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HAYMAN'S CONJECTURE IN A p-ADIC FIELD

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Abstract. In this paper we study the famous Hayman's conjecture for transcendental meromorphic functions in a p-adic field by using methods of p-adic analysis and particularly the p-adic Nevanlinna theory.

In $\mathbb{C}$, W. K. Hayman's stated that if f is a transcendental meromorphic function, then $f' + af^m$ has infinitely many zeros that are not zeros of f for each integer $m \geq 3$ and $a \in \mathbb{C} \setminus \{0\}$, which was proved in [2], [6], [8] and [11]. Here we examine the problem in an algebraically closed complete ultrametric field $\mathbb{K}$ of characteristic zero. Considering the function $f' + Tf^m$ with $T \in \mathbb{K}(x)$, we show that Hayman's statement holds for each $m \geq 5$ and $m = 1$. Further, if the residue characteristic of $\mathbb{K}$ is zero, then the statement holds for each positive integer m different from 2. We also examine the problem inside an "open" disc.

1. Introduction and Results

1.1 Definitions, Notations and Main Results

Throughout this paper, $\mathbb{K}$ will denote an algebraically closed field of characteristic zero, complete for an ultrametric absolute value. In $\mathbb{K}$, the valuation v is defined by a logarithm function $\log : \; v(x) = -\log|x|$.

We denote by $\mathcal{A}(\mathbb{K})$ the set of entire functions in $\mathbb{K}$ and by $\mathcal{M}(\mathbb{K})$ the set of meromorphic functions in $\mathbb{K}$, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$. Obviously, $\mathcal{M}(\mathbb{K})$ contains the field $\mathbb{K}(x)$ of rational functions. We remember that the elements in $\mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ are called *transcendental functions* and have infinitely many zeros or infinitely many poles.

Given $a \in \mathbb{K}$ and r_1, r_2 such that $0 < r_1 < r_2$, we denote by $\Gamma(a, r_1, r_2)$ the annulus

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$\{x \in \mathbb{K} : r_1 < |x-a| < r_2\}$, and given $r > 0$, we denote by $d(a, r^-)$ the open disc $\{x \in \mathbb{K} : |x-a| < r\}$, by $C(a, r)$ the circle $\{x \in \mathbb{K} : |x-a| = r\}$, and by $d(a, r) := d(a, r^-) \cup C(a, r)$ the closed disc. Consequently, we denote by $\mathcal{A}(d(a, r^-))$ the set of analytic functions in $d(a, r^-)$, i.e., the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a, r^-)$, and by $\mathcal{M}(d(a, r^-))$ the set of meromorphic functions inside $d(a, r^-)$, i.e., the field of fractions of $\mathcal{A}(d(a, r^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, r^-))$ the $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a, r^-))$ consisted of the bounded analytic functions $f \in \mathcal{A}(d(a, r^-))$, which satisfy $\sup_{n \in \mathbb{N}} |a_n| r^n < +\infty$, and by $\mathcal{M}_b(d(a, r^-))$ the field of fractions of $\mathcal{A}_b(d(a, r^-))$. Finally, we set $\mathcal{A}_u(d(a, r^-)) = \mathcal{A}(d(a, r^-)) \setminus \mathcal{A}_b(d(a, r^-))$ and $\mathcal{M}_u(d(a, r^-)) = \mathcal{M}(d(a, r^-)) \setminus \mathcal{M}_b(d(a, r^-))$.

The paper aims at studying Hayman's conjecture for transcendental meromorphic functions, first in a field of any residue characteristic and next in a field of residue characteristic zero. The problem is the following one: *let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and $T \in \mathbb{K}(x)$. Can we conclude that $f' + Tf^m$ has infinitely many zeros that are not zeros of f?*. Setting $g = \frac{1}{f}$, it is easily seen that the zeros of $f' + Tf^m$ which are not zeros of f are those of $g'g^{m-2} - T$. Thus, solving Hayman's conjecture is equivalent to answering the question whether, given $g \in \mathcal{M}(\mathbb{K})$ transcendental and $T \in \mathbb{K}(x)$, $g'g^n - T$ has infinitely many zeros.

Indeed, let

$$g(x) = \frac{1}{f(x)}.$$

Then,

$$\begin{aligned} f'(x) + Tf^m(x) &= \frac{-1}{[g(x)]^2}\, g'(x) + \frac{T}{[g(x)]^m} \\ &= \frac{-1}{[g(x)]^m}\, (g^{m-2} g'(x) - T), \end{aligned} \tag{1}$$

where we do $n = m - 2$.

The question has been studied in complex analysis for many years, considering $T = a \in \mathbb{C}$. In 1959, W. K. Hayman [8] proved that if g is a transcendental meromorphic function, $a \in \mathbb{C} \setminus \{0\}$ and $n \geq 3$, then $g'g^n - a$ has infinitely many zeros. Twenty years later, E. Mues [11] solved the case $n = 2$, and finally in 1995 W. Bergweiler and A. Eremenko [2], and independently H. H. Chen and M. L. Fang [6] proved that this also holds for $n = 1$, which completed the proof of Hayman's conjecture. Thus, in the complex case, we could deduce that $f' + af^m$ has infinitely many zeros which are not zeros of f when $m \geq 3$.

Remark 1. In $\mathbb{C}$, $f' + f^m$ may have no zero if $m = 1$ or $m = 2$ as shown by $f(x) = \exp(x)$ and $f(x) = \tan(-x)$ respectively.

In p-adic analysis, we can also obtain results in a similar problem. Before stating the main theorems, we have to recall some notations used in several works in p-adic analysis, particularly those used by A. Escassut in [7].

Given $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ $\big($resp. in $\mathcal{A}(d(0, R^-))\big)$ and $r > 0$ $\big($resp. $r \in]0, R[\big)$, we set

$$|f|(r) = \lim_{|x| \to r,\ |x| \neq r} |f(x)|.$$

Indeed, this limit exists and $|*|$ is an absolute value on $\mathcal{A}(\mathbb{K})$ $\big($resp. on $\mathcal{A}(d(0, R^-))\big)$. It has a natural continuation to $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0, R^-))\big)$ by setting $|f|(r) = \frac{|g|(r)}{|h|(r)}$ whenever $f = \frac{g}{h}$, $g,\ h \in \mathcal{A}(\mathbb{K})$ $\big($resp. $g,\ h \in \mathcal{A}(d(0, R^-))\big)$.

On the other hand, let $f = \sum_{n \in \mathbb{Z}} a_n x^n \in \mathcal{M}(\mathbb{K})$ and let $r > 0$. Consider f in the circle $C(0, r)$. We will denote by $\nu^+(f, r)$ $\big($resp. $\nu^-(f, r)\big)$ the biggest integer $i \in \mathbb{Z}$ $\big($resp. the smallest integer $i \in \mathbb{Z}\big)$ such that $v(a_i) - i \log r = \inf_{n \in \mathbb{Z}} v(a_n) - n \log r$. We will only write $\nu(f, r)$ when $\nu^+(f, r) = \nu^-(f, r)$.

Remark 2. We now have to recall certain classical properties of meromorphic functions $\big($see Chapter 23 [7]$\big)$. Let $f \in \mathcal{M}(d(0, R^-))$ and let $r \in]0, R[$.

(1) The difference between the number of zeros and that of poles of f in the circle $C(0, r)$, taking multiplicities into account, is equal to $\nu^+(f, r) - \nu^-(f, r)$.

(2) If f has zeros and poles in the closed disc $d(0, r')$, and has no zeros and no poles in the annuli $\Gamma(0, r', r'')$, then $\nu^+(f, r) = \nu^-(f, r)$ $\forall r \in]r', r''[$.

Throughout the paper we consider $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $R > 1$ an integer and $T = \frac{A}{B} \in \mathbb{K}(x)$ with $A, B \in \mathbb{K}[x]$ having no common zeros.

Theorem 1. *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental $\big($resp. Let $f \in \mathcal{M}_u(d(0, R^-))\big)$. If $\lim_{r \to +\infty} |T|(r) > 0$ $\Big($resp. $\lim_{r \to R} |T|(r) > \dfrac{1}{R}$ $\Big)$, then $f' + Tf$ has infinitely many zeros that are not zeros of f.*

Theorem 2. *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and $\deg(A) \geq \deg(B)$ $\big($resp. Let $f \in \mathcal{M}_u(d(0, R^-))\big)$. Let $m > 2$ be an integer. If $\limsup_{r \to +\infty} |f|(r) > 0$ $\big($resp. $\limsup_{r \to R} |f|(r) = +\infty\big)$, then $f' + Tf^m$ has infinitely many zeros that are not zeros of f.*

Corollary 1. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) \geq \deg(B)$ $\big($*resp. Let* $f \in \mathcal{M}_u(d(0, R^-))\big)$. *If* f *has a finite number of poles and* $m > 2$ *is an integer, then* $f' + Tf^m$ *has infinitely many zeros that are not zeros of* f.

Proof. Since f has a finite number of poles and f is a transcendental meromorphic function in $\mathbb{K}$ $\big($resp. $f \in \mathcal{M}_u(d(0, R^-))\big)$, then necessarily f has infinitely many zeros. Therefore, $\lim\limits_{r\to+\infty} |f|(r) = +\infty$ $\big($resp. $\lim\limits_{r\to R} |f|(r) = +\infty\big)$. So, by Theorem 2, we can deduce the corollary. ■

Corollary 2. *Let* $g \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) \geq \deg(B)$ $\big($*resp. Let* $g \in \mathcal{M}_u(d(0, R^-))\big)$. *If* g *has a finite number of zeros, then* $g'g^n - T$ *has infinitely many zeros for all* $n \in \mathbb{N}^*$.

Proof. Since g has a finite number of zeros, then $f = \frac{1}{g}$ has a finite number of poles. So, applying Theorem 2 to f with $m \geq 3$, and considering that $n = m - 2$, we can deduce the corollary. ■

Let $\widehat{\mathbb{K}}$ be an algebraic extension of the field $\mathbb{K}$. In the following lemma, which is very useful for the proofs of the following theorems, we will denote by $\widehat{d}(0, R^-)$ the open disc $\{x \in \widehat{\mathbb{K}} : |x| < R\}$ contained in $\widehat{\mathbb{K}}$.

Lemma 1. *Let* $f \in \mathcal{M}(d(0, R^-))$ *and let* $\widehat{f}$ *be the meromorphic function defined by* f *in* $\widehat{d}(0, R^-)$. *Then the zeros and the poles of* $\widehat{f}$ *in* $\widehat{d}(0, R^-)$ *are exactly the zeros and the poles of* f *in* $d(0, R^-)$*, taking multiplicities into account.*

Remark 3. We remember that, given a meromorphic function f in the open disc $d(0, R^-) \subset \mathbb{K}$, it is not always possible to find analytic functions $h,\ l$ in $d(0, R^-)$ without common zeros such that $f = \frac{h}{l}$, except if $\mathbb{K}$ is *spherically complete*, i.e., every decreasing filter on $\mathbb{K}$ has a center in $\mathbb{K}$ $\big($see Chapter 3 [7] and [10]$\big)$. In our case, $\mathbb{K}$ is an algebraically closed complete ultrametric field, therefore it admits a spherically complete algebraically closed extension $\widehat{\mathbb{K}}$ $\big($see Chapter 7 [7]$\big)$.

Now, in the field $\mathbb{K}$, consider $f \in \mathcal{M}(d(0, R^-))$. It obviously defines a function $\widehat{f} \in \mathcal{M}(\widehat{d}(0, R^-))$ in the field $\widehat{\mathbb{K}}$. And then, we may write $\widehat{f}$ in the form $\frac{h_0}{l_0}$ with $h_0,\ l_0 \in \mathcal{A}(\widehat{d}(0, R^-))$ having no common zeros. Moreover, by Lemma 1, all zeros and poles of $\widehat{f}$ in $\widehat{\mathbb{K}}$ actually lie in $\mathbb{K}$. So, by Theorem 25.5 [7], there exists $h \in \mathcal{A}(d(0, R^-))$ such that the function $\widehat{h} \in \mathcal{A}(\widehat{d}(0, R^-))$ defined in $\widehat{\mathbb{K}}$ satisfies the following:

(1) h_0 divides $\widehat{h}$ in $\mathcal{A}(\widehat{d}(0, R^-))$.

(2) The function $u = \frac{\widehat{h}}{h_0}$ belongs to $\mathcal{A}_b(\widehat{d}(0, R^-))$.

Then we may set $l = ul_0 \in \mathcal{A}(\widehat{d}(0, R^-))$. Moreover, we check that l has coefficients in $\mathbb{K}$ because $f = \frac{h}{l}$, hence $l = fh$ belongs to $\mathcal{M}(d(0, R^-))$ and has no pole in $d(0, R^-)$.

In the following theorems, when it is necessary, we shall consider $f \in \mathcal{M}(\widehat{d}(0, R^-))$ because clearly $\mathcal{M}(d(0, R^-)) \subset \mathcal{M}(\widehat{d}(0, R^-))$.

In the general p-adic context, the following theorem is the equivalence of this proved by W. K. Hayman (Theorem 9 [8]). In the proofs of this theorem and the following theorems, the previous Remark 3 and Lemma 1 will be useful.

Theorem 3. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) \geq \deg(B)$ (*resp. Let* $f \in \mathcal{M}_u(d(0, R^-))$). *If* $m \geq 5$ *is an integer, then* $f' + Tf^m$ *has infinitely many zeros that are not zeros of* f. *Moreover,* $f' + f^4$ *must have at least one zero in* $\mathbb{K}$ *that is not a zero of* f.

Considering (1) and the previous theorem, we obtain the following corollaries.

Corollary 3. *Let* $g \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) \geq \deg(B)$ (*resp. Let* $g \in \mathcal{M}_u(d(0, R^-))$). *If* $n \geq 3$ *is an integer, then* $g'g^n - T$ *has infinitely many zeros.*

Corollary 4. *Let* $g \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) \geq \deg(B)$. *Then* $g'g^2 - T$ *has at least one zero in* $\mathbb{K}$.

In order to state Theorem 4, we need to recall some classical definitions. Let $\mathcal{U}_{\mathbb{K}} = \{x \in \mathbb{K} : |x| \leq 1\}$ and $\mathcal{W}_{\mathbb{K}} = \{x \in \mathbb{K} : |x| < 1\}$ be the valuation ring and the valuation ideal of $\mathbb{K}$ respectively. The *residue characteristic* of $\mathbb{K}$ is the characteristic of the quotient of $\mathcal{U}_{\mathbb{K}}$ by $\mathcal{W}_{\mathbb{K}}$ (see Chapter 1 [7]).

Lemma 2. *Let* $f(x) = \sum_{-\infty}^{+\infty} a_n x^n$ *be a Laurent series converging for* $r' < |x| < r''$ *and have no zeros and no poles in* $\Gamma(0, r', r'')$. *Let* $q = \nu(f, r)\ \ \forall r \in]r', r''[$. *If the residue characteristic of* $\mathbb{K}$ *does not divide* q, *then*

$$|f'(x)| = \frac{|f(x)|}{|x|} \quad \forall x \in \Gamma(0, r', r'').$$

Corollary 5. *Let* $f \in \mathcal{M}(d(0, r''))$. *Assume that* f *has* s *zeros and* t *poles in* $d(0, r')$ *and has no zeros and no poles in* $\Gamma(0, r', r'')$. *If the residue characteristic of* $\mathbb{K}$ *does not divide* $s - t$, *then* $|f'(x)| = \frac{|f(x)|}{|x|}\ \ \forall x \in \Gamma(0, r', r'')$.

Proof. Indeed, by Theorem 23.4 [7], $\nu^+(f, r) = \nu^-(f, r)\ \ \forall r \in]r', r''[$. If we consider $f = \frac{h}{l}$ with $h, l \in \mathcal{A}(d(0, r''))$, we have

$$\nu(f, r) = \nu(h, r) - \nu(l, r) = s - t,$$

whenever $r \in]r', r''[$. So, by Lemma 2, we deduce the corollary. ∎

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{M}(d(0, R^-))\big)$.

A number $r \in]0, +\infty[$ $\big($resp. $r \in]0, R[\big)$ will be said to be *f-suitable* if the difference between the number of zeros and that of poles of f in $d(0, r)$, taking multiplicities into account, is not a multiple of the residue characteristic of $\mathbb{K}$.

A sequence $\{r_n\}_{n\in\mathbb{N}} \subset]0, +\infty[$ $\big($resp. $\{r_n\}_{n\in\mathbb{N}} \subset]0, R[\big)$ will be said to be *f-suitable* if each r_n is f-suitable and $\lim\limits_{n\to+\infty} r_n = +\infty$ $\big($resp. $\lim\limits_{n\to+\infty} r_n = R\big)$.

The function f will be said to be *optimal* if there exists a f-suitable sequence $\{r_n\}_{n\in\mathbb{N}}$ in $]0, +\infty[$ $\big($resp. in $]0, R[\big)$.

Theorem 4. *Let $g \in \mathcal{M}(\mathbb{K})$ be transcendental $\big($resp. Let $g \in \mathcal{M}_u(d(0, R^-))\big)$ and let $\{r_n\}_{n\in\mathbb{N}}$ be a g-suitable sequence. Then $\frac{g'}{g}$ has infinitely many zeros.*

Corollary 6. *Let $g \in \mathcal{M}(\mathbb{K})$ be transcendental $\big($resp. Let $g \in \mathcal{M}_u(d(0, R^-))\big)$ and let $\{r_n\}_{n\in\mathbb{N}}$ be a g-suitable sequence. Then $g'g^n$ and $\frac{g'}{g^n}$ have infinitely many zeros whenever $n \in \mathbb{N}^*$.*

Proof. Let $n \in \mathbb{N}$. Observe that $g'g^n = \left(\frac{g'}{g}\right)g^{n+1}$ and $\frac{g'}{g^n} = \left(\frac{g'}{g}\right)\frac{1}{g^{n-1}}$. Note that every zero of $\frac{g'}{g}$ is neither a zero nor a pole of g, every zero and every pole of g being a simple pole of $\frac{g'}{g}$. Thereby, since $\frac{g'}{g}$ has infinitely many zeros, we deduce that $g'g^n$ and $\frac{g'}{g^n}$ have infinitely many zeros in $\mathbb{K}$ (resp. in $d(0, R^-)$). ∎

Theorem 5. *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and optimal. If $\deg(A) = \deg(B)$ and if $m \geq 3$ is an integer, then $f' + Tf^m$ has infinitely many zeros that are not zeros of f.*

Thus, by (1) we may derive the following corollary.

Corollary 7. *Let $g \in \mathcal{M}(\mathbb{K})$ be transcendental and optimal. If $\deg(A) = \deg(B)$, then $g'g^n - T$ has infinitely many zeros for every $n \in \mathbb{N}^*$.*

Theorem 6. *Let $f \in \mathcal{M}_u(d(0, R^-))$ be an optimal function and let $U = \frac{\phi}{\psi} \in \mathcal{M}_b(d(0, R^-))$ have the same finite number of zeros and poles in $d(0, R^-)$. If $m \geq 3$ is an integer, then $f' + Uf^m$ has infinitely many zeros that are not zeros of f.*

By (1), the following corollary is immediate.

Corollary 8. *Let $g \in \mathcal{M}_u(d(0, R^-))$ be an optimal function and let $U \in \mathcal{M}_b(d(0, R^-))$ have the same finite number of zeros and poles in $d(0, R^-)$. Then $g'g^n - U$ has infinitely many zeros for every $n \in \mathbb{N}^*$.*

Since almost every meromorphic function in a field of residue characteristic zero is optimal, by Theorems 5 and 6, we can deduce Corollaries 9 - 10 and 11 - 12 respectively.

Corollary 9. *Let* $f \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) = \deg(B)$. *If* $\mathbb{K}$ *has residue characteristic zero and* $m \geq 3$ *is an integer, then* $f' + Tf^m$ *has infinitely many zeros that are not zeros of* f.

Corollary 10. *Let* $g \in \mathcal{M}(\mathbb{K})$ *be transcendental and* $\deg(A) = \deg(B)$. *If* $\mathbb{K}$ *has residue characteristic zero, then* $g'g^n - T$ *has infinitely many zeros for every* $n \in \mathbb{N}^*$.

Corollary 11. *Let* $f \in \mathcal{M}_u(d(0, R^-))$ *be such that* $|f|(r)$ *is not constant when* r *tends to R, and let* $U \in \mathcal{M}_b(d(0, R^-))$ *have the same finite number of zeros and poles in* $d(0, R^-)$. *If* $\mathbb{K}$ *has residue characteristic zero and* $m \geq 3$ *is an integer, then* $f' + Uf^m$ *has infinitely many zeros that are not zeros of* f.

Corollary 12. *Let* $g \in \mathcal{M}_u(d(0, R^-))$ *and let* $U \in \mathcal{M}_b(d(0, R^-))$ *have the same finite number of zeros and poles in* $d(0, R^-)$. *If* $\mathbb{K}$ *has residue characteristic zero, then* $g'g^n - U$ *has infinitely many zeros for all* $n \in \mathbb{N}^*$.

Remark 4. If $\mathbb{K}$ has residue characteristic $p \neq 0$, the problem remains unsolved when $m = 2$, 3 and 4. In particular, in a p-adic field, we don't know how to construct a counter-exemple such as $f(x) = \tan(-x)$ showing that Hayman's statement does not hold when $m = 2$.

1.2. Nevanlinna Theory, Preliminary Results

We must now introduce some notations and results used in the p-adic Nevanlinna theory that we will employ for proving the previous theorems.

Let $\alpha \in d(0, R^-)$ and $h \in \mathcal{M}(d(0, R^-))$. If h has a zero of order n at α, we set $\omega_\alpha(h) = n$, if h has a pole of order n at α, we set $\omega_\alpha(h) = -n$, and finally, if $h(\alpha) \neq 0$ and ∞, we set $\omega_\alpha(h) = 0$.

Let $f \in \mathcal{M}(d(0, R^-))$ be such that 0 is neither a zero nor a pole of f. Let $r \in]0, R[$. We denote by $Z(r, f)$ the *counting function of zeros* of f in $d(0, R^-)$

$$Z(r, f) = \sum_{\omega_\alpha(f)>0 \ |\alpha|\leq r} \omega_\alpha(f)(\log r - \log|\alpha|),$$

and similarly, we set

$$\overline{Z}(r, f) = \sum_{\omega_\alpha(f)>0 \ |\alpha|\leq r} (\log r - \log|\alpha|).$$

We shall also consider the *counting functions of poles* of f in $d(0,R^-)$

$$N(r,f)=Z(r,\frac{1}{f}) \qquad \text{and} \qquad \overline{N}(r,f)=\overline{Z}(r,\frac{1}{f}).$$

The *Nevanlinna function* $T(r,f)$ is defined by

$$T(r,f)=\max\{Z(r,f)+\log|f(0)|;\ N(r,f)\}.$$

A. Boutabaa and A. Escassut in [5], A. Escassut in [7] and P. C. Hu and C. C. Yang in [9] give us results related to the p-adic Nevanlinna theory which we will use in the later proofs. Some of them are the followings.

Lemma 3. *If $f\in\mathcal{A}(\mathbb{K})\setminus\mathbb{K}[x]$ $\big($resp. If $f\in\mathcal{A}_u(d(0,R^-))\big)$, then f has infinitely many zeros.*

Lemma 4. *Let $f\in\mathcal{A}(\mathbb{K})$ $\big($resp. Let $f\in\mathcal{A}(d(0,R^-))\big)$ be such that $f(0)\neq 0$ and let $r>0$ (resp. let $r\in]0,R[$). For any $b\in\mathbb{K}$, we have*

$$Z(r,f-b)=Z(r,f)+O(1).$$

Lemma 5. *Let $f\in\mathcal{A}(\mathbb{K})$ $\big($resp. Let $f\in\mathcal{A}(d(0,R^-))\big)$ be such that $f(0)\neq 0$ and let $r>0$ $\big($resp. let $r\in]0,R[\big)$. The functions $T(r,f)$ and $Z(r,f)$ are equivalent up to an additive constant.*

Proposition 1. *Let $f_i\in\mathcal{M}(\mathbb{K})$ $\big($resp. Let $f_i\in\mathcal{M}(d(0,R^-))\big)$ be such that $f_i(0)\neq 0,\ \infty$ for $i=1,...,k$. Then, for $r>0$ $\big($resp. for $r\in]0,R[\big)$, we have*

$$Z\Big(r,\prod_{i=1}^{k}f_i\Big)\leq\sum_{i=1}^{k}Z(r,f_i),$$

$$T\Big(r,\sum_{i=1}^{k}f_i\Big)\leq\sum_{i=1}^{k}T(r,f_i)\ , \qquad T\Big(r,\prod_{i=1}^{k}f_i\Big)\leq\sum_{i=1}^{k}T(r,f_i),$$

and $T(r,f)$ is an increasing function of r.

As a corollary of Lemma 2.1 [5], considering the previous notations, we obtain the following Lemma 6 that also is known as the version p-adic of Jensen's formula.

Lemma 6. *Let $f\in\mathcal{M}(\mathbb{K})$ $\big($resp. Let $f\in\mathcal{M}(d(0,R^-))\big)$ be such that 0 is neither a zero nor a pole of f. Then,*

$$\log|f|(r)=Z(r,f)-N(r,f)+\log|f(0)|.$$

Proposition 2. *Let $f \in \mathcal{M}(d(0,R^-))$ be such that $f(0) \neq 0,\ \infty$. Then, $f \in \mathcal{M}_b(d(0,R^-))$ if and only if $T(r,f)$ is bounded in $]0,R[$.*

Let $f \in \mathcal{M}(d(0,R^-))$ be such that 0 is neither a zero nor a pole of f' and let S be a finite subset of $\mathbb{K}$. We denote by $Z_0^S(r,f')$ the counting function of zeros of f' in $d(0,r)$ which are not zeros of any $f-s$ for $s \in S$. Then,

$$Z_0^S(r,f') = \sum_{s\in S,\ w_\alpha(f-s)=0,\ |\alpha|\le r} w_\alpha(f')(\log r - \log|\alpha|).$$

Now we can state the *ultrametric Nevanlinna Second Main Theorem* in a basic form.

Theorem N. *Let $\beta_1,...,\beta_n \in \mathbb{K}$ with $n \geq 2$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0,R^-))$). Let $S = \{\beta_1,...,\beta_n\}$. Assume that none of $f,\ f'$ and $f-\beta_j$ with $1 \leq j \leq n$ equals 0 or ∞ at the origin. Then, for all $r > 0$ $\big($resp. for all $r \in]0,R[\big)$, we have*

$$(n-1)T(r,f) \leq \sum_{j=1}^{n}\overline{Z}(r,f-\beta_j) + \overline{N}(r,f) - Z_0^S(r,f') - \log r + O(1).$$

In order to go on, we remember the interesting corollary of the Nevanlinna Second Main Theorem on three small functions for p-adic analytic functions $\big($see Theorem 4 [12]$\big)$, which we will use later in the proof of Theorem 3.

Theorem T. *Let $f \in \mathcal{A}(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{A}(d(0,R^-))\big)$ be non-constant such that $f(0) \neq 0$, and let $u_1,\ u_2 \in \mathcal{A}(\mathbb{K})$ $\big($resp. let $u_1,\ u_2 \in \mathcal{A}(d(0,R^-))\big)$ be small functions with respect to f and not zero at 0. Then,*

$$T(r,f) \leq \overline{Z}(r,f-u_1) + \overline{Z}(r,f-u_2) + S(r)$$

where $S(r) = 2T(r,u_1) + 3T(r,u_2) - \log r + O(1)$.

2. Proofs of the Main Lemmas and Theorems

2.1. Proof of Lemma 1

Proof. It is sufficient to show the claim whenever $f \in \mathcal{A}(d(0,R^-))$. Let $f(x) = \sum_{i=0}^{+\infty} c_i x^i$. Clearly we can notice that every zero of f in $d(0,R^-)$ is also a zero of $\widehat{f}$ in $\widehat{d}(0,R^-)$.

Let $r \in]0, R[$ and let $\alpha_1, ..., \alpha_q$ be the zeros of f in the circle $C(0, r)$ with $\omega_{\alpha_i}(f) = s_i$ for $i = 1, ..., q$. Thereby, f is factorized in the form $f = \prod_{i=1}^{q}(x - \alpha_i)^{s_i} g$, where $g \in \mathcal{A}(d(0, R^-))$ and $g(\alpha_i) \neq 0$ for $i = 1, ..., q$. Observe that this factorization also holds in $\mathcal{M}(\widehat{d}(0, R^-))$. Hence α_i is also a zero of order s_i of $\widehat{f}$ for $i = 1, ..., q$. Now, suppose that $\widehat{f}$ admits other zeros $\alpha_{q+1}, ..., \alpha_t$ with $\omega_{\alpha_i}(\widehat{f}) = s_i$ for $i = q+1, ..., t$. By Theorem 23.1 [7], for all $r \in]0, R[$, we have

$$\nu^+(f, r) - \nu^-(f, r) = \sum_{i=1}^{q} s_i,$$

and similarly, we have

$$\nu^+(\widehat{f}, r) - \nu^-(\widehat{f}, r) = \sum_{i=1}^{t} s_i.$$

But, we know that $\nu^+(f, r)$, $\nu^-(f, r)$, $\nu^+(\widehat{f}, r)$, $\nu^-(\widehat{f}, r)$ are only defined by the coefficients of f. So, for $r \in]0, R[$, we have $\nu^+(f, r) = \nu^+(\widehat{f}, r)$ and $\nu^-(f, r) = \nu^-(\widehat{f}, r)$. Consequently $t = q$, which finishes the proof. ∎

2.2. Proof of Lemma 2

Proof. Since f has no zeros in $\Gamma(0, r', r'')$, then by Theorem 23.4 [7], $\nu^+(f, r) = \nu^-(f, r)$ $\forall r \in]r', r''[$. Moreover, since $q = \nu(f, r)$ $\forall r \in]r', r''[$, we have

$$|f(x)| = |a_q||x|^q \quad \forall x \in \Gamma(0, r', r'')$$

with $|a_q||x|^q > |a_n||x|^n$ $\forall q \neq n$. Consequently, since $|q| = 1$ by our assumption that the residue characteristic of $\mathbb{K}$ does not divide q, we have

$$|f'(x)| = \Big|\sum_{-\infty}^{+\infty} n a_n x^{n-1}\Big| = |a_q||x|^{q-1} = \frac{1}{|x|}|a_q||x|^q.$$

Therefore, we may deduce that $|f'(x)| = \frac{|f(x)|}{|x|}$. ∎

2.3. Proof of Theorem 1

Proof. Let $r > 0$ (resp. Let $r \in [1, R[$). By Lemma 4 [3], we know that $|f'|(r) \leq \frac{1}{r}|f|(r)$. We shall check that there exists a $\rho \in]0, +\infty[$ (resp. $\rho \in [1, R[$) such that $|f'|(r) < |Tf|(r)$ $\forall r \in]\rho, +\infty[$ (resp. $\forall r \in]\rho, R[$). Indeed, if $f \in \mathcal{M}(\mathbb{K})$ the existence of ρ is immediate because $\lim_{r \to +\infty} |T|(r) > 0$. Now, suppose that

$f \in \mathcal{M}_u(d(0,R^-))$. Since $\lim\limits_{r\to R}|T|(r) > \dfrac{1}{R}$, by continuity, we can find a $\rho \in [1,R[$ such that $|T|(r) > \frac{1}{r}\ \ \forall r \in]\rho,R[$. So we have proved the existence of $\rho \in]0,+\infty[$ $\big($resp. $\rho \in [1,R[\big)$ such that $|f'|(r) \le \frac{1}{r}|f|(r) < |Tf|(r)$. Consequently

$$|f' + Tf|(r) = |Tf|(r) \quad \forall r > \rho \qquad \big(\text{resp. } \forall\ r \in]\rho,R[\big).$$

Suppose first that f has a finite number of poles. Then, f has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $d(0,R^-)\big)$ because f is transcendental in $\mathbb{K}$ $\big($resp. is unbounded in $d(0,R^-)\big)$. Moreover, there exists an increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ with $\lim\limits_{n\to+\infty} r_n = +\infty$ $\big($resp. $\lim\limits_{n\to+\infty} r_n = R\big)$, such that f admits zeros and no poles in $C(0,r_n)$, such that T has no zeros and no poles in $C(0,r_n)$ and such that

$$|f' + Tf|(r) = |Tf|(r) \quad \forall r \ge r_1.$$

Since $|f' + Tf|(r) = |Tf|(r)$ in a neighborhood of r_n, we have

$$\nu^+(f'+Tf, r_n) - \nu^-(f'+Tf, r_n) = \nu^+(f, r_n) - \nu^-(f, r_n), \tag{2}$$

where $\nu^+(f,r_n) - \nu^-(f,r_n)$ is the number of zeros of f in $C(0,r_n)$ and $\nu^+(f' + Tf, r_n) - \nu^-(f'+Tf, r_n)$ is the number of zeros of $f'+Tf$ in $C(0,r_n)$ $\big($counting multiplicities$\big)$. Hence, we may deduce that $f' + Tf$ has zeros in $C(0,r_n)$ and the number of zeros of $f' + Tf$ is equal to the number of zeros of f in $C(0,r_n)$ $\big($counting multiplicities$\big)$.

On the other hand, since each zero of f in $C(0,r_n)$ either is not a zero of $f' + Tf$ or is a zero of $f' + Tf$ of order strictly lower than its order as a zero of f, by (2) there does exist at least a zero of $f' + Tf$ that is not a zero of f in $C(0,r_n)$. Since this is true for all $n \in \mathbb{N}$, we obtain that $f' + Tf$ has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $d(0,R^-)\big)$ that are not zeros of f.

Now, suppose that f has infinitely many poles. Then, there exists an increasing sequence $\{r_n\}_{n\in\mathbb{N}}$ with $\lim\limits_{n\to+\infty} r_n = +\infty$ $\big($resp. $\lim\limits_{n\to+\infty} r_n = R\big)$, such that f admits poles in $C(0,r_n)$, such that T has no zeros and no poles in $C(0,r_n)$ and such that

$$|f' + Tf|(r) = |Tf|(r) \quad \forall r \ge r_1.$$

Let $n \in \mathbb{N}$. Let s_n and t_n be the number of zeros and that of poles of f in $C(0,r_n)$ respectively, and let γ_n and τ_n be the number of zeros and that of poles of $f' + Tf$ in $C(0,r_n)$ respectively. Then, we deduce that

$$\nu^+(f,r_n) - \nu^-(f,r_n) = s_n - t_n \quad \text{and} \quad \nu^+(f'+Tf,r_n) - \nu^-(f'+Tf,r_n) = \gamma_n - \tau_n.$$

Since $|f' + Tf|(r) = |Tf|(r)$ in a neighborhood of r_n, we have again

$$\nu^+(f,r_n) - \nu^-(f,r_n) = \nu^+(f'+Tf,r_n) - \nu^-(f'+Tf,r_n).$$

Consequently, $\gamma_n - \tau_n = s_n - t_n$ in $C(0, r_n)$. But we may observe that τ_n is the number of poles of f' in $C(0, r_n)$ $\big($counting multiplicities$\big)$. So, since T has no zeros and no poles in $C(0, r_n)$, we have $\tau_n > t_n$ which implies that $\gamma_n > s_n$. Thus, $f' + Tf$ must have at least one zero in $C(0, r_n)$ that is not a zero of f. Since this is true for all $n \in N$, we deduce that $f' + Tf$ has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$ which are not zeros of f. ∎

2.4. Proof of Theorem 2

Proof. Assume, without loss of generality, that 0 is neither a zer nor a pole Tf^m and $f' + Tf^m$. We shall prove that f has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$. First we suppose $f \in \mathcal{M}(\mathbb{K})$. By hypothesis $\limsup\limits_{r \to +\infty} |f|(r) > 0$, i.e., there exist a sequence $\{\Gamma(0, r'_n, r''_n)\}_{n \in \mathbb{N}}$ with $\lim\limits_{n \to +\infty} r''_n = +\infty$, and a constant $C > 0$, such that $Z(r, f) \geq N(r, f) + C \quad \forall r \in \bigcup\limits_{n \in \mathbb{N}}]r'_n, r''_n[$. If f has a finite number of zeros, say, q, then $Z(r, f) = q \log r$ and so $N(r, f) + C \leq q \log r$. Consequently f has a finite number of poles, a contradiction because f is transcendental.

Now, suppose $f \in \mathcal{M}_u(d(0, R^-))$. If f has a finite number of zeros in $d(0, R^-)$, then $\lim\limits_{r \to R} Z(r, f) < +\infty$ and hence $\limsup\limits_{r \to R} |f|(r) < +\infty$, a contradiction to our hypothesis.

Suppose that the set of zeros of $f' + Tf^m$ which are not zeros of f is finite. Then, there exists a $\rho > 0$ $\big($resp. $\rho \in [1, R[\big)$ such that $f' + Tf^m$ has no zeros other than the multiple zeros of f in $\mathbb{K} \setminus d(0, \rho)$ $\big($resp. in $\Gamma(0, \rho, R)\big)$ and such that T has no zeros and no poles in $\mathbb{K} \setminus d(0, \rho)$ $\big($resp. in $\Gamma(0, \rho, R)\big)$. So, each pole of $f' + Tf^m$ is a pole of f^m of the same multiplicity. Hence,

$$\begin{aligned} &N(r, f' + Tf^m) - N(\rho, f' + Tf^m) \\ &= N(r, f^m) - N(\rho, f^m) \quad \forall r \in \mathbb{K} \setminus d(0, \rho) \quad (\text{resp. } \forall r \in]\rho, R[). \end{aligned} \tag{3}$$

Let $\sigma > \rho$ be such that $C(0, \sigma)$ contains at least one zero of f. Each zero of f, say, of order q, either is not a zero of $f' + Tf^m$ or is a zero of $f' + Tf^m$ with order $q - 1$. Since $f' + Tf^m$ has no zeros in $C(0, r)$ other than the zeros of f and T has no zeros and no poles in $C(0, r)$, clearly the number of zeros of $f' + Tf^m$ in $C(0, r)$ $\big($counting multiplicities$\big)$ is strictly inferior to the number of zeros of Tf^m $\big($counting multiplicities$\big)$. So, the function

$$\Psi(r) = Z(r, f^m) - Z(\rho, f^m) - \Big[Z(r, f' + Tf^m) - Z(\rho, f' + Tf^m)\Big]$$

is strictly increasing in $[\sigma, +\infty[$ $\big($resp. in $[\sigma, R[\big)$.

Now, we will show that there exists an increasing sequence of intervals $]r'_n, r''_n[$ with
$\rho < r'_n < r''_n < r'_{n+1}$ and $\lim\limits_{n\to+\infty} r''_n = +\infty$ $\big($resp. $\lim\limits_{n\to+\infty} r''_n = R\big)$, such that $|f' + Tf^m|(r) = |Tf^m|(r)$ $\forall r \in]r'_n, r''_n[$. Suppose first that $f \in \mathcal{M}(\mathbb{K})$. Since $\limsup\limits_{r\to+\infty} |f|(r) > 0$, there exist a sequence of annulus $\{\Gamma(0, r'_n, r''_n)\}_{n\in\mathbb{N}}$ with $\rho < r'_n < r''_n$ and $\lim\limits_{n\to+\infty} r''_n = +\infty$, and a constant $C > 0$ such that

$$|f|(r) > C \quad \forall r \in]r'_n, r''_n[\quad \forall n \in \mathbb{N}.$$

Since T has no zeros and no poles in $]r'_n, r''_n[$ and $\deg(A) \geq \deg(B)$, then there exists a constant $\lambda > 0$ such that $|T|(r) \geq \lambda$ $\forall r \in]r'_n, r''_n[$. So

$$|Tf^m|(r) > C^m\lambda \quad \forall r \in]r'_n, r''_n[\quad \forall n \in \mathbb{N}.$$

On the other hand, by Lemma 4 [3], $|f'|(r) \leq \frac{1}{r}|f|(r)$. So, if we consider the previous observation, we can deduce that

$$\Big|\frac{f'}{Tf^m}\Big|(r) \leq \frac{1}{r}\frac{1}{|Tf^{m-1}|(r)} < \frac{1}{\lambda r}\Big(\frac{1}{C}\Big)^{m-1}.$$

However, for r sufficiently large, we have $\frac{1}{\lambda r}(\frac{1}{C})^{m-1} < 1$. Hence $|f'|(r) < |Tf^m|(r)$. Thereby, $|f' + Tf^m|(r) = |Tf^m|(r)$. Thus, this equality holds in all annulus $\Gamma(0, r'_n, r''_n)$ when r'_n is sufficiently large. Consequently, without loss of generality, we may assume that $|f' + Tf^m|(r) = |Tf^m|(r)$ $\forall r \in]r'_n, r''_n[$ $\forall n \in \mathbb{N}$.

Now, we suppose that $f \in \mathcal{M}_u(d(0, R^-))$. Since $\limsup\limits_{r\to R} |f|(r) = +\infty$ there exists a sequence of annulus $\{\Gamma(0, r'_n, r''_n)\}_{n\in\mathbb{N}}$ with $\rho < r'_n < r''_n$ and $\lim\limits_{n\to+\infty} r''_n = R$, such that $|f|(r) \geq n$ $\forall r \in]r'_n, r''_n[$ and $n \in \mathbb{N}$. Since $T \in \mathbb{K}(x)$, there exists a constant $\lambda > 0$ such that $\inf\limits_{r\in[1,R[} |T|(r) = \lambda$. Then, $|Tf^m|(r) \geq \lambda |f|(r) n^{m-1}$ $\forall r \in]r'_n, r''_n[$ and $n \in \mathbb{N}$. Moreover, we can see that $|f'|(r) < |f|(r)$ $\forall r \in]r'_n, r''_n[$ because $r'_n > 1$. Consequently, when n is sufficiently large, we have $|f'|(r) < |f|(r) < \lambda n^{m-1}|f|(r) \leq |Tf^m|(r)$ $\forall r \in]r'_n, r''_n[$, which implies that $|f' + Tf^m|(r) = |Tf^m|(r)$ $\forall r \in]r'_n, r''_n[$.

Therefore, by Lemma 6, we obtain

$$\begin{aligned} &Z(r, Tf^m + f') - N(r, Tf^m + f') \\ &= Z(r, f^m) - N(r, f^m) + \chi \quad \forall r \in]r'_n, r''_n[, \end{aligned} \tag{4}$$

where χ is defined as $m \log |f(0)| - \log |T(0)f^m(0) + f'(0)|$. And by (3) and (4), we can check that

$$\Psi(r) = Z(\rho, f' + Tf^m) - N(\rho, f' + Tf^m) - \big[Z(\rho, f^m) - N(\rho, f^m) + \chi\big].$$

Consequently Ψ is constant in $[\sigma, +\infty[$ $\big($resp. in $[\sigma, R[\big)$, a contradiction because we have showed that it is strictly increasing. ■

2.5. Proof of Theorem 3

Proof. In order to prove Theorem 3, thanks to Lemma 1, we can place ourselves in $\widehat{d}(0, R^-) \subset \widehat{\mathbb{K}}$ in the case when $f \in \mathcal{M}_u(d(0, R^-))$. Since f is a transcendental meromorphic function in $\mathbb{K}$ $\big($resp. unbounded in $\widehat{d}(0, R^-)\big)$, there exist entire functions $h,\ l \in \mathcal{A}(\mathbb{K})$ $\big($resp. $h,\ l \in \mathcal{A}(\widehat{d}(0, R^-))\big)$ without common zeros and at least one of them being transcendental $\big($resp. unbounded$\big)$ such that $f = \frac{h}{l}$. We can write h in the form $\overline{h}\,\widetilde{h}$, where the zeros of $\overline{h}$ are exactly the different zeros of h but all with multiplicity 1. Then, necessarily, h' is multiple of $\widetilde{h}$ in $\mathcal{A}(\mathbb{K})$ $\big($resp. in $\mathcal{A}\widehat{d}(0, R^-)\big)$. So $h' = u\,\widetilde{h}$ with $u \in \mathcal{A}(\mathbb{K})$ $\big($resp. $u \in \mathcal{A}(\widehat{d}(0, R^-))\big)$.

Suppose that $f' + Tf^m$ has a finite number of zeros in $\mathbb{K}$ $\big($resp. in $\widehat{d}(0, R^-)\big)$ which are not zeros of f. Then, there exists a polynomial $P \in \mathbb{K}[x]$ of degree q, having no common zeros with Bl, such that

$$f' + Tf^m = \frac{P\,\widetilde{h}}{Bl^m}.$$

This implies

$$\frac{f'}{f^m} = \frac{P\widetilde{h} - Ah^m}{Bh^m} = \frac{P - A\overline{h}h^{m-1}}{B\,\overline{h}^m\,\widetilde{h}^{m-1}}. \tag{5}$$

On the other hand, we note that

$$\frac{f'}{f^m} = \frac{l^{m-2}(h'l - hl')}{h^m} = \frac{l^{m-2}(ul - \overline{h}l')}{\overline{h}^m\widetilde{h}^{m-1}}. \tag{6}$$

So, by (5) and (6),

$$Bl^{m-2}(ul - \overline{h}l') = P - A\overline{h}h^{m-1}.$$

Let $F = Bl^{m-2}(ul - \overline{h}l')$ and $s = \deg(A)$. Let $r > 0$ $\big($resp. Let $r \in [1, R[\big)$. Applying Theorem T to F, and noting that $\overline{Z}(r, h) = \overline{Z}(r, \overline{h}h^{m-1}) = Z(r, \overline{h})$, we obtain

$$\begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}(r, F - P) + 3T(r, P) - \log r + O(1) \\ &\leq \overline{Z}(r, B) + \overline{Z}(r, l^{m-2}) + \overline{Z}(r, ul - \overline{h}l')) \\ &\quad + \overline{Z}(r, A) + \overline{Z}(r, h) + (3q - 1)\log r + O(1) \\ &\leq Z(r, B) + Z(r, l) \\ &\quad + Z(r, ul - \overline{h}l') + Z(r, h) + (3q + s - 1)\log r + O(1). \end{aligned} \tag{7}$$

Moreover, we have

$$
\begin{aligned}
T(r,F) &= T(r,B)+T(r,l^{m-2})+T(r,ul-\overline{h}l')+O(1) \\
&= Z(r,B)+(m-2)Z(r,l)+Z(r,ul-\overline{h}l')+O(1).
\end{aligned} \tag{8}
$$

Let $d = 3q+s-1$. By (7) and (8), we deduce that

$$(m-3)Z(r,l) \le Z(r,h) + d\log r + O(1). \tag{9}$$

Since we assume that the set of zeros of $f' + Tf^m$ that are not zeros of f is finite, by Theorem 2, we can restrict ourselves to the assumption $\limsup\limits_{r\to+\infty} |f|(r) = 0$ $\big($resp. $\limsup\limits_{r\to R} |f|(r) < +\infty\big)$ and therefore $\limsup\limits_{r\to+\infty}[Z(r,l) - Z(r,h)] = +\infty$ $\big($resp. $\limsup\limits_{r\to R}[Z(r,h) - Z(r,l)] < +\infty\big)$. Consequently, there exist a sequence $\{r_n\}_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} r_n = +\infty$ $\big($resp. $\lim\limits_{n\to+\infty} r_n = R\big)$, and a constant $C > 0$ such that $Z(r_n,h) < Z(r_n,l) + C \quad \forall n \in \mathbb{N}$. So, by (9), we have

$$(m-4)Z(r_n,l) < d\log r_n + O(1). \tag{10}$$

If we assume $f \in \mathcal{M}(\mathbb{K})$, then by hypothesis, $\limsup\limits_{r\to+\infty}|f|(r) = 0$ and so l is a transcendental function. Thereby, when $m \ge 5$, we have $\lim\limits_{n\to+\infty} Z(r_n,l) = +\infty$, a contradiction to (10). Now, if we assume $f \in \mathcal{M}_u(\widehat{d}(0,R^-))$, then at least one of the two functions h, l belongs to $\mathcal{A}_u(\widehat{d}(0,R^-))$. Since, by hypothesis, $\limsup\limits_{r\to R} |f|(r) < +\infty$, we deduce that l must lie in $\mathcal{A}_u(\widehat{d}(0,R^-))$ because if $l \in \mathcal{A}_b(\widehat{d}(0,R^-))$, then $h \in \mathcal{A}_u(\widehat{d}(0,R^-))$ and in this case $\limsup\limits_{r\to R} |f|(r) = +\infty$, a contradiction. Hence $\lim\limits_{n\to+\infty} Z(r_n,l) = +\infty$, a contradiction to (10) again.

Thus, when $m \ge 5$, $f' + Tf^m$ has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $\widehat{d}(0,R^-)\big)$ which are not zeros of f. Consequently, by Lemma 1, $f' + Tf^m$ has infinitely many zeros in $d(0,R^-)$ that are not zeros of f.

Now, consider $T \equiv 1$ and suppose that $f' + f^4$ has no zeros in $\mathbb{K}$ which are not zeros of f. Then $d = -1$. So, by (10), we obtain

$$0 < -\log r_n + O(1) \quad \forall n \in \mathbb{N},$$

and hence we have a contradiction when $n \to +\infty$. Consequently, $f' + f^4$ has at least one zero in $\mathbb{K}$ that is not a zero of f. ■

2.6. Proof of Theorem 4

Proof. Let $\{r_n\}_{n\in\mathbb{N}}$ be a g-suitable sequence. For each $n \in \mathbb{N}^*$, there exists $r'_n \in]r_n, r_{n+1}[$ such that g has no zero and no pole in the annulus $\Gamma(0, r_n, r'_n)$. Consequently, by Lemma 2, we have $\frac{|g'(x)|}{|g(x)|} = \frac{1}{|x|} \quad \forall x \in \Gamma(0, r_n, r'_n)$. So, $\nu\Big(\frac{g'}{g}, r\Big) = -1 \quad \forall r \in]r_n, r'_n[$.

Observe that the poles of $\frac{g'}{g}$ are all simple ones and correspond to the zeros and the poles of g. Since g is a transcendental meromorphic function in $\mathbb{K}$ $\big($resp. g is unbounded in $d(0, R^-)\big)$, we derive that $\frac{g'}{g}$ has infinitely many poles in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$. Moreover, since $\nu\Big(\frac{g'}{g}, r\Big) = -1$ whenever $r \in]r_n, r'_n[$, by Corollary 5, the difference between the number of poles and the number of zeros of $\frac{g'}{g}$ in $d(0, r)$ is just 1. Then clearly, $\frac{g'}{g}$ has infinitely many zeros in $\mathbb{K}$ $\big($resp. in $d(0, R^-)\big)$. ■

2.7. Proof of Theorem 5

Proof. Here we assume $\deg(A) = \deg(B)$. Let f be of the form $\frac{h}{l}$ with $h,\ l \in \mathcal{A}(\mathbb{K})$ having no common zeros. As in the proofs of Theorem 3, we can write h in the form $\overline{h}\ \widetilde{h}$, where the zeros of $\overline{h}$ are exactly the different zeros of h but all with multiplicity 1, and h' is of the form $\widetilde{h}u$ with $u \in \mathcal{A}(\mathbb{K})$.

Suppose that $f' + Tf^m$ only has finitely many zeros which are not zeros of f. There exists a $P \in \mathbb{K}[x]$ such that $f' + Tf^m = \frac{P\widetilde{h}}{Bl^m}$ with $P\widetilde{h}$ and Bl^m having no common zeros in $\mathbb{K}$.

On the other hand, we have

$$f' + Tf^m = \frac{\big[Bl^{m-2}(ul - \overline{h}\ l') + A\overline{h}^m\ \widetilde{h}^{m-1}\big]\ \widetilde{h}}{Bl^m}.$$

Since h, l have no common zeros and since $A,\ B$ have no common zeros either, each zero α of $\big[Bl^{m-2}(ul - \overline{h}\ l') + A\overline{h}^m\ \widetilde{h}^{m-1}\big]$ that is not a zero of $f' + Tf^m$ must be a zero of A or a zero of B or l. But note that if α is a zero of l then it is a zero of A. Thus the zeros of $\big[Bl^{m-2}(ul - \overline{h}\ l') + A\overline{h}^m\ \widetilde{h}^{m-1}\big]$ which are not zeros of $f' + Tf^m$ must be zeros of A or B and therefore are a finite number. Moreover, we notice that a zero α of $\big[Bl^{m-2}(ul - \overline{h}\ l') + A\overline{h}^m\ \widetilde{h}^{m-1}\big]$ is not a zero of f except if it is a zero of B, because a zero of f cannot be a zero of u. Consequently, the zeros of $f' + Tf^m$ that are not zeros of f are the zeros of $Bl^{m-2}(ul - \overline{h}l') + A\overline{h}^m\ \widetilde{h}^{m-1}$ $\big($counting multiplicities$\big)$, except a finite number.

Next, we may notice that $h \notin \mathbb{K}[x]$. Indeed, suppose $h \in \mathbb{K}[x]$. Since $f \notin \mathbb{K}(x)$, then $l \notin \mathbb{K}[x]$ and hence $Bl^{m-2}(ul - \overline{h}l') + A\overline{h}^m\ \widetilde{h}^{m-1} \notin \mathbb{K}[x]$. Therefore, $Bl^{m-2}(ul - \overline{h}l') + A\overline{h}^m\ \widetilde{h}^{m-1}$ has infinitely many zeros which are not zeros of f, a contradiction to our initial supposition.

Now, we consider $H = \frac{f'}{f^m} = -\frac{A}{B} + \frac{P\widetilde{h}}{Bh^m} = -\frac{A}{B} + \frac{P}{Bh^{m-1}\overline{h}}$. Since h is not a polynomial in $\mathbb{K}$ we have $\lim_{|x|\to+\infty} \left|\frac{P(x)}{(Bh^{m-1}\overline{h})(x)}\right| = 0$. Moreover, since $\deg(A) = \deg(B)$, A and B have the same number of zeros, taking multiplicities into account and hence we may derive that $\lim_{|x|\to+\infty} \left|\frac{A(x)}{B(x)}\right| = a$ with $a \in \mathbb{R}_+$. Hence, $\lim_{|x|\to+\infty} |H(x)| = a$. Consequently, there exists a $\rho > 0$ such that $\nu(H, r) = 0\ \forall r \geq \rho$.

On the other hand, since f is an optimal function, there exists a f-suitable sequence $\{r_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} r_n = +\infty$. Let $\{s_n\}_{n\in\mathbb{N}}$ be another sequence such that $r_n < s_n < r_{n+1}$ and such that $\nu(f, r)$ is constant inside $]r_n, s_n[$. By Proposition 20.9 [7], we have $\nu(\frac{f'}{f^m}, r) = \nu(f', r) - m\nu(f, r)$, and by Corollary 5, we have $\nu(f', r) = \nu(f, r) - 1\ \ \forall r \in]r_n, s_n[$. Consequently,

$$0 = \nu(H, r) = (1 - m)\nu(f, r) - 1 \quad \forall r \in]r_n, s_n[,$$

a contradiction when $m \geq 3$. Thus, $f' + Tf^m$ has infinitely many zeros in $\mathbb{K}$ which are not zeros of f. ∎

2.8. Proof of Theorem 6

Proof. As in the proof of Theorem 3, without loss of generality, we may place ourselves in the spherically complete field $\widehat{\mathbb{K}}$ and consider $f \in \mathcal{M}_u(\widehat{d}(0, R^-))$. So, there exist functions $h,\ l \in \mathcal{A}(\widehat{d}(0, R^-))$ having no common zeros, such that $f = \frac{h}{l}$. Moreover, at least one of them is unbounded. As in the proofs of Theorems 3 and 5, we can write h in the form $\overline{h}\ \widetilde{h}$, where the zeros of $\overline{h}$ are exactly the different zeros of h but all with order 1. Then $h' = \widetilde{h}u$ with $u \in \mathcal{A}(\widehat{d}(0, R^-))$.

Suppose that $f' + Uf^m$ only has a finite number of zeros which are not zeros of f. The proof now is similar to this of Theorem 5. There exists $P \in \widehat{\mathbb{K}}[x]$ such that $f' + Uf^m$ is of the form $\frac{P\widetilde{h}}{\psi l^m}$ with $P\widetilde{h}$ and ψl^m having no common zeros in $\widehat{d}(0, R^-)$. Consequently, $\frac{f'}{f^m} = \frac{P\widetilde{h} - \phi h^m}{\psi h^m}$. Since f is an optimal function, there exists a f-suitable sequence $\{r_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} r_n = R$. Let $\{s_n\}$ be another sequence such that $r_n < s_n < r_{n+1}$ and such that $\nu(f, r)$ is constant inside $]r_n, s_n[$. By Corollary 5, we have $\nu(f', r) = \nu(f, r) - 1\ \ \forall r \in]r_n, s_n[$, and by Proposition 20.9 [7], we have $\nu(\frac{f'}{f^m}, r) = \nu(f', r) - m\nu(f, r)$. Consequently, as in the proof of the previous theorem, we have

$$\nu(\tfrac{f'}{f^m}, r) = -(m - 1)\nu(f, r) - 1\ \forall r \in]r_n, s_n[. \tag{11}$$

On the other hand, considering that f is of the form $\frac{\overline{h}\,\widetilde{h}}{l}$ we deduce that

$$f' + Uf^m = \frac{\left[\psi l^{m-2}(ul - \overline{h}l') + \phi\overline{h}^m\ \widetilde{h}^{m-1}\right]\ \widetilde{h}}{\psi l^m}.$$

Let $H = \frac{P\widetilde{h} - \phi h^m}{\psi h^m}$. We shall prove that h is unbounded in $\widehat{d}(0, R^-)$. Suppose that $h \in \mathcal{A}_b(\widehat{d}(0, R^-))$. Since ϕ and psi belons to $\mathcal{A}_b(\widehat{d}(0, R^-))$, then H belong to $\mathcal{M}_b(\widehat{d}(0, R^-))$, hence $l \in \mathcal{A}_u(\widehat{d}(0, R^-))$. Thereby, $\psi l^{m-2}(ul - \overline{h}l') + \phi\overline{h}^m\ \widetilde{h}^{m-1}$ is an unbounded analytic function in $\widehat{d}(0, R^-)$. So, by Lemma 3, $\psi l^{m-2}(ul - \overline{h}l') + \phi\overline{h}^m\ \widetilde{h}^{m-1}$ has infinitely many zeros in $\widehat{d}(0, R^-)$, but these zeros are the zeros of $f' + Uf^m$ that are not zeros of f except a finite number of them (see arguments in the proof of Theorem 5), a contradiction to our supposition. Hence, we may deduce that $\lim\limits_{r \to R^-} \left|\dfrac{P}{\psi h^{m-1}\overline{h}}\right|(r) = 0$.

Now, since ϕ and ψ have the same finite number of zeros in $\widehat{d}(0, R^-)$ $\big($counting multiplicities$\big)$, there exists a $\rho < R$ such that $\nu(\frac{\phi}{\psi}, r) = 0\ \ \forall r \in [\rho, R[$, and therefore $|U|(r)$ is a constant c in $[\rho, R[$. Consequently, there exists a $\rho' \in [\rho, R[$ such that $|H|(r) = |U|(r) = c\ \ \forall r \in [\rho', R[$. Thus, $\nu(H, r) = 0\ \ \forall r \in [\rho', R[$.

The end of the proof is then similar to that of the previous theorem. By (11) and the previous observation, we have $(m-1)\nu(f, r) = -1\ \ \forall r \in]r_n, s_n[\ \ \forall n \in \mathbb{N}$, which is absurd because $m \geq 3$. Hence $f' + Uf^m$ has infinitely many zeros which are not zeros of f whenever $m \geq 3$. ■

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References

1. Y. Amice, *Les nombres p-adic*, Presses Universitaires de France, Collection SUP, Le mathématicien, 14, 1975.

2. W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana*, **11** (1995), 355-373.

3. A. Boutabaa and A. Escassut, On uniqueness of p-adic meromorphic functions, *Proc. Amer. Math. Soc.*, **126(9)** (1998), 2557-2568.

4. A. Boutabaa and A. Escassut, Applications of the p-adic Nevanlinna Theory to functional equations, *Ann. Inst. Fourier*, **50** (2000), 751-766.

5. A. Boutabaa and A. Escassut, URS and URSIMS for p-adic meromorphic functions inside a disc, *Proc. Edinb. Math. Soc.*, **44** (2001), 485-504.

6. H. H. Chen and M. L. Fang, On the value distribution of $f'f^n$, *Science in China* (*Serie A*), **38(7)** (1995), 789-798.

7. A. Escassut, *Analytic Elements in p-adic Analysis*, World Scientific Publishing, 1995.

8. W. K. Hayman, Picard values of meromorphic functions and their derivates, *Ann. of Math.*, **70** (1959), 9-42.

9. P. C. Hu and C. C. Yang, *Meromorphic Functions over Non-Archimedean Fields*, Kluwer Academic Publishers, 2000.

10. M. Lazard, Les zéros des fonctions analytiques d'une variable sur un corp valué complet, *Publ. Math., Inst. Hautes tud. Sci.*, **14** (1962), 223-251.

11. E. Mues, Uber ein Problem von Hayman, *Math. Z*, **164** (1979), 239-259.

12. J. Ojeda, Applications of the p-adic Nevanlinna theory to problems of uniqueness, *Journal of Analysis, Madras.*, 2006, (to appear).

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