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SOME CHARACTERIZATIONS FOR VECTOR VARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, a class of η-PPM mappings F where F and $-F$ are both η-pseudomonotone are introduced, which are proper generalizations of the PPM mappings considered by Bianchi and Schaible. The solution sets of two kinds of vector variational-like inequality problems involving η-PPM mappings are characterized in Banach spaces. Furthermore, the solution sets of two classes of vector variational-like inequalities involving set-valued mappings are also characterized via the scalarization approach due to Konnov.

1. INTRODUCTION

The concept of vector variational inequality (for short, VVI) was first introduced by Giannessi [9] in the setting of finite-dimensional Euclidean space. In recent years, (VVIs) have been studied extensively by many authors (see, for example, [1, 3, 4, 7, 10, 11, 13, 19-23, 28] and the references therein).

At the same time, there has been an increasing interest in generalizations of monotonicity in connection with variational inequality problems. It has been found that the rigid assumption of monotonicity can be relaxed in different ways without losing some of the valuable properties of these models (see, for example, [2, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20, 24-27]). In [15], Karamardian discussed the pseudomonotone mapping F for which $-F$ is also pseudomonotone. In case of a gradient mapping $F = \nabla f$, such mappings characterize pseudolinear functions f

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which have been studied extensively (see [5, 14, 18, 24]). Recently, Bianchi and Schaible [2] extended results for pseudolinear functions to mappings where both F and $-F$ are pseudomonotone and studied variational inequality problems involving such mappings.

Motivated and inspired by the above works, in this paper, we introduce a class of η-PPM mappings F where F and $-F$ are both η-pseudomonotone, which are proper generalizations of the PPM mappings considered by Bianchi and Schaible [2]. We give some characterizations for the solution sets of two kinds of vector variational-like inequality problems involving η-PPM mappings in Banach spaces. We also show some characterizations for the solution sets of two classes of vector variational-like inequalities involving set-valued mappings via the scalarization approach due to Konnov [19]. The results presented in this paper extend and improve some recent results in the literature.

2. Preliminaries

Let Y be a real Banach space. A nonempty subset P of Y is said to be a cone if $\lambda P \subseteq P$ for all $\lambda > 0$. A cone P is said to be convex if $P + P = P$. We say that a cone P is pointed if $P \cap (-P) = \{0\}$. An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a closed, convex and pointed cone $P \subseteq Y$ with apex at the origin, in the form of

$$x \geq y \Leftrightarrow x - y \in P, \quad \forall x, y \in Y,$$

and

$$x \ngeq y \Leftrightarrow x - y \notin P, \quad \forall x, y \in Y.$$

If the interior of P, say int$P$, is nonempty, then a weak ordering in Y is also defined by

$$y < x \Leftrightarrow x - y \in \text{int}P, \quad \forall x, y \in Y,$$

and

$$y \nless x \Leftrightarrow x - y \notin \text{int}P, \quad \forall x, y \in Y.$$

Remark that, for any $x, y \in Y$,

$$x \geq y \Leftrightarrow y \leq x; \quad x \ngeq y \Leftrightarrow y \nleq x; \quad y < x \Leftrightarrow x > y; \quad y \nless x \Leftrightarrow x \ngtr y.$$

It is easy to see the following lemma is true.

Lemma 2.1. *Let (Y, P) be an ordered Banach space induced by a closed, convex and pointed cone P. Then,*

(i) $x \geq 0$ *and* $x \leq 0$ *imply that* $x = 0$, *for all* $x \in Y$*;*

(ii) $x \geq y$ *and* $y \geq 0$ *imply that* $x \geq 0$, *for all* $x \in Y$.

Throughout this paper, let X be a real Banach space and X^* its dual space. Let K be a nonempty, closed and convex set of X, and let (Y, P) be an ordered Banach space induced by a closed, convex and pointed cone P. Denote by $L(X, Y)$ the space of all the continuous linear mappings from X to Y, and by $\langle l, x \rangle$ the value of $l \in L(X, Y)$ at $x \in X$. Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping, $F : K \to L(X, Y)$ a mapping, and $\eta : K \times K \to X$ a bifunction. In this paper, we consider the following four kinds of vector variational inequality problems:

Generalized Hartman and Stampacchia Vector Variational-like Inequality Problem (for short, GHSVVLIP) is the problem of finding $x^* \in K$ such that

$$\exists t^* \in T(x^*) : \langle t^*, \eta(y, x^*) \rangle \not< 0, \quad \forall y \in K;$$

Generalized Vector Variational-like Inequality Problem (for short, GVVLIP) is the problem of finding $x^* \in K$ such that

$$\forall y \in K, \exists t^* \in T(x^*) : \langle t^*, \eta(y, x^*) \rangle \not< 0;$$

Hartman and Stampacchia Vector Variational-like Inequality Problem (for short, HSVVLIP) is the problem of finding $x^* \in K$ such that

$$\langle F(x^*), \eta(y, x^*) \rangle \geq 0, \quad \forall y \in K;$$

Minty Vector Variational-like Inequality Problem (for short, MVVLIP) is the problem of finding $x^* \in K$ such that

$$\langle F(y), \eta(y, x^*) \rangle \geq 0, \quad \forall y \in K.$$

We denote by S_{GHSV}, S_{GVV}, S_{HSV} and S_{MV} the solution set of (GHSVVLIP), (GVVLIP), (HSVVLIP) and (MVVLIP), respectively. Clearly, $S_{GHSV} \subseteq S_{GVV}$. If $\eta(y, x) = y - x$ for all $x, y \in K$, then (HSVVLIP) and (MVVLIP) reduce to the following vector variational inequality problems, respectively:

Hartman and Stampacchia Vector Variational Inequality Problem (for short, HSVVIP) is the problem of finding $x^* \in K$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K;$$

Minty Vector Variational Inequality Problem (for short, MVVIP) is the problem of finding $x^* \in K$ such that

$$\langle F(y), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

We first recall some definitions and lemmas which will be needed in the main results of this paper.

Definition 2.1. Let $F : K \to L(X, Y)$ be a mapping, and $\eta : K \times K \to X$ a bifunction. F is said to be

(i) η-monotone on K if for any $x, y \in K$, we have

$$\langle F(y) - F(x), \eta(y, x) \rangle \geq 0;$$

(ii) η-pseudomonotone on K if for any $x, y \in K$, we have

$$\langle F(x), \eta(y, x) \rangle \geq 0 \Rightarrow \langle F(y), \eta(y, x) \rangle \geq 0;$$

(iii) η-PPM on K if both F and $-F$ are η-pseudomonotone on K;

(iv) η-G on K if it is η-PPM on K, and there exists a positive function $k(x, y)$ on $K \times K$ such that, for any $x, y \in K$,

$$\langle F(x), \eta(y, x) \rangle = 0 \Rightarrow F(x) = k(x, y)F(y).$$

Remark 2.1. It is easy to see that η-monotonicity implies that η-pseudomonotonicity, and by Definition 2.1 (iv), η-G mappings are η-PPM mappings. However, the reverse is not true.

Example 2.1. Let $X = R, K = [0, +\infty), Y = R^2, P = [0, +\infty) \times [0, +\infty)$. Let

$$F(x) = \begin{pmatrix} x \\ x^2 + 1 \end{pmatrix}$$

and $\eta(y, x) = x^2 - y^2$ for all $x, y \in K$. Let

$$\langle F(x), \eta(y, x) \rangle = \begin{pmatrix} x(x^2 - y^2) \\ (x^2 + 1)(x^2 - y^2) \end{pmatrix}$$

for all $x, y \in K$. It is easy check that F is η-pseudomonotone on K. However, F is not η-monotone on K. For instance, if let $x = 1$ and $y = 0$, then,

$$\langle F(0) - F(1), \eta(0, 1) \rangle = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \ngeq 0.$$

Example 2.2. Let $X = R, K = [0, +\infty), Y = R^2, P = [0, +\infty) \times [0, +\infty)$. Let

$$F(x) = \begin{pmatrix} 0 \\ -x \end{pmatrix}$$

and $\eta(y,x) = xy$ for all $x, y \in K$. Let

$$\langle F(x), \eta(y,x)\rangle = \begin{pmatrix} 0 \\ -x^2 y \end{pmatrix}$$

for all $x, y \in K$. It is easy check that F is η-PPM on K. However, F is not η-G on K. In fact, letting $x = 0$ and $y = 1$, then

$$\langle F(0), \eta(1,0)\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$F(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad F(1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

If $L(X,Y) = X^*$, then we have the following conclusion.

Theorem 2.1. *Let $F : K \to X^*$ be a mapping, and let $\eta : K \times K \to X$ be a bifunction such that $\eta(y,x) + \eta(x,y) = 0$ for all $x, y \in K$. Then F is η-pseudomonotone on K if and only if for any distinct $x, y \in K$, $\langle F(x), \eta(y,x)\rangle > 0 \Rightarrow \langle F(y), \eta(y,x)\rangle > 0$.*

Proof. It is easy and so we omit it.

Remark 2.2. (i) It is clear that, η-monotone, η-pseudomonotone, η-PPM, and η-G mappings reduce to monotone, pseudomonotone, PPM, and G mappings, respectively, provided $X = L(X,Y) = R^n$ and $\eta(y,x) = y - x$, for all $x, y \in K$, which have been studied by Bianchi and Schaible [2]; (ii) In the case of $X = L(X,Y) = R^n$, the η-pseudomonotonicity on K of F reduces to the pseudo invex monotonicity on K of F defined by Garzon, Gomez and Lizana [8]; (iii) If $\eta(y,x) = y - x$, then the pseudo invex monotonicity on K of F collapses to the pseudomonotonicity on K of F defined by Karamardian and Schaible [16].

Remark 2.3. Theorem 2.1 is a generalization of the corresponding result of [16].

Lemma 2.2. [17] *Let A be a nonempty convex set in a vector space and let B be a nonempty compact convex set in a Hausdorff topological vector space. Suppose that g is a real-valued function on $A \times B$ such that for each fixed $a \in A$, $g(a,\cdot)$ is lower semicontinuous and convex on B, and for each fixed $b \in B$, $g(\cdot,b)$ is concave on A. Then*

$$\min_{b\in B}\sup_{a\in A} g(a,b) = \sup_{a\in A}\min_{b\in B} g(a,b).$$

3. Characterizations of (HSVVLIP) and (MVVLIP) Involving η-PPM Mappings

In this section, we derive some characterizations of solution sets for (HSVVLIP) and (MVVLIP) involving η-PPM mappings. We first present some necessary conditions for a mapping to be η-PPM.

Theorem 3.1. *Let F be an η-PPM mapping. Then for any $x, y \in K$,*

$$\langle F(x), \eta(y, x)\rangle = 0 \Rightarrow \langle F(y), \eta(y, x)\rangle = 0.$$

Proof. Let F be an η-PPM mapping and $x, y \in K$ such that $\langle F(x), \eta(y, x)\rangle = 0$. Then η-pseudomonotonicity on K of F and $-F$ imply that

$$\langle F(y), \eta(y, x)\rangle \geq 0 \quad \text{and} \quad \langle -F(y), \eta(y, x)\rangle \geq 0.$$

It folows from Lemma 2.1 (i) that $\langle F(y), \eta(y, x)\rangle = 0$. This completes the proof.

Corollary 3.1. *Let F be an η-PPM mapping. If $F(x^*) = 0$, then $\langle F(y), \eta(y, x^*)\rangle = 0$ for all $y \in K$.*

Now, we define a set-valued mapping $M : K \times K \to 2^K$ by

$$M(x, y) = \{z \in K : \eta(z, x) = t\eta(y, x), \quad t \in R \quad \text{with} \quad t \neq 0\}, \quad \forall (x, y) \in K \times K.$$

Clearly, $M(x, y) \neq \emptyset$ for all $(x, y) \in K \times K$. In fact, let $t = 1$, we have $y \in M(x, y)$.

Corollary 3.2. *Let F be an η-PPM mapping. Then for any $x, y \in K$ and $z \in M(x, y)$,*

$$\langle F(x), \eta(y, x)\rangle = 0 \Rightarrow \langle F(z), \eta(y, x)\rangle = 0.$$

Proof. Let F be an η-PPM mapping. Let $x, y \in K$ such that $\langle F(x), \eta(y, x)\rangle = 0$. For any $z \in M(x, y)$, we have $\eta(z, x) = t\eta(y, x)$ for some $t \in R$ with $t \neq 0$. Then, $\langle F(x), \frac{1}{t}\eta(z, x)\rangle = 0$, and hence $\langle F(x), \eta(z, x)\rangle = 0$. Since F is an η-PPM mapping, from Theorem 3.1, it follows that $\langle F(z), \eta(z, x)\rangle = 0$, that is, $\langle F(z), t\eta(y, x)\rangle = 0$ and consequently $\langle F(z), \eta(y, x)\rangle = 0$. This proof is complete.

Next, we characterize the solution sets of (HSVVLIP) and (MVVLIP) involving η-PPM mappings.

Theorem 3.2. *Let F be an η-PPM mapping with $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Then $S_{HSV} = S_{MV}$.*

Proof. If $x^* \in S_{HSV}$, then $\langle F(x^*), \eta(y, x^*)\rangle \geq 0$ for all $y \in K$. From η-pseudomonotonicity on K of F, we have $\langle F(y), \eta(y, x^*)\rangle \geq 0$ for all $y \in K$. Thus,

$x^* \in S_{MV}$. Conversely, if $x^* \in S_{MV}$, then $\langle F(y), \eta(y, x^*)\rangle \geq 0$ for all $y \in K$. Since $\eta(x^*, y) + \eta(y, x^*) = 0$ for all $y \in K$, it follows that

$$\langle F(y), -\eta(x^*, y)\rangle \geq 0, \quad \forall y \in K$$

or equivalently,

$$\langle -F(y), \eta(x^*, y)\rangle \geq 0, \quad \forall y \in K.$$

Now the η-pseudomonotonicity on K of $-F$ implies that $\langle -F(x^*), \eta(x^*, y)\rangle \geq 0$ for all $y \in K$ and so $\langle F(x^*), \eta(y, x^*)\rangle \geq 0$ for all $y \in K$. Thus, $x^* \in S_{HSV}$. This proof is complete.

Remark 3.1. As is well known, the equivalence between (HSVVLIP) and (MVVLIP) can be established in the case of the η-pseudomonotonicity on K and hemicontinuity of F. It is clear that the method adopted in Theorem 3.2 is much different. In detail, the hemicontinuity of F is replaced by the η-pseudomonotonicity on K of $-F$ in Theorem 3.2.

Example 3.1. Let $X = R$ and $K = R$. Let $\eta : K \times K \to X$ be a mapping defined by

$$\eta(x, y) = \begin{cases} x - y, & \text{if} \quad x \geq 0, y \geq 0, \\ -x + y, & \text{if} \quad x < 0, y < 0, \\ x + y, & \text{if} \quad x \geq 0, y < 0, \\ -x - y, & \text{if} \quad x < 0, y \geq 0 \end{cases}$$

for all $x, y \in K$. Then it is easy to verify that $\eta(x, y) + \eta(y, x) = 0$ holds for all $x, y, z \in K$.

Theorem 3.3. *Let F be an η-PPM mapping with $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Then*

$$S_{HSV} \subseteq S(x^*) = \{x \in K : \langle F(x^*), \eta(x, x^*)\rangle = 0\}, \quad \forall x^* \in S_{HSV}.$$

Furthermore,

$$S(x^*) = \{x \in K : \langle F(z), \eta(x, x^*)\rangle = 0, \quad \forall z \in M(x, x^*)\}.$$

Proof. If $x \in S_{HSV}$, then $\langle F(x), \eta(x^*, x)\rangle \geq 0$. By η-pseudomonotonicity on K of F, we obtain $\langle F(x^*), \eta(x^*, x)\rangle \geq 0$. Since $\eta(x, x^*) + \eta(x^*, x) = 0$, it follows that $-\langle F(x^*), \eta(x, x^*)\rangle \geq 0$. It is obvious that $\langle F(x^*), \eta(x, x^*)\rangle \geq 0$ for all $x^* \in S_{HSV}$. Now Lemma 2.1 (i) implies that $\langle F(x^*), \eta(x, x^*)\rangle = 0$ and so $x \in S(x^*)$. It follows that

$$S_{HSV} \subseteq S(x^*) = \{x \in K : \langle F(x^*), \eta(x, x^*)\rangle = 0\}, \quad \forall x^* \in S_{HSV}.$$

Furthermore, let $x \in S(x^*)$. Then $\langle F(x^*), \eta(x, x^*)\rangle = 0$ and Theorem 3.1 implies that

$$\langle F(x), \eta(x, x^*)\rangle = 0.$$

Again since $\eta(x, x^*) + \eta(x^*, x) = 0$, it follows that $\langle F(x), \eta(x^*, x)\rangle = 0$ and Corollary 3.2 implies that

$$\langle F(z), \eta(x^*, x)\rangle = 0, \quad \forall z \in M(x, x^*),$$

or equivalently,

$$\langle F(z), \eta(x, x^*)\rangle = 0, \quad \forall z \in M(x, x^*).$$

Thus, $x \in \{x \in K : \langle F(z), \eta(x, x^*)\rangle = 0, \quad \forall z \in M(x, x^*)\}$ and so

$$S(x^*) \subseteq \{x \in K : \langle F(z), \eta(x, x^*)\rangle = 0, \quad \forall z \in M(x, x^*)\}.$$

Since $x^* \in M(x, x^*)$, we have

$$\begin{aligned}&\{x \in K : \langle F(z), \eta(x, x^*)\rangle = 0,\\ &\forall z \in M(x, x^*)\} \subseteq \{x \in K : \langle F(x^*), \eta(x, x^*)\rangle = 0\} = S(x^*).\end{aligned}$$

Thus the conclusion holds. This proof is complete.

From Theorems 3.2 and 3.3, we obtain the following result.

Corollary 3.3. *Let F be an η-PPM mapping with $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$. Then $S_{MV} \subseteq S(x^*)$ for all $x^* \in S_{HSV}$.*

Now, we characterize the solution sets of (HSVVLIP) and (MVVLIP) involving η-G mappings, respectively.

Theorem 3.4. *Let F be an η-G mapping with $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{HSV} = S(x^*)$ for all $x^* \in S_{HSV}$.*

Proof. Since $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$, it is easy to check that $\eta(x, x) = 0$ for all $x \in K$ and so $\eta(x, y) + \eta(y, x) = 0$ holds for all $x, y \in K$. Now from Theorem 3.3, we have $S_{HSV} \subseteq S(x^*)$ for all $x^* \in S_{HSV}$. We only need to show that $S(x^*) \subseteq S_{HSV}$ for all $x^* \in S_{HSV}$. If $x \in S(x^*)$, then $\langle F(x^*), \eta(x, x^*)\rangle = 0$. By considering Theorem 3.1, we have $\langle F(x), \eta(x, x^*)\rangle = 0$. It follows from $\eta(x, x^*)+\eta(x^*, x) = 0$ that $\langle F(x), \eta(x^*, x)\rangle = 0$. Since F is an η-G mapping, we deduce that $F(x^*) = k(x^*, x)F(x)$. Since $\langle F(x^*), \eta(y, x^*)\rangle \geq 0$ for all $y \in K$, we obtain $\langle F(x), \eta(y, x^*)\rangle \geq 0$. Again since $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$,

$$\begin{aligned}\langle F(x), \eta(y, x)\rangle &= \langle F(x), \eta(y, x^*)\rangle + \langle F(x), \eta(x^*, x)\rangle \\ &= \langle F(x), \eta(y, x^*)\rangle + 0 \\ &= \langle F(x), \eta(y, x^*)\rangle \\ &\geq 0\end{aligned}$$

for all $y \in K$, i.e., $x \in S_{HSV}$. This proof is complete.

Example 3.2. Let $X = R^2$, $K = [0, +\infty) \times [0, +\infty)$, and $\eta : K \times K \to X$ be defined by

$$\eta(x, y) = (x_1 - y_1, x_2 - y_2)$$

for all $x = (x_1, x_2) \in K$ and $y = (y_1, y_2) \in K$. Then it is easy to verify that $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$.

From Theorems 3.2 and 3.4, we have the following conclusion.

Corollary 3.4. *Let F be an η-G mapping with $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{MV} = S(x^*)$ for all $x^* \in S_{HSV}$.*

Theorem 3.5. *Let F be an η-G mapping with $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{HSV} = S_1(x^*)$ for all $x^* \in S_{HSV}$, where*

$$\begin{aligned}S_1(x^*) &= \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\} \\ &= \{x \in K : \langle F(x), \eta(x^*, x)\rangle \geq 0\} \\ &= \{x \in K : \langle F(z), \eta(x^*, x)\rangle \geq 0, \quad \forall z \in M(x, x^*)\}.\end{aligned}$$

Proof. From Theorem 3.4, we know that $S_{HSV} = S(x^*)$ for all $x^* \in S_{HSV}$. Since $\eta(x, x^*) + \eta(x^*, x) = 0$, from Theorem 3.3, we have

$$\begin{aligned}S(x^*) &= \{x \in K : \langle F(x^*), \eta(x, x^*)\rangle = 0\} \\ &= \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle = 0\} \\ &\subseteq \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\}\end{aligned}$$

and

$$\begin{aligned}S(x^*) &= \{x \in K : \langle F(x), \eta(x, x^*)\rangle = 0\} \\ &= \{x \in K : \langle F(x), \eta(x^*, x)\rangle = 0\} \\ &\subseteq \{x \in K : \langle F(x), \eta(x^*, x)\rangle \geq 0\}\end{aligned}$$

and

$$\begin{aligned} S(x^*) &= \{x \in K : \langle F(z), \eta(x, x^*)\rangle = 0, \quad \text{for all} \quad z \in M(x, x^*)\} \\ &= \{x \in K : \langle F(z), \eta(x^*, x)\rangle = 0, \quad \text{for all} \quad z \in M(x, x^*)\} \\ &\subseteq \{x \in K : \langle F(z), \eta(x^*, x)\rangle \geq 0, \quad \text{for all} \quad z \in M(x, x^*)\}. \end{aligned}$$

By choosing $z = x^*$ and considering the η-pseudomonotonicity on K of $-F$ and $\eta(x, x^*) + \eta(x^*, x) = 0$, we have

$$\begin{aligned} &\{x \in K : \langle F(z), \eta(x^*, x)\rangle \geq 0, \quad \text{for all} \quad z \in M(x, x^*)\} \\ \subseteq\ &\{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\} \\ =\ &\{x \in K : \langle -F(x^*), \eta(x, x^*)\rangle \geq 0\} \\ \subseteq\ &\{x \in K : \langle -F(x), \eta(x, x^*)\rangle \geq 0\} \\ =\ &\{x \in K : \langle F(x), \eta(x^*, x)\rangle \geq 0\}. \end{aligned}$$

From the η-pseudomonotonicity on K of F, we obtain

$$\begin{aligned} &\{x \in K : \langle F(x), \eta(x^*, x)\rangle \geq 0\} \\ \subseteq\ &\{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\}. \end{aligned}$$

It follows that

$$\begin{aligned} S(x^*) &\subseteq \{x \in K : \langle F(z), \eta(x^*, x)\rangle \geq 0, \quad \text{for all} \quad z \in M(x, x^*)\} \\ &\subseteq \{x \in K : \langle F(x), \eta(x^*, x)\rangle \geq 0\} \\ &= \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\}. \end{aligned}$$

Since $x^* \in S_{HSV}$, that is, $\langle F(x^*), \eta(x, x^*)\rangle \geq 0$ for all $x \in K$, and $\eta(x, x^*) + \eta(x^*, x) = 0$, one has $-\langle F(x^*), \eta(x^*, x)\rangle \geq 0$. By Lemma 2.1 (i), it follows that

$$\{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq 0\} \subseteq S(x^*)$$

and thus the conclusion holds. This proof is complete.

From Theorems 3.2 and 3.5, we obtain the following.

Corollary 3.5. *Let F be an η-G mapping, $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{MV} = S_1(x^*)$ for all $x^* \in S_{HSV}$.*

Theorem 3.6. *Let F be an η-G mapping, $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{HSV} = S_2(x^*)$ for all $x^* \in S_{HSV}$, where*

$$\begin{aligned} S_2(x^*) &= \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle = \langle F(x), \eta(x, x^*)\rangle\} \\ &= \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq \langle F(x), \eta(x, x^*)\rangle\}. \end{aligned}$$

Proof. From Theorem 3.4, we know that $S_{HSV} = S(x^*)$ for all $x^* \in S_{HSV}$. Since $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$, from Theorem 3.3, we have that

$$\begin{aligned} S(x^*) &\subseteq \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle = \langle F(x), \eta(x, x^*)\rangle\} \\ &\subseteq \{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq \langle F(x), \eta(x, x^*)\rangle\}. \end{aligned}$$

To complete the proof, we show that any element $\bar{x}$ of $\{x \in K : \langle F(x^*), \eta(x^*, x)\rangle \geq \langle F(x), \eta(x, x^*)\rangle\}$ is contained in S_{HSV}. Since $x^* \in S_{HSV}$, we know that $\langle F(x^*), \eta(\bar{x}, x^*)\rangle \geq 0$ and so $\langle F(\bar{x}), \eta(\bar{x}, x^*)\rangle \geq 0$ because of η-pseudomonotonicity on K of F. It follows from Lemma 2.1 (ii) that $\langle F(x^*), \eta(x^*, \bar{x})\rangle \geq 0$. Since $\eta(\bar{x}, x^*) + \eta(x^*, \bar{x}) = 0$ and $-\langle F(x^*), \eta(\bar{x}, x^*)\rangle \geq 0$, again by Lemma 2.1 (i), we have $\langle F(x^*), \eta(\bar{x}, x^*)\rangle = 0$. Thus $\bar{x} \in S(x^*)$ and so $S(x^*) = S_2(x^*)$ for all $x^* \in S_{HSV}$. In view of Theorem 3.4, this implies that $S_{HSV} = S_2(x^*)$. This completes the proof.

Theorems 3.2 and 3.6 imply the following conclusion holds.

Corollary 3.6. *Let F be an η-G mapping, $\eta(x, y) = \eta(x, z) + \eta(z, y)$ for all $x, y, z \in K$. Then $S_{MV} = S_2(x^*)$ for all $x^* \in S_{HSV}$.*

Remark 3.2. The results presented in this section extend and improve some corresponding results of Jeyakumar and Yang [14], Bianchi and Schaible [2].

4. Characterizations of (GHSVVLIP) and (GVVLIP) via Scalarization Approach

In this section, we set $Y = R^n$,

$$P = R^n_+ = \{y = (y_i)_{i=1,\cdots,n} \in R^n : y_i \geq 0, i = 1, \cdots, n\},$$

$$\text{int}P = \text{int}R^n_+ = \{y = (y_i)_{i=1,\cdots,n} \in R^n : y_i > 0, i = 1, \cdots, n\},$$

and

$$T(x) = \prod_{i=1}^{n} T_i(x), \quad \text{where} \quad T_i : K \to 2^{X^*}.$$

We define the set-valued mapping $T_0 : K \to 2^{X^*}$ as follows:

$$T_0(x) = \text{conv}\{T_i(x)\}_{i=1,\cdots,n},$$

where the conv$\{T_i(x)\}_{i=1,\cdots,n}$ denotes the convex hull of $\{T_i(x)\}_{i=1,\cdots,n}$. Now, we consider two kinds of the scalar variational inequality problems:

Generalized Hartman and Stampacchia Variational-like Inequality Problem (for short, GHSVLIP) is the problem of finding $x^* \in K$ such that

$$\exists f^* \in T_0(x^*) : \langle f^*, \eta(y, x^*)\rangle \geq 0, \quad \forall y \in K;$$

Generalized Variational-like Inequality Problem (for short, GVLIP) is the problem of finding $x^* \in K$ such that

$$\forall y \in K, \exists f^* \in T_0(x^*) : \langle f^*, \eta(y, x^*)\rangle \geq 0.$$

We denote by S_{GHS} and S_{GV} the solution set of (GHSVLIP) and (GVLIP), respectively. Clearly, $S_{GHS} \subseteq S_{GV}$. We first establish an equivalent result between (GHSVLIP) and (GVLIP).

Theorem 4.1. *Assume that, for each $x \in K$, $T_i(x)$ is nonempty, convex and weakly* compact for $i = 1, \cdots, n$. Assume that, for each $x \in K$ and $f \in T_0(x)$, $\langle f, \eta(x, \cdot)\rangle$ is concave on K and assume that $\eta(x, y)+\eta(y, x) = 0$ for all $x, y \in K$. Then $S_{GHS} = S_{GV}$.*

Proof. Clearly, $S_{GHS} \subseteq S_{GV}$. Since $T_i(x)$, $i = 1, \cdots, n$, are nonempty, convex and weakly* compact, so is $T_0(x)$. If $x^* \in S_{GV}$, then we have

$$\forall y \in K, \exists f^* \in T_0(x^*) : \langle f^*, \eta(y, x^*)\rangle \geq 0.$$

Since $\eta(x^*, y) + \eta(y, x^*) = 0$, it follows that

$$\forall y \in K, \exists f^* \in T_0(x^*) : \langle f^*, \eta(x^*, y)\rangle \leq 0.$$

If set $g(a, b) = \langle b, \eta(x^*, a)\rangle$, $A = K$ and $B = T_0(x^*)$, then

$$\sup_{a\in A} \min_{b\in B} g(a, b) = \sup_{a\in K} \min_{b\in T_0(x^*)} \langle b, \eta(x^*, a)\rangle \leq 0.$$

It is clear that for each fixed $a \in A$, $g(a, \cdot)$ is continuous in the weak* topology of X^* and convex on B, and for each fixed $b \in B$, $g(\cdot, b)$ is concave on A. By Lemma 2.2, we have

$$\min_{b\in T_0(x^*)} \sup_{a\in K} \langle b, \eta(x^*, a)\rangle = \min_{b\in B} \sup_{a\in A} g(a, b) = \sup_{a\in A} \min_{b\in B} g(a, b) \leq 0.$$

Thus,

$$\exists f^* \in T_0(x^*) : \langle f^*, \eta(x^*, y)\rangle \leq 0, \quad \forall y \in K.$$

Considering again $\eta(x^*, y) + \eta(y, x^*) = 0$, we obtain

$$\exists f^* \in T_0(x^*) : \langle f^*, \eta(y, x^*)\rangle \geq 0, \quad \forall y \in K$$

and so $x^* \in S_{GHS}$. This completes the proof.

Now, we characterize the solution sets of (GHSVVLIP) and (GVVLIP) via (GHSVLIP) and (GVLIP).

Theorem 4.2. *Assume that, for each $x \in K$, $T_i(x)$ is nonempty, convex and weakly* compact for $i = 1, \cdots, n$. Assume that, for each $x \in K$ and $f \in T_0(x)$, $\langle f, \eta(x, \cdot)\rangle$ is concave on K and assume that $\eta(x, y)+\eta(y, x) = 0$ for all $x, y \in K$. Then $S_{GHSV} = S_{GVV} = S_{GHS} = S_{GV}$.*

Proof. From Theorem 4.1, we have $S_{GHS} = S_{GV}$. And clearly, $S_{GHSV} \subseteq S_{GVV}$. If $x^* \in S_{GVV}$, then for each $y \in K$, we have

$$\langle t^*, \eta(y, x^*)\rangle \not< 0$$

for some $t^* \in T(x^*)$, i.e., for some i, there is $t_i^* \in T_i(x^*)$ such that

$$\langle t_i^*, \eta(y, x^*)\rangle \geq 0$$

and so

$$\langle f^*, \eta(y, x^*)\rangle \geq 0$$

with $f^* = t_i^* \in T_0(x^*)$. Thus, $x^* \in S_{GV} = S_{GHS}$. If $x^* \in S_{GHS}$, then there exists $f^* \in T_0(x^*)$ such that

$$\langle f^*, \eta(y, x^*)\rangle \geq 0, \quad \forall y \in K.$$

Since $T_0(x) = \text{conv}\{T_i(x)\}_{i=1,\cdots,n}$, there exists a subset $I \subseteq \{1, \cdots, n\}$ and $t_i^* \in T_i(x), i \in I$ such that

$$f^* \in \text{conv}\{t_i^*\}_{i\in I}.$$

For each $y \in K$, there exists at least one index $i \in I$ such that

$$\langle t_i^*, \eta(y, x^*)\rangle \geq 0.$$

Choose arbitrary elements $t_j^* \in T_j(x^*), j \notin I$, and set $t^* = (t_k^*)_{k=1,\cdots,n} \in T(x^*)$. It follows that

$$\langle t^*, \eta(y, x^*)\rangle \not< 0 \qquad \text{for all} \qquad y \in K$$

and so $x^* \in S_{GHSV}$. Hence, $S_{GHSV} = S_{GVV} = S_{GHS} = S_{GV}$. This completes the proof.

Remark 4.1. If $\eta(y,x) = y - x$ for all $x, y \in K$, then Theorems 4.1 and 4.2 reduce to the corresponding results of Konnov [19].

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