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TWO EXTRAGRADIENT APPROXIMATION METHODS FOR VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS OF STRICT PSEUDO-CONTRACTIONS

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Abstract. Let $\{S_i\}_{i=1}^N$ be N strict pseudo-contractions defined on a nonempty closed convex subset C of a real Hilbert space H. Consider the problem of finding a common element of the set of common fixed points of these mappings $\{S_i\}_{i=1}^N$ and the set of solutions of the variational inequality for a monotone Lipschitz continuous mapping of C into H, and consider the parallel-extragradient and cyclic-extragradient algorithms for solving this problem. We will derive the weak convergence of these algorithms. Moreover, these weak convergence results will be applied to finding a common zero point of a finite family of maximal monotone mappings. Further we prove that these algorithms can be modified to have strong convergence by virtue of additional projections. Our results represent the improvement, generalization and development of the previously known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let $A : C \to H$ be a mapping of C into H. The variational inequality problem ($\mathrm{VI}(A, C)$) is formulated as finding an element $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

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Let P_C be the metric projection of H onto C. It is known that the $\mathrm{VI}(A, C)$ is equivalent to the fixed-point equation

$$u = P_C(u - \lambda Au),$$

where $\lambda > 0$ is an arbitrary fixed constant. The set of solutions of the VI(A,C) is denoted by Ω. Variational inequalities were initially studied by Stampacchia [1] and ever since have been widely studied and generalized in various directions, because they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance; see, e.g., [1-5]. Existence and uniqueness of solutions are important problems in the study of the variational inequality theory. At the same time, an equally important problem is how to develop efficient and implementable algorithms for solving variational inequality and its generalizations if any. A great deal of effort has gone into this problem; see, e.g., [2,8,10-15,20,25].

Definition 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called

(i) monotone if
$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) α-inverse-strongly monotone (see, e.g., [10]) if there exists a constant $\alpha > 0$ such that
$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C;$$

(iii) β-strongly monotone if there exists a constant $\beta > 0$ such that
$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C;$$

(iv) k-Lipschitz continuous if there exists a constant $k > 0$ such that
$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C.$$

It is clear that every α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous.

Definition 1.2. Let C be a nonempty closed convex subset of a real Hilbert space H. A self-mapping $S : C \to C$ is called a strict pseudo-contraction [6] if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

(If the last inequality holds, we also say that S is a κ-strict pseudo-contraction.) These mappings are extensions of nonexpansive mappings which satisfy the last inequality with $\kappa = 0$.

Iterative methods for nonexpansive mappings have been extensively investigated; see [9,18,20-22,24,26,27] and the references therein. However iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [6] initiated their work in 1967. Here the reason is probably that the second term appearing in the right-hand side of the last inequality impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction S. However, we remind the reader of an important fact that strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see Scherzer [23]). Therefore it is interesting to develop the theory of iterative methods for strict pseudo-contractions.

Quite recently, motivated by Browder and Petryshyn [6] Marino and Xu [17] defined the following Mann's algorithm (see [7])

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n,$$

and proved that the sequence $\{x_n\}$ generated by the algorithm converges weakly to a fixed point of S, provided the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the conditions that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1 - \alpha_n) = \infty$. Such a result can also be viewed as the Hilbert space version for strict pseudo-contractions of Reich's Banach space result [26] for nonexpansive mappings which states that if S is a nonexpansive self-mappings, with a fixed point, of a closed convex subset C of a uniformly convex Banach space with a Fréchet differentiable norm, then the sequence $\{x_n\}$ generated by the above Mann's algorithm converges weakly to a fixed point of S provided the sequence $\{\alpha_n\}_{n=0}^{\infty}$ of parameters satisfies the conditions that $0 < \alpha_n < 1$ for all n and that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. We remark that if S is nonexpansive, then S is κ-strict pseudo-contraction with $\kappa = 0$. In this case, the condition $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1 - \alpha_n) = \infty$ reduces to $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Very recently, Acedo and Xu [19] introduced and considered the problem of finding a point x such that

$$x \in \bigcap_{i=1}^{N} F(S_i), \tag{1.1}$$

where $N \geq 1$ is a positive integer and $\{S_i\}_{i=1}^{N}$ are N strict pseudo-contractions defined on a nonempty closed convex subset C of a Hilbert space H. Here $F(S_i) = \{z \in C : S_i z = z\}$ is the set of fixed points of S_i, $1 \leq i \leq N$. Let S be defined by

$$S = \sum_{i=1}^{N} \lambda_i S_i$$

where $\lambda_i > 0$ for all i such that $\sum_{i=1}^{N} \lambda_i = 1$. We will see that S is a strict pseudo-contraction on C and $F(S) = \bigcap_{i=1}^{N} F(S_i)$. Marino and Xu's result [17] applies to S and hence the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{N} \lambda_i S_i x_n \tag{1.2}$$

converge weakly to a solution to the problem (1.1). Moreover, they considered a more general situation by allowing the weights $\{\lambda_i\}_{i=1}^{N}$ in (1.2) to depend on n, the number of steps of the iteration. That is, they considered the algorithm which generates a sequence $\{x_n\}$ in the following way

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{N} \lambda_i^{(n)} S_i x_n. \tag{1.3}$$

Under appropriate assumptions on the sequences of the weights $\{\lambda_i^{(n)}\}_i^N$ they also proved the weak convergence, to a solution of the problem (1.1), of the algorithm (1.3).

Another approach to the problem (1.1) is the cyclic algorithm. (For convenience, the mappings $\{S_i\}_{i=1}^{N}$ are relabeled as $\{S_i\}_{i=0}^{N-1}$.) This means that beginning with an x_0 in C, the sequence $\{x_n\}$ is defined cyclically by

$$\begin{aligned}
x_1 &= \alpha_0 x_0 + (1-\alpha_0) S_0 x_0, \\
x_2 &= \alpha_1 x_1 + (1-\alpha_1) S_1 x_1, \\
&\vdots \\
x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1}) S_{N-1} x_{N-1}, \\
x_{N+1} &= \alpha_N x_N + (1-\alpha_N) S_0 x_N, \\
&\vdots
\end{aligned}$$

In a more compact form, x_{n+1} can be written as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_{[n]} x_n, \tag{1.4}$$

where $S_{[n]} = S_i$, with $i = n(\mathrm{mod} N),\ 0 \le i \le N-1$. They proved that this cyclic algorithm (1.4) is also weakly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen.

Furthermore, Acedo and Xu [19] proposed the modification for the algorithm (1.3) as follows

$$x_{n+1} = P_{C_n \cap Q_n} x_0, \tag{1.5}$$

where C_n and Q_n are given by

$$C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa)\|x_n - A_n x_n\|^2\}$$

where $A_n = \sum_{i=1}^{N} \lambda_i^{(n)} S_i$ and $y_n = \alpha_n x_n + (1 - \alpha_n) A_n x_n$, and

$$Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}. \tag{1.6}$$

As for the algorithm (1.4), they proposed the following modification that produces the sequence $\{x_n\}$ given by the same formula (1.5) with C_n given by

$$C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa)\|x_n - S_{[n]} x_n\|^2\}$$

where $y_n = \alpha_n x_n + (1 - \alpha_n) S_{[n]} x_n$, and with Q_n given by the same formula (1.6). They proved the strong convergence of the algorithm (1.5) for strict pseudo-contractions.

On the other hand, recently, by combining Korpelevich's extragradient method [8] with Takahashi and Toyoda's iterative algorithm [10], Nadezhkina and Takahashi [11] introduced the following iterative scheme for finding an element of $F(S) \cap \Omega$ and proved its weak convergence.

Theorem 1.1. [11, Theorem 3.1]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz-continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by any given $x_0 \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n) \end{cases} \tag{1.7}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$ where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n$.

Very recently, inspired by Nadezhkina and Takahashi [11, Theorem 3.1], Zeng and Yao [13] introduced the following iterative process for finding an element of $F(S) \cap \Omega$ and established the following strong convergence theorem.

Theorem 1.2. [13, Theorem 3.1]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be sequences generated by any given $x_0 \in C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases}$$

for every $n \geq 0$, where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*;*

(b) $\{\alpha_n\} \subset (0,1),\ \sum_{n=0}^{\infty} \alpha_n = \infty,\ \lim_{n\to\infty} \alpha_n = 0.$

Then the sequences $\{x_n\}, \{y_n\}$ *converge strongly to the same element* $P_{F(S)\cap\Omega} x_0$ *provided* $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$

On the other hand, motivated by Nadezhkina and Takahashi [11, Theorem 3.1], Ceng and Yao [14] introduced the following extragradient-like approximation method for finding an element of $F(S) \cap \Omega$ and established the following weak convergence theorem.

Theorem 1.3. [14, Theorem 3.1]. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to C$ *be a contractive mapping with a contractive constant* $\alpha \in (0,1),$ $A : C \to H$ *be a monotone,* k*-Lipschitz-continuous mapping and* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \cap \Omega \neq \emptyset$*. Let* $\{x_n\}, \{y_n\}$ *be sequences generated by any given* $x_0 \in C$ *and*

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda_n A y_n), \end{cases}$$

for every $n \geq 0$*, where* $\{\lambda_n\}$ *is a sequence in* $(0,1)$ *with* $\sum_{n=0}^{\infty} \lambda_n < \infty$*, and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *are three sequences in* $[0,1]$ *satisfying the conditions:*

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \liminf_{n\to\infty} \beta_n < 1.$

Then the sequences $\{x_n\}, \{y_n\}$ *converge strongly to the same point* $q = P_{F(S)\cap\Omega} f(q)$ *if and only if* $\{Ax_n\}$ *is bounded and* $\liminf_{n\to\infty} \langle Ax_n, y - x_n\rangle \geq 0$ *for all* $y \in C$*.*

Let $\{S_i\}_{i=1}^N$ be N strict pseudo-contractions defined on a nonempty closed convex subset C of a real Hilbert space H. In this paper, consider the problem of finding a common element of the set of common fixed points of these mappings $\{S_i\}_{i=1}^N$ and the set of solutions of the variational inequality $\mathrm{VI}(A, C)$ for a monotone Lipschitz continuous mapping A of C into H, and consider the parallel-extragradient and cyclic-extragradient algorithms for solving this problem. We will derive the weak convergence of these algorithms. Moreover, these weak convergence results will be applied to finding a common zero point of a finite family of maximal monotone mappings. Further we prove that these algorithms can be modified to have strong convergence by virtue of additional projections. Our results represent the improvement, generalization and development of the previously known results in the literature.

Notation:

1. $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

We need some facts and tools in a real Hilbert space H which are listed as lemmas below (see [18] for necessary proofs of Lemmas 2.2 and 2.4).

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities*

(*i*) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,\ \forall x, y \in H$;

(*ii*) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2,\ \forall t \in [0,1], \forall x, y \in H$;

(*iii*) *If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2,\ \forall y \in H.$$

Lemma 2.2. *Let H be a real Hilbert space. Given a nonempty closed convex subset $C \subset H$ and points $x, y, z \in H$ and given also a real number $a \in R = (-\infty, \infty)$, the set*

$$\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that given a nonempty closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted as $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x-y\|, \quad \forall y \in K.$$

Lemma 2.3. *Let K be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation:*

$$\langle x-z, y-z\rangle \leq 0, \quad \forall y \in K.$$

Remark 2.1. It is easy to see that the last inequality is equivalent to

$$\|x-y\|^2 \geq \|x-P_K x\|^2 + \|y - P_K x\|^2 \tag{2.1}$$

for all $x \in H$ and all $y \in K$; see [9] for more derails.

Lemma 2.4. *Let K be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. Suppose $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n.$$

Then $x_n \to q$.

Lemma 2.5. [19, Lemma 2.5]. *Let K be a nonempty closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that the weak ω-limit set $\omega_w(x_n) \subset K$ and for each $z \in K$, $\lim_{n\to\infty} \|x_n - z\|$ exists. Then $\{x_n\}$ is weakly convergent to a point in K.*

The following proposition lists some useful properties for strict pseudo-contractions; see also [6,23].

Proposition 2.6. [19, Proposition 2.6]. *Assume C is a nonempty closed convex subset of a real Hilbert space H.*

(i) *If $S : C \to C$ is a κ-strict pseudo-contraction, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

(ii) *If $S : C \to C$ is a κ-strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$.*

(iii) *If $S : C \to C$ is a κ-strict pseudo-contraction, then the fixed point set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.*

(iv) *Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $S_i : C \to C$ is a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i S_i$ is a κ-strict pseudo-contraction, with $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$.*

(v) *Let $\{S_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ be given as in (iv) above. Suppose that $\{S_i\}_{i=1}^N$ has a common fixed point. Then*

$$F(\sum_{i=1}^N \lambda_i S_i) = \bigcap_{i=1}^N F(S_i).$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$, then $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - y, w\rangle \geq 0,\ \forall y \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see [16].

3. Parallel-extragradient Algorithm

Mann's algorithm has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [26] in a uniformly convex Banach space with a Fréchet differentiable norm. Recently Marino and Xu [17, Theorem 3.1] extended Reich's result to strict pseudo-contractions in the Hilbert space setting. Very recently, Acedo and Xu [19, Theorem 3.3] also extended Marino and Xu's result to a finite family of strict pseudo-contractions. In this section, by combining the iterative scheme in [19, Theorem 3.3] with the iterative one in [11, Theorem 3.1], we propose a parallel-extragradient algorithm for finding an element of $\bigcap_{i=1}^N F(S_i) \cap \Omega$ where for each $1 \leq i \leq N$, $S_i : C \to C$ is a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$.

Lemma 3.1. (see [10]). *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall n \geq 0.$$

Then the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ be a monotone, k-Lipschitz continuous mapping. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$ such that $\bigcap_{i=1}^N F(S_i) \cap \Omega \neq \emptyset$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that for each n, $\{\lambda_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ for each $n \geq 0$ where $\lambda_i^{(n)} > 0$ for all $n \geq 0$ and $1 \leq i \leq N$. Given any $x_0 \in C$, let $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ be the sequences* *generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1-\alpha_n) \sum\limits_{i=1}^{N} \lambda_i^{(n)} S_i t_n, \quad n \geq 0, \end{cases} \tag{3.1}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$*;*

(*ii*) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$.

Then, the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$*, where* $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$.

Proof. We divide the proof into several steps.

Step 1. We claim that the following hold:

(i) $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$ and all $n \geq 0$;

(ii) $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Indeed, put $t_n = P_C(x_n - \lambda_n A y_n)$ for each $n \geq 0$. Let $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$ be an arbitrary element. Then, from (2.1), monotonicity of A, and $u \in \Omega$, we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 \\ &\quad + 2\lambda_n(\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle) \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further, since $y_n = P_C(x_n - \lambda_n A x_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned} &\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\ &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \lambda_n \langle A x_n - A y_n, t_n - y_n\rangle \\ &\leq \lambda_n \langle A x_n - A y_n, t_n - y_n\rangle \\ &\leq \lambda_n k \|x_n - y_n\| \|t_n - y_n\|. \end{aligned}$$

So, we have

$$\begin{aligned}\|t_n - u\|^2 &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad +2\lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\ &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad +\lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\ &= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\le \|x_n - u\|^2.\end{aligned} \tag{3.2}$$

Write, for each $n \ge 1$,

$$T_n = \sum_{i=1}^{N} \lambda_i^{(n)} S_i.$$

By Proposition 2.6 (iv), each T_n is a κ-strict pseudo-contraction on C, and from the algorithm (3.1) we obtain

$$x_{n+1} = \alpha_n t_n + (1 - \alpha_n) T_n t_n. \tag{3.3}$$

Hence from (3.2), $u = S_i u$ $(1 \le i \le N)$, and $\{\alpha_n\} \subset (\kappa, 1)$, we have

$$\begin{aligned}&\|x_{n+1} - u\|^2 \\ &= \|\alpha_n(t_n - u) + (1-\alpha_n)(T_n t_n - u)\|^2 \\ &= \alpha_n\|t_n - u\|^2 + (1-\alpha_n)\|T_n t_n - u\|^2 - \alpha_n(1-\alpha_n)\|t_n - T_n t_n\|^2 \\ &\le \alpha_n\|t_n - u\|^2 + (1-\alpha_n)(\|t_n - u\|^2 \\ &\quad +\kappa\|t_n - T_n t_n\|^2) - \alpha_n(1-\alpha_n)\|t_n - T_n t_n\|^2 \\ &= \|t_n - u\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|t_n - T_n t_n\|^2 \\ &\le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|t_n - T_n t_n\|^2 \\ &\le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\le \|x_n - u\|^2.\end{aligned} \tag{3.4}$$

Therefore, there exists

$$c = \lim_{n\to\infty} \|x_n - u\|$$

and the sequences $\{x_n\}, \{t_n\}$ are bounded.

Step 2. We claim that the following hold:

(i) $\lim_{n\to\infty} \|x_n - y_n\| = 0$;

(ii) $\lim_{n\to\infty} \|y_n - t_n\| = 0$;

(iii) $\lim_{n\to\infty} \|x_n - t_n\| = 0$.

Indeed, from (3.4), we get

$$(1-\lambda_n^2 k^2)\|x_n - y_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

So we have

$$\begin{aligned}\|x_n - y_n\|^2 &\le \frac{1}{1-\lambda_n^2 k^2}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2)\\ &\le \frac{1}{1-b^2k^2}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2).\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Further, it follows that

$$\|t_n - y_n\| = \|P_C(x_n - \lambda_n A y_n) - P_C(x_n - \lambda_n A x_n)\| \le \lambda_n k \|x_n - y_n\|.$$

This implies that $\lim_{n\to\infty} \|y_n - t_n\| = 0$. From $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$, we also have $\lim_{n\to\infty} \|x_n - t_n\| = 0$. Since A is k-Lipschitz continuous, we have $\lim_{n\to\infty} \|Ay_n - At_n\| = 0$.

Step 3. We claim that the following hold:

(i) $\lim_{n\to\infty} \|t_n - T_n t_n\| = 0$;

(ii) $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Indeed, since $\kappa < \alpha \le \alpha_n \le \beta < 1$ for all $n \ge 0$, from (3.4) it follows that

$$\begin{aligned}(\alpha-\kappa)(1-\beta)\|t_n - T_n t_n\|^2 &\le (\alpha_n - \kappa)(1-\alpha_n)\|t_n - T_n t_n\|^2\\ &\le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.\end{aligned}$$

From Step 1 (ii) we deduce that $\lim_{n\to\infty} \|t_n - T_n t_n\| = 0$. Furthermore, utilizing the Lipschitz continuity of T_n, we have

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - t_n\| + \|t_n - T_n t_n\| + \|T_n t_n - T_n x_n\|\\ &\le \|x_n - t_n\| + \|t_n - T_n t_n\| + \frac{1+\kappa}{1-\kappa}\|t_n - x_n\|\\ &= \|t_n - T_n t_n\| + \frac{2}{1-\kappa}\|t_n - x_n\|,\end{aligned}$$

which hence implies that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Step 4. We claim that $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Indeed, first, let us show that

$$\omega_w(x_n) \subset \bigcap_{i=1}^{N} F(S_i). \tag{3.5}$$

To see this, we take $z \in \omega_w(x_n)$ arbitrarily and assume that $x_{n_l} \rightharpoonup z$ as $l \to \infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. Without loss of generality, we may assume that

$$\lambda_i^{(n_l)} \to \lambda_i \quad (\text{as } l \to \infty),\ 1 \le i \le N. \tag{3.6}$$

It is readily seen that each $\lambda_i > 0$ and $\sum_{i=1}^{N} \lambda_i = 1$. We also have

$$T_{n_l}x \to \Gamma x \quad (\text{as } l \to \infty) \text{ for all } x \in C,$$

where $\Gamma = \sum_{i=1}^{N} \lambda_i S_i$. Note that by Proposition 2.6, Γ is a κ-strict pseudo-contraction and $F(\Gamma) = \bigcap_{i=1}^{N} F(S_i)$. Since

$$\begin{aligned} \|x_{n_l} - \Gamma x_{n_l}\| &\le \|x_{n_l} - T_{n_l}x_{n_l}\| + \|T_{n_l}x_{n_l} - \Gamma x_{n_l}\| \\ &\le \|x_{n_l} - T_{n_l}x_{n_l}\| + \sum_{i=1}^{N} |\lambda_i^{(n_l)} - \lambda_i| \|S_i x_{n_l}\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$, from (3.6) we conclude that $\lim_{l\to\infty} \|x_{n_l} - \Gamma x_{n_l}\| = 0$. So by the demiclosedness principle (Proposition 2.6 (ii)), it follows that $z \in F(\Gamma) = \bigcap_{i=1}^{N} F(S_i)$ and hence (3.5) holds.

Second, let us show that $\omega_w(x_n) \subset \Omega$. Indeed, take $z \in \omega_w(x_n)$ arbitrarily and assume still that $x_{n_l} \rightharpoonup z$ as $l \to \infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $t_{n_l} \rightharpoonup z$ and $y_{n_l} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see [16]. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have

$$\langle v - u, w - Av \rangle \ge 0, \quad \forall u \in C.$$

On the other hand, from

$$t_n = P_C(x_n - \lambda_n A y_n) \quad \text{and} \quad v \in C,$$

we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v \rangle \ge 0,$$

and hence

$$\langle v - t_n, (t_n - x_n)/\lambda_n + Ay_n \rangle \geq 0.$$

Therefore from

$$w - Av \in N_C v \quad \text{and} \quad t_{n_l} \in C,$$

we have

$$\begin{aligned}
\langle v - t_{n_l}, w \rangle &\geq \langle v - t_{n_l}, Av \rangle \\
&\geq \langle v - t_{n_l}, Av \rangle - \langle v - t_{n_l}, (t_{n_l} - x_{n_l})/\lambda_{n_l} + Ay_{n_l} \rangle \\
&= \langle v - t_{n_l}, Av - At_{n_l} \rangle + \langle v - t_{n_l}, At_{n_l} - Ay_{n_l} \rangle \\
&\quad - \langle v - t_{n_l}, (t_{n_l} - x_{n_l})/\lambda_{n_l} \rangle \\
&\geq \langle v - t_{n_l}, At_{n_l} - Ay_{n_l} \rangle - \langle v - t_{n_l}, (t_{n_l} - x_{n_l})/\lambda_{n_l} \rangle.
\end{aligned}$$

Hence we obtain

$$\langle v - z, w \rangle \geq 0, \quad \text{as } l \to \infty.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \Omega$. Thus, we conclude that $\omega_w(x_n) \subset \Omega$. Therefore, $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Step 5. We claim that $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$.

Indeed, we first show that $\omega_w(x_n)$ is a single-point set. We take $z_1, z_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup z_1$ and $x_{m_j} \rightharpoonup z_2$, respectively. Since $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$ and since $z_1, z_2 \in \bigcap_{i=1}^N F(S_i) \cap \Omega$, by Lemma 2.1 (iii) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - z_1\|^2 \\
&= \lim_{j\to\infty} \|x_{m_j} - z_2\|^2 + \|z_2 - z_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - z_2\|^2 + \|z_2 - z_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - z_1\|^2 + 2\|z_2 - z_1\|^2 \\
&= \lim_{n\to\infty} \|x_n - z_1\|^2 + 2\|z_2 - z_1\|^2.
\end{aligned}$$

Hence $z_1 = z_2$. This shows that $\omega_w(x_n)$ is a single-point set. Without loss of generality, we may write $\omega_w(x_n) = \{z\}$. This implies that $x_n \rightharpoonup z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$. Since $x_n - y_n \to 0$ as $n \to \infty$, we also have $y_n \rightharpoonup z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Now, put $u_n = P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$. Let us show that $\lim_{n\to\infty} \|u_n - z\| = 0$. Since $u_n = P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$ and $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$, we have

$$\langle z - u_n, u_n - x_n \rangle \geq 0.$$

By Lemma 3.2, $\{u_n\}$ converges strongly to some $z_0 \in \bigcap_{i=1}^N F(S_i) \cap \Omega$. Then we have $\langle z - z_0, z_0 - z \rangle \geq 0$ and hence $z = z_0$. This completes the proof. ∎

Utilizing Theorem 3.1, we derive two corollaries in a real Hilbert space.

Corollary 3.1. *Let H be a real Hilbert space and $A : H \to H$ be a monotone, k-Lipschitz continuous mapping. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $S_i : H \to H$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$ such that $\bigcap_{i=1}^N F(S_i) \cap A^{-1}0 \neq \emptyset$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that for each n, $\{\lambda_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ for each $n \geq 0$ where $\lambda_i^{(n)} > 0$ for all $n \geq 0$ and $1 \leq i \leq N$. Given any $x_0 \in H$, let $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ be the sequences generated by*

$$\begin{cases} y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{i=1}^N \lambda_i^{(n)} S_i t_n, \quad n \geq 0, \end{cases}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$;

(*ii*) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$.

Then, the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in \bigcap_{i=1}^N F(S_i) \cap A^{-1}0$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^N F(S_i) \cap A^{-1}0} x_n$.

Proof. We have $C = H$, $A^{-1}0 = \Omega$ and $P_H = I$. By Theorem 3.1 we obtain the desired result. ∎

Corollary 3.2. *Let H be a real Hilbert space and $A : H \to H$ be a monotone, k-Lipschitz continuous mapping. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $B_i : H \to 2^H$ be a maximal monotone mapping such that $\bigcap_{i=1}^N B_i^{-1}0 \cap A^{-1}0 \neq \emptyset$. Let $J_r^{B_i}$ be the resolvent of B_i for each $r > 0$. Assume that for each n, $\{\lambda_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ for each $n \geq 0$ where $\lambda_i^{(n)} > 0$ for all $n \geq 0$ and $1 \leq i \leq N$. Given any $x_0 \in H$, let $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ be the sequences generated by*

$$\begin{cases} y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{i=1}^N \lambda_i^{(n)} J_r^{B_i} t_n, \quad n \geq 0, \end{cases}$$

where there hold the following conditions

(i) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$*;*

(ii) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (0, 1)$.

Then, the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in \bigcap_{i=1}^{N} B_i^{-1}0 \cap A^{-1}0$*, where* $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^{N} B_i^{-1}0 \cap A^{-1}0} x_n$.

Proof. We have $C = H,\ A^{-1}0 = \Omega,\ F(J_r^{B_i}) = B_i^{-1}0$ and $\kappa = 0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. ■

4. Cyclic-extragradient Algorithm

Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{S_i\}_{i=0}^{N-1}$ be N κ-strict pseudo-contractions on C such that the intersection set $\bigcap_{i=0}^{N-1} F(S_i) \cap \Omega \neq \emptyset$. In this section, we propose a cyclic-extragradient algorithm for finding an element of $\bigcap_{i=0}^{N-1} F(S_i) \cap \Omega$.

Algorithm 4.1. Let$\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(\kappa, 1)$ and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence in $(0, 1/k)$. Given any $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ be the sequences generated via the iterative scheme

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1-\alpha_n) S_{[n]} t_n, \quad n \geq 0, \end{cases} \tag{4.1}$$

where $S_{[n]} = S_i$, with $i = n(\mathrm{mod} N),\ 0 \leq i \leq N-1$, i.e., if $n = jN + i$ for some integers $j \geq 0$ and $0 \leq i \leq N-1$, then $S_{[n]} = S_0$ if $i = 0$ and $S_{[n]} = S_i$ if $0 < i \leq N-1$.

We are now in a position to discuss the convergence analysis for Algorithm 4.1.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $A : C \to H$ *be a monotone,* k*-Lipschitz continuous mapping. Let* $N \geq 1$ *be an integer. Let, for each* $0 \leq i \leq N-1$, $S_i : C \to C$ *be a* κ_i*-strict pseudo-contraction for some* $0 \leq \kappa_i < 1$ *such that* $\bigcap_{i=0}^{N-1} F(S_i) \cap \Omega \neq \emptyset$*. Let* $\kappa = \max\{\kappa_i : 0 \leq i \leq N-1\}$*. Given any* $x_0 \in C$*, let* $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ *be the sequences generated by the cyclic-extragradient algorithm (4.1). Assume that the sequences* $\{\alpha_n\} \subset (\kappa, 1)$ *and* $\{\lambda_n\} \subset (0, 1/k)$ *satisfy the following conditions*

(i) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$*;*

(ii) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$*, where* $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$.

Proof. We divide the proof into several steps.

Step 1. We claim that the following hold:

(i) $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$ and all $n \ge 0$;

(ii) $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Indeed, utilizing the same argument as in the proof of (3.2), we obtain

$$\|t_n - u\|^2 \le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \le \|x_n - u\|^2. \tag{4.2}$$

From (4.2), $u = S_i u$ $(0 \le i \le N-1)$, and $\{\alpha_n\} \subset (\kappa, 1)$, it follows that

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&= \|\alpha_n(t_n - u) + (1-\alpha_n)(S_{[n]}t_n - u)\|^2 \\
&= \alpha_n\|t_n - u\|^2 + (1-\alpha_n)\|S_{[n]}t_n - u\|^2 - \alpha_n(1-\alpha_n)\|t_n - S_{[n]}t_n\|^2 \\
&\le \alpha_n\|t_n - u\|^2 + (1-\alpha_n)(\|t_n - u\|^2 \\
&\quad +\kappa\|t_n - S_{[n]}t_n\|^2) - \alpha_n(1-\alpha_n)\|t_n - S_{[n]}t_n\|^2 \\
&= \|t_n - u\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|t_n - S_{[n]}t_n\|^2 \\
&\le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 + (1-\alpha_n)(\kappa-\alpha_n)\|t_n - S_{[n]}t_n\|^2 \\
&\le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned} \tag{4.3}$$

Therefore, there exists

$$c = \lim_{n\to\infty} \|x_n - u\|$$

and the sequences $\{x_n\}, \{t_n\}$ are bounded.

Step 2. We claim that the following hold:

(i) $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$;

(ii) $\lim\limits_{n\to\infty} \|y_n - t_n\| = 0$;

(iii) $\lim\limits_{n\to\infty} \|x_n - t_n\| = 0$.

Indeed, utilizing the same argument as in Step 2 of the proof of Theorem 3.1, we can obtain the assertions.

Step 3. We claim that the following hold:

(i) $\lim_{n\to\infty} \|t_n - S_{[n]}t_n\| = 0$;

(ii) $\lim_{n\to\infty} \|x_n - S_{[n]}x_n\| = 0$.

Indeed, utilizing the same argument as in Step 3 of the proof of Theorem 3.1, we can obtain the assertions, where T_n is replaced by $S_{[n]}$.

Step 4. We claim that $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i) \cap \Omega$.

Indeed, first, let us show that

$$\omega_w(x_n) \subset \bigcap_{i=0}^{N-1} F(S_i). \tag{4.4}$$

To see this, we take $z \in \omega_w(x_n)$ arbitrarily and assume that $x_{n_l} \rightharpoonup z$ as $l \to \infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. We may further assume that $n_l = i(\mathrm{mod} N)$ for all l. Observe that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|\alpha_n(t_n - x_n) + (1-\alpha_n)(S_{[n]}t_n - x_n)\|^2 \\ &\le \alpha_n\|t_n - x_n\|^2 + (1-\alpha_n)\|S_{[n]}t_n - x_n\|^2 \\ &\le \alpha_n\|t_n - x_n\|^2 + (1-\alpha_n)[\|S_{[n]}t_n - t_n\| + \|t_n - x_n\|]^2. \end{aligned}$$

Thus we deduce that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and hence we also have $x_{n_l+j} \rightharpoonup z$ for all $j \ge 0$. Consequently, we conclude that

$$\|x_{n_l+j} - S_{[i+j]}x_{n_l+j}\| = \|x_{n_l+j} - S_{[n_l+j]}x_{n_l+j}\| \to 0.$$

Then the demiclosedness principle (Proposition 2.6 (ii)) implies that $z \in F(S_{[i+j]})$ for all j. This ensures that $z \in \bigcap_{i=0}^{N-1} F(S_i)$. Therefore (4.4) holds. Second, let us show that $\omega_w(x_n) \subset \Omega$. Indeed, the argument is the same as in Step 4 of the proof of Theorem 3.1. Thus we omit it.

Step 5. We claim that $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$, where $z = \lim_{n\to\infty} P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_n$.

Indeed, the argument is the same as in Step 5 of the proof of Theorem 3.1. Thus we omit it. This completes the proof. ∎

Utilizing Theorem 4.1, we derive two corollaries in a real Hilbert space.

Corollary 4.1. *Let H be a real Hilbert space and $A : H \to H$ be a monotone, k-Lipschitz continuous mapping. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $S_i : H \to H$ be a κ_i-strict pseudo-contraction for some $0 \le \kappa_i < 1$ such* *that* $\bigcap_{i=0}^{N-1} F(S_i) \cap A^{-1}0 \neq \emptyset$. *Let* $\kappa = \max\{\kappa_i : 0 \leq i \leq N-1\}$. *Given any* $x_0 \in H$, *let* $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ *be the sequences generated by*

$$\begin{cases} y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1-\alpha_n) S_{[n]} t_n, \quad n \geq 0, \end{cases}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$;

(*ii*) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (\kappa, 1)$.

Then, the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in \bigcap_{i=0}^{N-1} F(S_i) \cap A^{-1}0$, *where* $z = \lim_{n\to\infty} P_{\bigcap_{i=0}^{N-1} F(S_i) \cap A^{-1}0} x_n$.

Proof. We have $C = H,\ A^{-1}0 = \Omega$ and $P_H = I$. By Theorem 4.1 we obtain the desired result. ∎

Corollary 4.2. *Let* H *be a real Hilbert space and* $A : H \to H$ *be a monotone,* k*-Lipschitz continuous mapping. Let* $N \geq 1$ *be an integer. Let, for each* $0 \leq i \leq N-1$, $B_i : H \to 2^H$ *be a maximal monotone mapping such that* $\bigcap_{i=0}^{N-1} B_i^{-1}0 \cap A^{-1}0 \neq \emptyset$. *Let* $J_r^{B_i}$ *be the resolvent of* B_i *for each* $r > 0$. *Given any* $x_0 \in H$, *let* $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ *be the sequences generated by*

$$\begin{cases} y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1-\alpha_n) J_r^{B_{[n]}} t_n, \quad n \geq 0, \end{cases}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$;

(*ii*) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (0, 1)$.

Then, the sequences $\{x_n\}, \{y_n\}$ *converge weakly to the same point* $z \in \bigcap_{i=0}^{N-1} B_i^{-1}0 \cap A^{-1}0$, *where* $z = \lim_{n\to\infty} P_{\bigcap_{i=0}^{N-1} B_i^{-1}0 \cap A^{-1}0} x_n$.

Proof. We have $C = H,\ A^{-1}0 = \Omega,\ F(J_r^{B_i}) = B_i^{-1}0$ and $\kappa = 0$. Putting $P_H = I$, by Theorem 4.1 we obtain the desired result. ∎

5. Strong Convergence

In an infinite-dimensional Hilbert space, the previous two algorithms have only weak convergence (see Theorems 3.1 and 4.1). Hence in order to have strong convergence, we have to modify these two algorithms. Recently, a modification of Mann's algorithm for finding a fixed point of a single strict pseudo-contraction, which has strong convergence, was obtained in [17]. Subsequently, a modification of Mann's algorithm for finding a common fixed point of N strict pseudo-contractions, which has strong convergence, was considered in [19], where $N \geq 1$ is an integer.

Inspired by Acedo and Xu [19], below we purpose and analyze an iterative algorithm for finding a common element of the set of common fixed points of N strict pseudo-contractions and the set of solutions of the variational inequality $(\mathrm{VI}(A, C))$.

Theorem 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ be a monotone, k-Lipschitz continuous mapping. Given an integer $N \geq 1$, let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$ such that $\bigcap_{i=1}^N F(S_i) \cap \Omega \neq \emptyset$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume that for each n, $\{\lambda_i^{(n)}\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i^{(n)} = 1$ and $\inf_{n \geq 1} \lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping T_n be defined by*

$$T_n x = \sum_{i=1}^N \lambda_i^{(n)} S_i x, \quad \forall x \in C.$$

Given any $x_0 \in C$, let $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ be the sequences generated by

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ z_n = \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{i=1}^N \lambda_i^{(n)} S_i t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|t_n - z\|^2 \\ \qquad\quad -(1 - \alpha_n)(\alpha_n - \kappa)\|t_n - T_n t_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases} \tag{5.1}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$*;*

(*ii*) $0 \leq \alpha_n < 1$ *for all* $n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{\bigcap_{i=1}^N F(S_i) \cap \Omega} x_0$ provided $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Proof. We divide the proof into several steps.

First observe that C_n is convex by Lemma 2.2. Next, let us show that $\bigcap_{i=1}^N F(S_i)\cap \Omega \subset C_n$ for all n. Indeed, let $u \in \bigcap_{i=1}^N F(S_i) \cap \Omega$ be an arbitrary element. As in the proof of (3.4), we can derive

$$\|z_n - u\|^2 \leq \|t_n - u\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|t_n - T_n t_n\|^2. \tag{5.2}$$

So $u \in C_n$ for all n. Next let us show that

$$\bigcap_{i=1}^N F(S_i) \cap \Omega \subset Q_n \quad \text{for all } n \geq 0. \tag{5.3}$$

We prove this by induction. For $n = 0$, we have $\bigcap_{i=1}^N F(S_i) \cap \Omega \subset C = Q_0$. Assume that $\bigcap_{i=1}^N F(S_i) \cap \Omega \subset Q_n$ for some $n > 0$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $\bigcap_{i=1}^N F(S_i) \cap \Omega \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in \bigcap_{i=1}^N F(S_i) \cap \Omega$. This together with the definition of Q_{n+1} implies that $\bigcap_{i=1}^N F(S_i) \cap \Omega \subset Q_{n+1}$. Hence (5.3) holds for all $n \geq 0$.

Notice that the definition of Q_n actually implies $x_n = P_{Q_n} x_0$. This together with that fact $\bigcap_{i=1}^N F(S_i) \cap \Omega \subset Q_n$ further implies

$$\|x_n - x_0\| \leq \|u - x_0\|, \quad \forall u \in \bigcap_{i=1}^N F(S_i) \cap \Omega.$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q = P_{\bigcap_{i=1}^N F(S_i)\cap\Omega} x_0. \tag{5.4}$$

The fact that $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. This together with Lemma 2.1 (i) implies

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

It turns out that

$$\|x_{n+1} - x_n\| \to 0. \tag{5.5}$$

By the fact $x_{n+1} \in C_n$ we get

$$\|x_{n+1} - z_n\|^2 \leq \|x_{n+1} - t_n\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|t_n - T_n t_n\|^2. \tag{5.6}$$

Moreover, since $z_n = \alpha_n t_n + (1-\alpha_n)T_n t_n$, we deduce that

$$\begin{aligned} \|x_{n+1} - z_n\|^2 &= \alpha_n \|x_{n+1} - t_n\|^2 \\ +(1-\alpha_n)\|x_{n+1} - T_n t_n\|^2 &- \alpha_n(1-\alpha_n)\|t_n - T_n t_n\|^2. \end{aligned} \tag{5.7}$$

Substituting (5.7) into (5.6) we get

$$(1-\alpha_n)\|x_{n+1} - T_n t_n\|^2 \leq (1-\alpha_n)\|x_{n+1} - t_n\|^2 + (1-\alpha_n)\kappa\|t_n - T_n t_n\|^2.$$

Since $\alpha_n < 1$ for all n, the last inequality becomes

$$\|x_{n+1} - T_n t_n\|^2 \leq \|x_{n+1} - t_n\|^2 + \kappa\|t_n - T_n t_n\|^2. \tag{5.8}$$

But, on the other hand, we compute

$$\|x_{n+1} - T_n t_n\|^2 = \|x_{n+1} - t_n\|^2 + 2\langle x_{n+1} - t_n, t_n - T_n t_n\rangle + \|t_n - T_n t_n\|^2. \tag{5.9}$$

Combining (5.9) with (5.8) we obtain

$$(1-\kappa)\|t_n - T_n t_n\|^2 \leq -2\langle x_{n+1} - t_n, t_n - T_n t_n\rangle.$$

Therefore,

$$\|t_n - T_n t_n\| \leq \frac{2}{1-\kappa}\|x_{n+1} - t_n\|. \tag{5.10}$$

Furthermore, from $\|x_n - y_n\| \to 0$ it follows that

$$\|t_n - y_n\| = \|P_C(x_n - \lambda_n A y_n) - P_C(x_n - \lambda_n A x_n)\| \leq \lambda_n k \|x_n - y_n\|.$$

This implies that $\lim_{n\to\infty} \|y_n - t_n\| = 0$. From $\|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\|$, we also have $\lim_{n\to\infty} \|x_n - t_n\| = 0$. Consequently, from (5.10) we derive

$$\|t_n - T_n t_n\| \leq \frac{2}{1-\kappa}\|x_{n+1} - t_n\| \leq \frac{2}{1-\kappa}[\|x_{n+1} - x_n\| + \|x_n - t_n\|] \to 0.$$

Utilizing the Lipschitz continuity of T_n, we have

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - t_n\| + \|t_n - T_n t_n\| + \|T_n t_n - T_n x_n\| \\ &\leq \|t_n - T_n t_n\| + \frac{2}{1-\kappa}\|t_n - x_n\|, \end{aligned}$$

which hence implies that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

As in Step 4 of the proof of Theorem 3.1 we can deduce that $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i) \cap \Omega$. Then by virtue of (5.4) and Lemma 2.4, we conclude that $x_n \to q$ as $n \to \infty$, where $q = P_{\bigcap_{i=1}^N F(S_i) \cap \Omega} x_0$. ∎

Regarding the cyclic-extragradient algorithm (4.1), we have the following modification which has strong convergence.

Theorem 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $A : C \to H$ be a monotone, k-Lipschitz continuous mapping. Given a positive integer $N \geq 1$, let, for each $0 \leq i \leq N-1$, $S_i : C \to C$ be a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$ such that $\bigcap_{i=0}^{N-1} F(S_i) \cap \Omega \neq \emptyset$. Let $\kappa = \max\{\kappa_i : 0 \leq i \leq N-1\}$. Given any $x_0 \in C$, let $\{x_n\}_{n=0}^\infty, \{y_n\}_{n=0}^\infty$ be the sequences generated by the following algorithm*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ z_n = \alpha_n t_n + (1-\alpha_n) S_{[n]} t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|t_n - z\|^2 \\ \qquad\qquad -(1-\alpha_n)(\alpha_n - \kappa)\|t_n - S_{[n]} t_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases} \tag{5.11}$$

where there hold the following conditions

(*i*) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$*;*

(*ii*) $0 \leq \alpha_n < 1$ *for all* $n \geq 0$.

Then, the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{\bigcap_{i=1}^N F(S_i) \cap \Omega} x_0$ provided $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. The proof of this theorem is similar to that of Theorem 5.1. The main points include

(i) x_n is well defined for all $n \geq 1$;

(ii) $\|x_n - x_0\| \leq \|q - x_0\|$ for all n, where $q = P_{\bigcap_{i=0}^{N-1} F(S_i) \cap \Omega} x_0$;

(iii) $\|x_{n+1} - x_n\| \to 0$;

(iv) $\|t_n - S_{[n]} t_n\| \to 0$ and $\|x_n - S_{[n]} x_n\| \to 0$;

(v) $\omega_w(x_n) \subset \bigcap_{i=0}^{N-1} F(S_i) \cap \Omega$;

(vi) $x_n \to q$.

To prove (i)-(iv), one simply replaces T_n with $S_{[n]}$ in the proof of Theorem 5.1. One can prove (v) by repeating the argument in Step 4 of the proof of Theorem 4.1. Finally the strong convergence to q of $\{x_n\}$ is the consequence of (ii), (v) and Lemma 2.4. ∎

Remark 5.1. As in Sections 3 and 4, we can derive the corresponding corollaries from Theorems 5.1 and 5.2, respectively.

References

1. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
2. R. Glowinski, J. L. Lions and R. Tremoliere, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, New York, 1981.
3. P. Jaillet, D. Lamberton and B. Lapeyre, Variational inequalities and the pricing of American options, *Acta Appl. Math.*, **21** (1990), 263-289.
4. J. T. Oden, *Qualitative Methods on Nonlinear Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey, 1986.
5. E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, III: Variational Methods and Applications, Springer, New York, 1985.
6. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.*, **20** (1967), 197-228.
7. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.
8. G. M. Korpelevich, An extragradient method for finding saddle points and for other problems, *Ekonomika i Matematicheskie Metody*, **12** (1976), 747-756.
9. K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, in: *Cambridge Studies in Advanced Mathematics*, Vol. 28, Cambridge University Press, 1990.
10. W. Takahashi and M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **118** (2003), 417-428.
11. N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **128** (2006), 191-201.
12. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, **13** (2003), 938-945.
13. L. C. Zeng and J. C. Yao, Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems, *Taiwan. J. Math.*, **10** (2006), 1293-1303.

14. L. C. Ceng and J. C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, *Appl. Math. Comput.*, **190** (2007), 205-215.

15. L. C. Zeng, N. C. Wong and J. C. Yao, Convergence analysis of modified hybrid steepest-descent methods with variable parameters for variational inequalities, *J. Optim. Theory Appl.*, **132** (2007), 51-69.

16. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Tran. Amer. Math. Soc.*, **149** (1970), 75-88.

17. G. Marino and H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, *J. Math. Anal. Appl.*, **329** (2007), 336-346.

18. C. Matinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point processes, *Nonlinear Anal.*, **64** (2006), 2400-2411.

19. G. L. Acedo and H. K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.*, **67** (2007), 2258-2271.

20. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240-256.

21. L. C. Zeng and J. C. Yao, Implicit iteration process with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.*, **64** (2006), 2507-2515.

22. L. C. Ceng, P. Cubiotti and J. C. Yao, Strong convergence theorems for finitely many nonexpansive mappings and applications, *Nonlinear Anal.*, **67** (2007), 1464-1473.

23. O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, *J. Math. Anal. Appl.*, **194** (1991), 911-933.

24. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems*, **20** (2004), 103-120.

25. I. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer, Berlin, Germany, 2001.

26. S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.*, **67** (1979), 274-276.

27. L. C. Ceng and H. K. Xu, Strong convergence of a hybrid viscosity approximation method with perturbed mappings for nonexpansive and accretive operators, *Taiwan. J. Math.*, **11** (2007), 661-682.

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