{where} \quad 0 < m < 1, \tag{3}$$

on an open bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $p > 1$, and $g(x, u)$ is a Carathéodory function. This problem without the constraint (3) has been studied in [3]. The study of eigenvalues and positive solutions of radial p-Lapacian equation can be found in [1, 5].

There are numerous literature existing on this type of eigenvalue problems when $g(x, u) = \lambda|u|^{p-1}u + h(x)$. In this paper, we consider a special type of a double well potential function $g(x, u) = \frac{1}{\epsilon}W'(u)$ and $W(u) = \frac{1}{4}u^2(1-u)^2$ and its corresponding energy functional is described as

$$E_\epsilon(u) = \int_\Omega (W(u(x)) + \epsilon|\nabla u(x)|^p)\, dx. \tag{4}$$

Equation (4) can be rewritten into

$$E_\epsilon(u) = \int_\Omega f(\epsilon^{1/p}|\nabla u(x)|, u(x))dx \tag{5}$$

where $f(s, u) = s^p + W(u)$ for $p > 1$. For $p = 2$, the method of Γ-convergence was introduced by De Giorgi in the early 1970's and the Γ-convergence of energy functionals in (4) to a perimeter functional was conjectured by De Giorgi and proved by Modica and Mortola [19] in 1977. This convergence implies that global minimizers of (4) converge to global minimizers of the perimeter functional. In 1987, Modica has applied the Γ-convergence theory to solve the minimal interface problem in the Van der Waals-Cahn-Hilliard theory of phase transitions [12]. The mathematical problem is thus to study the asymptotic behavior, as $\epsilon \to 0^+$, of solution u_ϵ to the minimization problem

$$\min\{E_\epsilon(u) : \int_\Omega u(x)dx = m\}.$$

Furthermore, Modica has shown that $\{u_\epsilon\}$ converges to a function u_0 which takes only the values 0 and 1 with interface between the set $\{x : u_0(x) = 0\}$ and $\{x : u_0(x) = 1\}$ having minimal area [18]. There have been numerous fantastic works on the subject of phase transitions by using De Giorgi's notion of Γ-convergence. For instance, R.V. Kohn and P. Sternberg [15], N. C. Owen [20], I. Fonseca and L. Tartar [11], P. Sternberg [22], S. Conti, I. Fonseca and G. Leoni [6], S. Baldo [2], W. Jin and R.V. Kohn [14], and E. Sandier and S. Serfaty [21].

In this paper we establish the existence of minimizers and Γ-convergence of the functionals $\{E_\epsilon\}_{\epsilon>0}$ defined in (4) with the constraint (3) for all $p > 1$. The technique for the proof of the existence of minimizers is based on the direct method [7, 16] involved the general theory on Sobolev space $\mathcal{W}^{1,p}(\Omega)$. We show that functions defined in (4) is weakly lower semicontinuous and coercive over $\mathcal{W}^{1,p}(\Omega)$. With the constraint (3), the existence of minimizers on a subset of $\mathcal{W}^{1,p}(\Omega)$ is considered and the general theory can not be directly applied. Therefore, we combine the two methods to show the existence.

For the case of a general type of function f is considered by Owen [20] and the result cannot be applied to $f(s,u) = s^p + W(u)$ for each $p > 1$ since the condition (H3) in the paper: $f(s,u) \geq cs^2$ with $c > 0$. Our result holds for the case of the special function $f(s,u)$ and $p > 1$ and Owen's is for the general function $f(s,u)$ but $p = 2$. With the help of the special function $f(s,u)$, our proof takes advantage of Young's inequality. For a general function $f(s,u)$, Owen should impose certain conditions in order to control the behavior of the function f. We conclude that the methods of Owen's and ours are different but the both results are the same for some cases. More precisely, Owen establishes the same result as ours whenever $f(s,u) = s^p + W(u)$ and $p = 2$. Our work combined with Owen's technique (by constructing a scalar function) can be expected to extend to more general problems.

We would like to point out the Γ-convergence for the functionals (considered here) which are associated with the Euler-Lagrange p-Laplacian equation. The kinematic energy has been modified by $p > 1$ and the potential and kinematic energy have more sensitive interaction and balance between themselves. In other words, the relation and structure of these two energies can be judged by Young's inequality.

2. The Existence of Minimizers

We will show the existence of minimizers of the functional (4) with the constraint (3) on A_m,

$$A_m \equiv \left\{ v \in \mathcal{W}^{1,p}(\Omega) : \int_\Omega v dx = m|\Omega| \right\}.$$

We first show the functionals in (4) have the properties of weakly lower semicontinuous and coercivity over $\mathcal{W}^{1,p}(\Omega)$.

Definition 2.1. Let X be a Banach space and $I : X \to \mathbb{R} \cup \{+\infty\}$ be a functional. I is weakly lower semicontinuous over X if

$$\liminf_{n\to\infty} I(u_n) \geq I(u) \textit{ whenever } u_n \rightharpoonup u \textit{ in } X$$

Definition 2.2. Let $(X, \|\cdot\|_X)$ be a Banach space and $I : X \to \mathbb{R} \cup \{+\infty\}$ be a functional. If there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$I(u) \geq \alpha \|u\|_X + \beta$$

then I is coercive over X.

We assume that

$$u_n \rightharpoonup u \text{ in } \mathcal{W}^{1,p}(\Omega), \tag{6}$$

that is,

$$u_n \rightharpoonup u \text{ in } L^p(\Omega) \tag{7}$$

and

$$\nabla u_n \rightharpoonup \nabla u \text{ in } L^p(\Omega : \mathbb{R}^n). \tag{8}$$

By the continuity of potential function W and Fatou's lemma yield,

$$\begin{aligned} E_\epsilon(u) &= \int_\Omega W(u(x)) + \epsilon |\nabla u(x)|^p dx \\ &\leq \int_\Omega \liminf_{n\to\infty} \left(W(u_n(x)) + \epsilon |\nabla u_n(x)|^p \right) dx \\ &\leq \liminf_{n\to\infty} \int_\Omega W(u_n(x)) + \epsilon |\nabla u_n(x)|^p dx \\ &= \liminf_{n\to\infty} E_\epsilon(u_n). \end{aligned}$$

The above argument leads to

Lemma 2.3. *The functional E_ϵ defined in (4) is weakly lower semicontinuous over $\mathcal{W}^{1,p}(\Omega)$.*

Next, we are going to show the lemma below.

Lemma 2.4. *The functionals E_ϵ defined in (4) are coercive over $\mathcal{W}^{1,p}(\Omega)$.*

Proof. By the Poincaré inequality [9], there is a constant $C > 0$ only depending on n, p and Ω such that

$$C^p \int_\Omega |\nabla u(x)|^p dx \geq \int_\Omega |u(x) - m|^p dx = \|u - m\|^p_{L^p(\Omega)} \geq \left| \|u\|_{L^p(\Omega)} - \|m\|_{L^p(\Omega)} \right|^p$$

The last equality is by Minkowski's inequality. We therefore deduce that, $\tilde{C}$ denoting a generic constant independent of u,

$$\|u\|^p_{\mathcal{W}^{1,p}(\Omega)} \leq \tilde{C}(1 + \| \, |\nabla u| \, \|^p_{L^p(\Omega)}).$$

Since

$$E_\epsilon(u) \geq \epsilon \int_\Omega |\nabla u(x)|^p dx = \epsilon \| \, |\nabla u| \, \|^p_{L^p(\Omega)},$$

we get

$$E_\epsilon(u) \geq \alpha \|u\|^p_{\mathcal{W}^{1,p}(\Omega)} + \beta,$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. Lemma 2.4 has been shown. ∎

Let us define the set S_m by

$$S_m = \{E_\epsilon(u) \mid u \in A_m\}.$$

Since the set S_m is bounded below by 0, the infimum of the set S_m exists and there is a sequence $\{u_k\}_{k=1}^\infty$ in A_m such that

$$\lim_{k\to\infty} E_\epsilon(u_k) = \inf S_m < +\infty. \tag{9}$$

We call $\{u_k\}_{k=1}^\infty$ a minimizing sequence for E_ϵ on A_m. By the coercivity of E_ϵ over $\mathcal{W}^{1,p}(\Omega)$, Lemma 2.4 and (9), $\{u_k\}_{k=1}^\infty$ is a bounded sequence in $\mathcal{W}^{1,p}(\Omega)$. Since $\mathcal{W}^{1,p}(\Omega)$ is compactly contained in $L^p(\Omega)$, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ of bounded sequence $\{u_k\}_{k=1}^\infty$ such that

$$u_{k_j} \to u \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad j \to \infty, \tag{10}$$

for some $u \in L^p(\Omega)$. Moreover, there is a subsequence $\{u_{k_{j_\ell}}\}_{\ell=1}^\infty$ of $\{u_{k_j}\}_{j=1}^\infty$ such that

$$u_{k_{j_\ell}} \to u \text{ a.e. in} \quad \Omega \quad \text{as} \quad \ell \to \infty. \tag{11}$$

We denote $\{u_{k_{j_\ell}}\}$ by $\{u_{k_j}\}$ for convenience. Note that for $p > 1$, $\mathcal{W}^{1,p}(\Omega)$ is reflexive and uniformly convex. For bounded sequence $\{u_{k_j}\}_{j=1}^\infty$ in $\mathcal{W}^{1,p}(\Omega)$, there is a subsequence $\{u_{k_{j_i}}\}_{i=1}^\infty$ and a function $\tilde{u} \in \mathcal{W}^{1,p}(\Omega)$ such that

$$u_{k_{j_i}} \rightharpoonup \tilde{u} \quad \text{in} \quad \mathcal{W}^{1,p}(\Omega) \quad \text{as} \quad i \to \infty. \tag{12}$$

Thus, $u = \tilde{u}$ a.e. in Ω which gives us that $u \in \mathcal{W}^{1,p}(\Omega)$, and

$$u_{k_{j_i}} \to u \quad \text{in} \quad L^p(\Omega) \quad \text{as} \quad i \to \infty, \tag{13}$$

$$u_{k_{j_i}} \rightharpoonup u \quad \text{in} \quad \mathcal{W}^{1,p}(\Omega) \quad \text{as} \quad i \to \infty. \tag{14}$$

By (13), we have

$$\begin{aligned}\left|\int_\Omega u(x)dx - m|\Omega|\right| &= \left|\int_\Omega (u(x) - u_{k_{j_i}}(x))dx\right| \\ &\le \int_\Omega |u(x) - u_{k_{j_i}}(x)|dx \\ &\le \|u - u_{k_{j_i}}\|_{L^p(\Omega)} \cdot \|1\|_{L^q(\Omega)},\end{aligned}$$

for each $i \in \mathcal{N}$, so that

$$\int_\Omega u(x)dx = m|\Omega|, \quad \text{and} \quad u \in A_m. \tag{15}$$

By (14), (15), Lemma 2.3 and Equation (9), we have established the existence of minimizers of E_ϵ over A_m.

Theorem 2.5. *Each functional E_ϵ over the space A_m has a minimizer, that is, there exists a $u_\epsilon \in A_m$ such that*

$$E_\epsilon(u_\epsilon) = \min\{E_\epsilon(v) : v \in A_m\}.$$

3. Γ-Convergence

We say that the functional E_0 is the $\Gamma(L^1(\Omega))$-limit of $\{E_\epsilon\}_{\epsilon>0}$ (denoted by $E_0 = \Gamma(L^1(\Omega))$-limit $\{E_\epsilon\}_{\epsilon>0}$) means that for each $u \in L^1(\Omega)$, we have

(i) If $u_\epsilon \to u$ in $L^1(\Omega)$, then $E_0(u) \le \liminf_{\epsilon\to 0^+} E_\epsilon(u_\epsilon)$.

(ii) There exists a family $\{v_\epsilon\}_{\epsilon>0}$ in $L^1(\Omega)$ such that $v_\epsilon \to u$ in $L^1(\Omega)$ and $\limsup_{\epsilon\to 0^+} E_\epsilon(v_\epsilon) \le E_0(u)$.

The references for the theory of Γ-convergence can be found in [4, 8].

For the reason of convenience, we use the following variational form

$$J_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{\epsilon}\frac{W(u(x))}{q} + \epsilon^{p-1}\frac{|\nabla u(x)|^p}{p}\right) dx, & \text{if } u \in \mathcal{W}^{1,p}(\Omega) \\ +\infty, & \text{otherwise.} \end{cases} \tag{16}$$

and define

$$J_0(u) = \begin{cases} \displaystyle\left(\int_0^1 W^{1/q}(t)dt\right) \mathrm{Per}_\Omega(A), & \text{if } W(u) = 0, \text{ a.e.}, \\ & \Phi \circ u \in \mathrm{BV}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases} \tag{17}$$

where $\frac{1}{q}+\frac{1}{p}=1$, $W(u)=\frac{1}{4}u^2(1-u)^2$ and $\mathrm{Per}_\Omega(A)$ is the perimeter of a set A in Ω,

$$A=\{x\in\Omega : u(x)=1\}, \tag{18}$$

and

$$\Phi(t)=\int_0^t W^{1/q}(s)ds \quad \text{on} \quad [0,1]. \tag{19}$$

We extend the function $\Phi(t)$ into $\mathbb{R}$ by

$$\Phi(t)=\Phi(1), \quad t>1, \quad \text{and} \quad \Phi(t)=0, \quad t<0.$$

Lemma 3.1. *If the functionals J_ϵ with $\liminf_{\epsilon\to 0^+} J_\epsilon(u_\epsilon)<\infty$ and $u_\epsilon\to u$ in $L^1(\Omega)$ then the values of the function u belong to $\{0,1\}$ almost everywhere, that is, $u(x)\in\{0,1\}$ a.e. Furthermore, the function $u(x)=\chi_A(x)$, where the set A is defined in (18).*

Proof. Since $u_\epsilon\to u$ in $L^1(\Omega)$, there is a subsequence $\{u_{\epsilon_j}\}$ such that $u_{\epsilon_j}\to u$ almost everywhere. It follows that $W\circ u_{\epsilon_j}\to W\circ u$ almost everywhere. By Fatou's lemma, we have

$$\begin{aligned}0\le\int_\Omega W(u(x))dx &= \int_\Omega \lim_{j\to\infty} W(u_{\epsilon_j}(x))dx\\ &\le \liminf_{j\to\infty}\int_\Omega W(u_{\epsilon_j}(x))dx\\ &\le \liminf_{j\to\infty}\epsilon_j q J_{\epsilon_j}(u_{\epsilon_j})=0.\end{aligned}$$

The last equality holds due to the assumption. The integral of the potential function W over Ω is vanished. That implies $u(x)\in\{0,1\}$ almost everywhere and this lemma has been proved. ∎

One can find the co-area formula in

Lemma 3.2. *(cf: [10, 17], Coarea formula for BV-functions) Let $U\subset\mathbb{R}^n$ be open.*

(**I**) *Suppose that $f\in BV(U)$. Define $E_t\equiv\{x\in U|f(x)>t\}$ for each $t\in\mathbb{R}$. Then*

(i) *$\mathrm{Per}_U(E_t)<+\infty$ for L^1 a.e. $t\in\mathbb{R}$.*

(ii) *The total variation of f on U, $|\nabla f|(U)(=\int_U|\nabla f(x)|dx)=\int_{-\infty}^{\infty}\mathrm{Per}_U(E_t)dt$.*

(**II**) *If* $f \in L^1(U)$ *and* $\int_{-\infty}^{\infty} Per_U(E_t)dt < +\infty$, *then* $f \in BV(U)$.

Lemma 3.3. *If the function* Φ *is defined in (19) and* $u(x) = \chi_A(x)$ *with a set* A *of finite perimeter in* Ω, *then*

$$J_0(u) = \int_{-\infty}^{\infty} Per_\Omega(\{x \in \Omega : (\Phi \circ u)(x) \leq t\})dt. \tag{20}$$

Proof. Let $S_t \equiv \{x \in \Omega | (\Phi \circ u)(x) \leq t\}$. By Lemma 3.2, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{Per}_\Omega(S_t)dt &= \int_{-\infty}^{\infty} \mathrm{Per}_\Omega(\{x \in \Omega : \Phi(1)\chi_A(x) \leq t\})dt \\
&= \int_0^{\Phi(1)} \mathrm{Per}_\Omega(\{x \in \Omega : \Phi(1)\chi_A(x) \leq t\})dt \\
&= \mathrm{Per}_\Omega(\Omega - A)(\Phi(1) - 0) = \Phi(1)\mathrm{Per}_\Omega(A) \\
&= \left(\int_0^1 W^{1/q}(t)dt\right) \mathrm{Per}_\Omega(A) \\
&= J_0(u).
\end{aligned}$$

■

Lemma 3.4. *Suppose* $v \in \mathcal{W}^{1,p}(\Omega)$ *and the function* Φ *is defined in (19). Then* $\Phi \circ v \in \mathcal{W}^{1,p}(\Omega)$ *and* $\Phi \circ v \in BV(\Omega)$.

Proof. Since

$$\begin{aligned}
\|\Phi \circ v\|^p_{L^p(\Omega)} &= \int_\Omega |\Phi(v(x))|^p dx \\
&= \int_\Omega |\Phi(v(x)) - \Phi(0)|^p dx \\
&\leq \int_\Omega (\mathrm{Lip}(\Phi))^p |v(x) - 0|^p dx \\
&= (\mathrm{Lip}(\Phi))^p \|v\|^p_{L^p(\Omega)} < +\infty,
\end{aligned}$$

where $\mathrm{Lip}(\Phi)$ is the Lipschitz constant of the function Φ. Next,

$$\begin{aligned}
\|\frac{\partial}{\partial x_i}(\Phi \circ v)\|^p_{L^p(\Omega)} &= \int_\Omega |\frac{\partial}{\partial x_i}(\Phi(v(x)))|^p dx \\
&= \int_\Omega |\Phi'(v(x))\frac{\partial v(x)}{\partial x_i}|^p dx
\end{aligned}$$

$$\leq \int_\Omega \left(\max_{t\in[0,1]} W^{1/q}(t)\right)^p |\frac{\partial v(x)}{\partial x_i}|^p dx$$
$$\leq \left(\max_{t\in[0,1]} W^{1/q}(t)\right)^p \|\frac{\partial v}{\partial x_i}\|^p_{L^p(\Omega)} < +\infty.$$

Therefore, $\frac{\partial}{\partial x_i}(\Phi \circ v) \in L^p(\Omega)$ for $i = 1, 2, \ldots, n$. That implies $\Phi \circ v \in \mathcal{W}^{1,p}(\Omega)$. Since Ω is open, bounded with $\partial\Omega$ Lipschitz, in addition, by the fact $W^{1,1}(\Omega) \subset$ BV(Ω), so $\Phi \circ v \in$ BV(Ω). We obtain Lemma 3.4. ■

Lemma 3.5. (cf: [10, 17], *Lower semicontinuous of variation measure.) Suppose that $U \subset \mathbb{R}^n$ is open. Suppose $f_k \in BV(U)$ for each $k \in N$ and $f_k \to f$ in $L^1(U)$. Then*

$$\int_U |\nabla f(x)|dx \leq \liminf_{k\to\infty} \int_U |\nabla f_k(x)|dx.$$

Theorem 3.6. *The functionals J_ϵ and J_0 are defined in (16) and (17). If $u_\epsilon \to u$ in $L^1(\Omega)$ then $J_0(u) \leq \liminf_{\epsilon\to 0^+} J_\epsilon(u_\epsilon)$.*

Proof. Since $u_\epsilon \in \mathcal{W}^{1,p}(\Omega)$ and Φ is Lipschitz on $\mathbb{R}$, $\Phi \circ u_\epsilon \to \Phi \circ u$ in $L^1(\Omega)$ whenever $u_\epsilon \to u$ in $L^1(\Omega)$. Since

$$\begin{aligned} J_\epsilon(u_\epsilon) &= \int_\Omega \left(\frac{1}{\epsilon}\frac{W(u_\epsilon(x))}{q} + \epsilon^{p-1}\frac{|\nabla u_\epsilon(x)|^p}{p}\right) dx \\ &\geq \int_\Omega W^{1/q}(u_\epsilon(x))|\nabla u_\epsilon(x)|dx \quad \text{(by Young's inequality)} \\ &\geq \int_\Omega |\nabla(\Phi \circ u_\epsilon)(x)|\, dx, \end{aligned}$$

that last inequality is by

$$\begin{aligned} |\nabla(\Phi \circ u_\epsilon)(x)| &= |\Phi'(u_\epsilon(x))|\, |\nabla u_\epsilon(x)| \\ &= \begin{cases} 0, & \text{if} \quad u_\epsilon(x) \geq 1 \text{ or } u_\epsilon(x) \leq 0, \\ W^{1/q}(u_\epsilon(x))|\nabla u_\epsilon(x)|, & \text{if} \quad u_\epsilon(x) \in [0,1]. \end{cases} \end{aligned}$$

By Lemma 3.4 and Lemma 3.5, we have

$$\int_\Omega |\nabla(\Phi \circ u)(x)|dx \leq \liminf_{\epsilon\to 0^+} \int_\Omega |\nabla(\Phi \circ u_\epsilon)(x)|dx \leq \liminf_{\epsilon\to 0^+} J_\epsilon(u_\epsilon).$$

We obtain the inequality (by Lemma 3.1, Lemma 3.2, and Lemma 3.3)

$$J_0(u) = \int_\Omega |\nabla(\Phi \circ u)(x)|dx \le \liminf_{\epsilon \to 0^+} J_\epsilon(u_\epsilon), \tag{21}$$

provided by the limiting infimum is finite, $\liminf_{\epsilon \to 0^+} J_\epsilon(u_\epsilon) < \infty$. If the limiting infimum is infinite, $\liminf_{\epsilon \to 0^+} J_\epsilon(u_\epsilon) = \infty$, then the equation

$$J_0(u) \le \liminf_{\epsilon \to 0^+} J_\epsilon(u_\epsilon)$$

is trivial. ■

Lemma 3.7. (cf [10, 17], Coarea formula). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a measurable function and that $h : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function. Then*

$$\int_\Omega f(h(x))|\nabla h(x)|dx = \int_{-\infty}^{\infty} f(t)H^{n-1}(\{x \in \Omega | h(x) = t\})dt,$$

for each measurable subset Ω of $\mathbb{R}^n$. In particular, we have

$$\int_\Omega |\nabla h(x)|dx = \int_{-\infty}^{\infty} H^{n-1}(\{x \in \Omega | h(x) = t\})dt,$$

for each measurable subset Ω of $\mathbb{R}^n$.

Lemma 3.8. ([2, 18]). *Let $\Omega \subset \mathbb{R}^n$ be open, and A be a polygonal domain in $\mathbb{R}^n$ with the compact boundary ∂A and $H^{n-1}(\partial A \cap \partial\Omega) = 0$. Then the following two things hold*

- (i) *there exists a constant $\eta > 0$ such that the function $h(x)$ defined in (30) is Lipshitz continuous on $D_\eta \equiv \{x \in \mathbb{R}^n | \, |h(x)| < \eta\}$ and $|\nabla h(x)| = 1$, a.e. on D_η.*
- (ii) *If $S_t \equiv \{x \in \mathbb{R}^n | \, h(x) = t\}$, then $\lim_{t \to 0} H^{n-1}(S_t \cap \Omega) = H^{n-1}(\partial A \cap \Omega)$.*

Remark. By Lemma 3.1 in [2] and Lemma 3.8, it permits us only require to prove Theorem 3.9 for polygonal domains.

Theorem 3.9. *Let the functionals J_ϵ and J_0 be defined in (16) and (17), respectively. There exists a family $\{v_\epsilon\}_{\epsilon>0}$ in $L^1(\Omega)$ such that $v_\epsilon \to u$ in $L^1(\Omega)$ and $\limsup_{\epsilon \to 0^+} J_\epsilon(v_\epsilon) \le J_0(u)$.*

Proof. The Euler-Lagrange equation for J_ϵ defined in (16) is

$$\frac{W'(u(x))}{q} - \epsilon^p \text{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0, \quad \text{for all } x \in \Omega. \tag{22}$$

For $n = 1$, (22) becomes

$$\frac{W'(u(t))}{q} - \epsilon^p (p-1)|u'(t)|^{p-2}u''(t) = 0, \quad \text{for all } t \in I. \tag{23}$$

Let $f(t) = |t|^{p-2}t$ on $\mathbb{R}$ and $F(t) = \int_0^t f(s)ds$. Multiplying $u'(t)$ to (23), we can rewrite it into

$$\frac{d}{dt}\left(\frac{W(u(t))}{q} - \epsilon^p (p-1)F(u'(t))\right) = 0, \quad \text{for all } t \in I. \tag{24}$$

That implies there is a positive constant $C_{\epsilon,p}$ which depends on ϵ and p and will be chosen latter such that

$$\frac{W(u(t))}{q} - \epsilon^p (p-1)F(u'(t)) = -C_{\epsilon,p}, \quad \text{for all } t \in I. \tag{25}$$

Since $F(t) = \displaystyle\int_0^t s^{p-1}ds = t^p/p$ is positive and strictly increasing on $[0,\infty)$, the function $F : [0,\infty) \to [0,\infty)$ has an inverse F^{-1}. By (25),

$$u'(t) = F^{-1}\left(\frac{1}{\epsilon^p(p-1)}\left(\frac{W(u(t))}{q} + C_{\epsilon,p}\right)\right). \tag{26}$$

The inverse of the function F is $F^{-1}(t) = (pt)^{1/p}$. Hence (26) can be simplified expressed as

$$u'(t) = \frac{1}{\epsilon}\left(W(u(t)) + qC_{\epsilon,p}\right)^{1/p}. \tag{27}$$

Therefore, u is a solution of (23) and is also one of the ordinary differential equation (27) [13].

Let us define

$$\Psi_\epsilon(t) \equiv \epsilon \int_0^t \left(\frac{1}{W(s) + qC_{\epsilon,p}}\right)^{1/p} ds, \quad t \in [0,1],$$

and $\eta_\epsilon \equiv \Psi_\epsilon(1)$. We have $\Psi_\epsilon(0) = 0$ and $\Psi'_\epsilon(t) = \epsilon(\frac{1}{W(t)+qC_{\epsilon,p}})^{1/p} > 0$ for all $t \in (0,1)$. It implies Ψ_ϵ is strictly increasing on $[0,1]$ and its inverse Ψ_ϵ^{-1} : $[0,\eta_\epsilon] \to [0,1]$ exists. The derivative of the inverse function is

$$\frac{d}{dt}(\Psi_\epsilon^{-1}(t)) = \frac{1}{\epsilon}\left(W(\Psi_\epsilon^{-1}(t)) + qC_{\epsilon,p}\right)^{1/p}, \quad \text{for all } t \in (0,\eta_\epsilon).$$

So far, the function $\Psi_\epsilon^{-1}(t)$ satisfies (27). We extend $\Psi_\epsilon^{-1} : [0, \eta_\epsilon] \to [0, 1]$ to $\widetilde{\Psi_\epsilon^{-1}} : \mathbb{R} \to [0, 1]$ by

$$\widetilde{\Psi_\epsilon^{-1}}(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ \Psi_\epsilon^{-1}(t), & \text{if} \quad t \in [0, \eta_\epsilon], \\ 1, & \text{if} \quad t > \eta_\epsilon. \end{cases} \tag{28}$$

For each $\epsilon > 0$, since Ψ_ϵ^{-1} is increasing and (28), we have

$$\widetilde{\Psi_\epsilon^{-1}}(t) \le \chi_0(t) \le \widetilde{\Psi_\epsilon^{-1}}(t + \eta_\epsilon), \qquad \text{for all } t \in \mathbb{R}, \tag{29}$$

where χ_0 is the Heaviside function given by

$$\chi_0(t) \equiv \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \ge 0. \end{cases}$$

Let

$$h(x) \equiv \begin{cases} -d(x, \partial A), & \text{if } x \notin A, \\ \;\; d(x, \partial A), & \text{if } x \in A, \end{cases} \tag{30}$$

where A defined in (18) and d is the Euclidean metric function. If we take $t = h(x)$ in (29) and also take integrations over the region Ω, then we get

$$\int_\Omega \widetilde{\Psi_\epsilon^{-1}}(h(x))dx \le \int_\Omega \chi_0(h(x))dx \le \int_\Omega \widetilde{\Psi_\epsilon^{-1}}(h(x) + \eta_\epsilon)dx. \tag{31}$$

For simplification of notations, we define $H_\epsilon : [0, \eta_\epsilon] \to \mathbb{R}$ by

$$H_\epsilon(t) \equiv \int_\Omega \widetilde{\Psi_\epsilon^{-1}}(h(x) + t)dx, \qquad \text{for all } t \in [0, \eta_\epsilon],$$

and (31) can be rewritten as

$$H_\epsilon(0) \le \int_\Omega \chi_0(h(x))dx \le H_\epsilon(\eta_\epsilon). \tag{32}$$

Since $\widetilde{\Psi_\epsilon^{-1}}(h(x) + t)$ is continuous for $x \in \Omega$ and $t \in [0, \eta_\epsilon]$, it follows that H_ϵ is continuous on $[0, \eta_\epsilon]$. Applying the Intermediate Value Theorem and for each $\epsilon > 0$, (32) implies that there is $\delta_\epsilon \in [0, \eta_\epsilon]$ such that

$$H_\epsilon(\delta_\epsilon) = \int_\Omega \chi_0(h(x))dx.$$

We observe that $u(x) = \chi_A(x) = \chi_0(h(x))$ for all $x \in \Omega$. Then

$$\int_\Omega \widetilde{\Psi_\epsilon^{-1}}(h(x) + \delta_\epsilon)dx = H_\epsilon(\delta_\epsilon) = \int_\Omega \chi_0(h(x))dx = \int_\Omega u(x)dx. \tag{33}$$

We denote $\chi_\epsilon(t) \equiv \widetilde{\Psi_\epsilon^{-1}}(t + \delta_\epsilon)$ for all $t \in \mathcal{R}$ and define function $u_\epsilon : \Omega \to \mathcal{R}$ by

$$u_\epsilon(x) \equiv \chi_\epsilon(h(x)), \quad \text{for all } x \in \Omega, \tag{34}$$

and by (33), it gives us

$$\int_\Omega u_\epsilon(x)dx = \int_\Omega u(x)dx. \tag{35}$$

Let

$$\Omega_{\delta_\epsilon} = \{x \in \Omega : -\delta_\epsilon \le h(x) \le \eta_\epsilon - \delta_\epsilon\}$$

and u_ϵ is constructed in (34), we have the following estimation

$$\begin{aligned}
&\int_\Omega |u_\epsilon(x) - u(x)|dx \\
= &\int_\Omega |\chi_\epsilon(h(x)) - \chi_0(h(x))|dx \\
= &\int_{\Omega_{\delta_\epsilon}} |\chi_\epsilon(h(x)) - \chi_0(h(x))|dx \\
= &\int_{\Omega_{\delta_\epsilon}} |\chi_\epsilon(h(x)) - \chi_0(h(x))|\,|\nabla h(x)|dx \quad \text{(By Lemma 3.7, 3.8)} \\
= &\int_{-\delta_\epsilon}^{\eta_\epsilon - \delta_\epsilon} |\chi_\epsilon(t) - \chi_0(t)| H^{n-1}(\{x \in \Omega |\, h(x) = t\})dt \quad \text{(By Lemma 3.7)} \\
\le &\ \eta_\epsilon \sup_{|t| \le \eta_\epsilon} H^{n-1}(\{x \in \Omega |\, h(x) = t\}) \\
\le &\ \frac{\epsilon}{(qC_{\epsilon,p})^{1/p}} \gamma_\epsilon,
\end{aligned}$$

where $\gamma_\epsilon = \sup_{|t| \le \eta_\epsilon} H^{n-1}(\{x \in \Omega |\, h(x) = t\})$ with

$$\lim_{\epsilon \to 0^+} \gamma_\epsilon = H^{n-1}(\partial A \cap \Omega) = \text{Per}_\Omega(A). \quad \text{(By Lemma 3.8)} \tag{36}$$

Once, we choose $C_{\epsilon,p} = \epsilon^{p/2}$ then

$$\lim_{\epsilon \to 0^+} \frac{\epsilon}{(qC_{\epsilon,p})^{1/p}} \gamma_\epsilon = \lim_{\epsilon \to 0^+} \sqrt{\epsilon} q^{-1/p} \gamma_\epsilon = 0. \tag{37}$$

It follows

$$u_\epsilon \to u \text{ in } L^1(\Omega) \quad \text{as } \epsilon \to 0^+. \tag{38}$$

The remaining work is to show

$$\limsup_{\epsilon\to 0} J_\epsilon(u_\epsilon) \le J_0(u). \tag{39}$$

By (34), we calculate

$$|\nabla u_\epsilon(x)|^p = \left[\sum_{i=1}^{n}(\frac{\partial}{\partial x_i}(u_\epsilon(x)))^2\right]^{p/2} = (\chi'_\epsilon(h(x)))^p|\nabla h(x)|^p \tag{40}$$

and substitute it into (16). Then

$$J_\epsilon(u_\epsilon) \equiv \int_\Omega \left(\frac{1}{\epsilon}\frac{W(u_\epsilon(x))}{q} + \epsilon^{p-1}\frac{(\chi'_\epsilon(h(x)))^p|\nabla h(x)|^p}{p}\right)dx$$

Let

$$\phi_\epsilon(t) \equiv \frac{1}{\epsilon}\frac{W(\chi_\epsilon(t))}{q} + \epsilon^{p-1}\frac{(\chi'_\epsilon(t))^p}{p},$$

and we compute

$$\begin{aligned}
J_\epsilon(u_\epsilon) &= \int_{\Omega_{\delta_\epsilon}} \phi_\epsilon(h(x))|\nabla h(x)|dx \quad \text{(By Lemma 3.7 and 3.8)}\\
&= \int_{-\delta_\epsilon}^{\eta_\epsilon-\delta_\epsilon} \phi_\epsilon(t)H^{n-1}(\{x\in\Omega_{\delta_\epsilon}|\ h(x)=t\})dt\\
&\le \gamma_\epsilon \int_{-\delta_\epsilon}^{\eta_\epsilon-\delta_\epsilon} \phi_\epsilon(t)dt\\
&= \gamma_\epsilon \int_0^{\eta_\epsilon}\left(\frac{1}{\epsilon}\frac{W(\Psi_\epsilon^{-1}(t))}{q} + \epsilon^{p-1}\frac{((\Psi_\epsilon^{-1})'(t))^p}{p}\right)dt\\
&\le \gamma_\epsilon \int_0^{\eta_\epsilon}\left(\frac{1}{\epsilon}\frac{W(\Psi_\epsilon^{-1}(t))}{q} + \frac{C_{\epsilon,p}}{\epsilon} + \epsilon^{p-1}\frac{((\Psi_\epsilon^{-1})'(t))^p}{p}\right)dt\\
&= \gamma_\epsilon \int_0^{\eta_\epsilon}\left(\frac{1}{\epsilon}(W(\Psi_\epsilon^{-1}(t)) + qC_{\epsilon,p})\right)^{1/q}\left(\epsilon^{1/q}(\Psi_\epsilon^{-1})'(t)\right)dt\\
&= \gamma_\epsilon \int_0^1 (W(t)+qC_{\epsilon,p})^{1/q}\,dt.
\end{aligned}$$

The last second equality follows from the Young's inequality with equality holds and Ψ_ϵ^{-1} is a solution of Equation (27). The last equality is by the change of variables formula. Since (36) and $\lim_{\epsilon\to 0^+} C_{\epsilon,p} = 0$, it implies

$$\begin{aligned}\limsup_{\epsilon\to0^+} J_\epsilon(u_\epsilon) &\le \limsup_{\epsilon\to0^+}\left[\gamma_\epsilon\int_0^1 (W(t)+qC_{\epsilon,p})^{1/q}\,dt\right]\\ &= \left(\int_0^1 W^{1/q}(t)dt\right)\mathrm{Per}_\Omega(A) = J_0(u),\end{aligned}$$

where the last equality followed by (17) for $u=\chi_A$ where A is a polygonal domain. If $u\in L^1(\Omega)$ and u is not of the form $u=\chi_A$ with a set A of finite perimeter in Ω, then $J_0(u)=\infty$ and $\limsup\limits_{\epsilon\to0^+} J_\epsilon(u_\epsilon)\le J_0(u)$ are trivial. ∎

Finally, by Theorem 3.6 and 3.9, the Γ-convergence has been achieved.

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