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MODULES WITH C^*-CONDITION

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Abstract. In this paper we investigate modules with the property $C^*:$ namely, M has C^* if every submodule N of M contains a direct summand K of M such that N/K is cosingular. We prove that every right R-module M satisfies C^* if and only if every right R-module is the direct sum of an injective module and a cosingular module.

1. INTRODUCTION

Through this paper R is an associative ring with identity and all modules are unitary right R-modules. A submodule N of a module M is called *small* in M (denoted by $Nl-/M$) if for every proper submodule L of M, $N + L \neq M$. M is called a small module if M is small in some modules [1]. In [1] Leonard has proved that a module M is a small module if and only if it is small in its injective hull. For a module M let $\overline{Z}(M) = Rej(M, \mathsf{S}) = \bigcap\{Kerf \mid f : M \to S, S \in \mathsf{S}\} = \bigcap\{U \subseteq M \mid M/U \in \mathsf{S}\}$ where S denotes the class of all small modules. M is called *cosingular* if $\overline{Z}(M) = 0$. It is obvious that every small module is cosingular but in general the converse is not true. A module M has C^* if every submodule N of M contains a direct summand K of M such that N/K is cosingular. Finally we recall that a module M is lifting if every submodule N of M contains a direct summand K of M such that $N/Kl-/M/K$.

2. MODULES WITH C^*

Lemma 2.1. *Suppose that A, B and $A_i, i \in I$, are R-modules. Then we have the following:*

(1) *If $A \subseteq B$, then $\overline{Z}(A) \subseteq \overline{Z}(B)$ and $\overline{Z}(B/A) \supseteq (\overline{Z}(B) + A)/A$;*

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(2) *If* $f : B \to A$ *is a homomorphism, then* $f(\overline{Z}(B)) \subseteq \overline{Z}(A)$*;*
(3) $\overline{Z}(A/\overline{Z}(A)) = 0$*;*
(4) $\overline{Z}(\oplus_{i\in I} A_i) = \oplus_{i\in I}\overline{Z}(A_i)$*;*
(5) $\overline{Z}(\prod_{i\in I} A_i) \subseteq \prod_{i\in I}\overline{Z}(A_i)$*;*
(6) *If* $A = B + S$ *where* S *is a small module, then* $\overline{Z}(A) = \overline{Z}(B)$*;*
(7) $\overline{Z}(A)$ *is the smallest submodule of* A *such that* $A/\overline{Z}(A)$ *is cosingular.*

Proof. See [3, Proposition 2.7]. ∎

Proposition 2.2. *Every cosingular module* (*and so every small module*) *satisfies* C^*.

Proof. By Lemma 2.1, submodule of a cosingular module is cosingular, Hence the Proposition follows. ∎

As every small submodule of a module M is a small module and hence a cosingular module, we get

Proposition 2.3. *Every lifting module satisfies* C^*.

Remarks. In general, the converses of the above two Propositions 2.2 and 2.3 are not true. For example let $M = Z_{p^\infty}$. It is easy to check that M has C^* but M is not cosingular. On the other hand Z has C^*, but it is not a lifting module [2, p.56].

Lemma 2.4. *A module* M *is lifting if and only if every submodule* A *of* M *can be written as* $A = N \oplus S$ *such that* N *is a direct summand of* M *and* $Sl{-}/M$.

Proof. See [2, Proposition 4.8]. ∎

Proposition 2.5. *For an* R*-module* M *the following statements are equivalent:*

(1) M *has* C^**;*
(2) *For every submodule* N *of* M *there exists a decomposition* $M = M_1 \oplus M_2$ *such that* $M_1 \leq N$ *and* $N \cap M_2$ *is cosingular;*
(3) *For every submodule* N *of* M*,* N *has a decomposition* $N = N_1 \oplus N_2$ *such that* N_1 *is a direct summand of* M *and* N_2 *is cosingular.*

Proof. (1) $\Rightarrow$ (2) Let $N \leq M$, by definition there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and N/M_1 is cosingular. We have $N = M_1 \oplus (N \cap M_2)$ and $N \cap M_2 \simeq N/M_1$ is cosingular.

$(2) \Rightarrow (3)$ Let $M = M_1 \oplus M_2$ with $M_1 \leq N$. Then $N = M_1 \oplus (N \cap M_2)$. Take $M_1 = N_1$ and $N \cap M_2 = N_2$.

$(3) \Rightarrow (1)$ Suppose $N \leq M$. We have $N = N_1 \oplus N_2$ with N_1 is a direct summand of M and N_2 cosingular. But $N/N_1 \simeq N_2$ is cosingular. M has C^*. ■

Proposition 2.6. *Suppose that M satisfies C^* and every cosingular submodule of M is small in M. Then M is lifting.*

Proof. Proof follows from Lemma 2.4 and Proposition 2.5. ■

Proposition 2.7. *The class of all modules having C^* is closed under submodules.*

Proof. Let M be an R-module satisfying C^* and $N \leq M$. For each $X \leq N$ we have $X \leq M$, hence there exists a direct summand K of M with $K \leq X$ and X/K cosingular. Since K is a direct summand of N, N has C^*. ■

Proposition 2.8. *Let R be a ring. An injective right R-module M satisfies C^* if and only if every submodule of M is a direct sum of an injective module and a cosingular module.*

Proof. Suppose that M is an injective module and satisfies C^*. By Proposition 2.5(3), every submodule of M is a direct sum of an injective module and a cosingular module. ■

Conversely, suppose that every submodule of M is a direct sum of an injective module and a cosingular module. Since an injective submodule is a direct summand, M has C^* by 2.5(3).

Theorem 2.9. *The following statements are equivalent for a ring R:*

(1) *Every right R-module satisfies C^*;*
(2) *Every injective right R-module satisfies C^*;*
(3) *Every right R-module is a direct sum of an injective module and a cosingular module.*

Proof. $(1) \Leftrightarrow (2)$ It is clear because every submodule of a module with C^* also has C^*.
$(2) \Leftrightarrow (3)$ It follows by Proposition 2.8. ■

Theorem 2.10. *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 satisfies C^*. Then M satisfies C^*.*

Proof. Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 satisfies C^*. Let $N \leq M$. Then $M_1 = (N \cap M_1) \oplus M'$ for some $M' \leq M_1$. Thus $M = (N \cap M_1) \oplus M' \oplus M_2$ and $N = (N \cap M_1) \oplus A$ where $A = N \cap (M' \oplus M_2)$. Since $(M_2 \oplus M')/M'$ satisfies C^*, it follows that $(A + M')/M' = K/M' \oplus L/M'$ for some submodule K and L containing M' such that K/M' is a direct summand of $(M_2 \oplus M')/M'$ and L/M' is cosingular. Thus K is a direct summand of M. As $K = M' \oplus (K \cap A)$, $K \cap A$ is also a direct summand of M. It is now clear that $(N \cap M_1) \oplus (K \cap A)$ is a direct summand of M. Moreover

$$N/((N \cap M_1) \oplus (K \cap A)) \simeq A/(K \cap A) \simeq (A + K)/K = (A + M')/K \simeq L/M'$$

is cosingular. It follows that M satisfies C^*. ∎

It may be noted that the above Theorem is not true if we replace 'C^*' by 'lifting'. For example, let $R = \mathbb{Z}$, $M_1 = \mathbb{Z}/2\mathbb{Z}$ and $M_2 = \mathbb{Z}/8\mathbb{Z}$.

Corollary 2.11. *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 is cosingular. Then M satisfies C^*.*

Let P and M be modules. P is said to be M-projective if for any module N with an epimorphism $\pi : M \to N$ and homomorphism $\theta : P \to N$, there exists a homomorphism $\theta' : P \to M$ such that $\pi\theta' = \theta$. P is called projective if it is M-projective for every module M. If P is P-projective, P is called quasi-projective. A class of modules $\{P_i\}_{i \in I}$ is called relatively projective if P_i is P_j-projective for all distinct $i, j \in I$.

Lemma 2.12. *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. The following are equivalent:*

(1) *M_1 is M_2-projective;*

(2) *For every submodule N of M such that $M = N + M_2$, there exists a submodule N' of N such that $M = N' \oplus M_2$.*

Proof. See [4, 41.14]. ∎

Proposition 2.13. *Suppose $M = M_1 \oplus M_2$ where M_i (i = 1,2), is M_2 projective and $M = L + M_2$.*

(1) *If M_2 is a module satisfying C^*, then $M = K \oplus M_2'$ where $M_2' \leq M_2$, $K \leq L$ and $(L \cap M_2')$ is cosingular.*

(2) *If M_2 is a lifting module, then $M = K \oplus M_2'$ where $M_2' \leq M_2$, $K \leq L$ and $(L \cap M_2')l{-}/M$.*

Proof. (1) As M_2 satisfies C^*, there exists a decomposition $M_2 = M_2'' \oplus M_2'$ with $M_2'' \leq L \cap M_2$ and $((L \cap M_2) \cap M_2') = (L \cap M_2')$ a cosingular module. Hence we have

$$M = L + M_2 = L + M_2'.$$

As $M_2'' \leq L$,

$$L = M_2'' \oplus ((M_2' \oplus M_1) \cap L).$$

Put $N = M_1 \oplus M_2'$. Then $L = M_2'' \oplus (N \cap L)$. Since $M = L + M_2'$, $N = (N \cap L) + M_1$. As M_1 is M_2'-projective, by 2.12, there exists a decomposition $N = T \oplus M_2'$ where $T \leq (N \cap L)$. Since $T \leq (N \cap L)$ and $L = M_2'' \oplus (N \cap L)$, we have $(M_2'' \oplus T) \leq L$. Put $K = M_2'' \oplus T$. Then $M = K \oplus M_2'$, $K \leq L$ and $L \cap M_2'$ is a cosingular module.

(2) Proof is similar to the proof of (1) and hence is omitted. ∎

Theorem 2.14. *Let M be a quasi-projective module and $M = M_1 \oplus M_2$. M satisfies C^* if and only if both M_1 and M_2 satisfy C^*.*

Proof. Suppose M satisfies C^*. By Proposition 2.7 both M_1 and M_2 satisfy C^*.

Suppose M_1 and M_2 satisfy C^*. Let $L \leq M$. In the sequel π will denote the obvoious projections. Since M_1 satisfies C^*, there exists a decomposition $M = M_1' \oplus M_1''$ such that $M_1' \leq \pi_{M_1}(L)$ and $\pi_{M_1}(L) \cap M_1''$ is a cosingular module. Put $L' = (M_1'' \oplus M_2) \cap L$. Since M_2 satisfies C^*, there exists a decomposition $M_2 = M_2' \oplus M_2''$ such that $M_2' \leq \pi_{M_2}(L')$ and $\pi_{M_2}(L') \cap M_2''$ is a cosingular module.

Set $L'' = (M_1'' \oplus M_2'') \cap L$. Then

$$\pi_{M_1''}(L'') \leq M_1'' \cap \pi_{M_1}(L) \text{ and } \pi_{M_2''}(L'') \leq M_2'' \cap \pi_{M_2}(L').$$

Also

$$L'' \leq \pi_{M_1''}(L'') \oplus \pi_{M_2''}(L'').$$

Now $M_1'' \cap \pi_{M_1}(L)$ and $M_2'' \cap \pi_{M_2}(L')$ are cosingular modules. As cosingular modules are closed under direct sums and submodules (see 2.1) L'' is a cosingular module. Also

$$\begin{aligned} M &= \pi_{M_1}(L) + M_1'' + M_2 \\ &= L + M_1'' + M_2 \\ &= L + M_1'' + \pi_{M_2}(L') + M_2'' \\ &= L + M_1'' + L' + M_2'' \\ &= L + (M_1'' \oplus M_2'') \end{aligned}$$

By hypothesis, $M_1' \oplus M_2'$ is $(M_1'' \oplus M_2'')$-projective. By Proposition 2.13, there is a decomposition $M = U \oplus V$ with $U \leq L$ and $L \cap V$ a cosingular module. Now Proposition 2.5 implies that M satisfies C^*. ■

Corollary 2.15. *Let $M = M_1 \oplus M_2$ be a projective module. M satisfies C^* if and only if M_1 and M_2 satisfy C^*.*

Corollary 2.16. *Let R be a ring. Then R as a right R-module satisfies C^* if and only if every finitely generated projective R-module satisfies C^*.*

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