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DELTA FUNCTION FOR AN AFFINE SUBSPACE

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Abstract. The Kubo–Yokoi and Donsker delta functions are well known generalized functions in infinite dimensional distribution theory. In this paper we develop the delta function for an affine subspace and show that it is a generalization of the Kubo–Yokoi and Donsker delta functions. The Wiener–Itô expansion of the delta function for an affine subspace is also given.

1. Introduction

In finite dimensions we have the delta function δ, which for a given point $a \in \mathbf{R^n}$ satisfies

$$\int_{\mathbf{R^n}} f(x)\delta(x-a)\,dx = f(a)$$

for every test function f. It can be realized by

$$\lim_{\sigma \searrow 0} e^{-|x-a|^2/2\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}} = \delta(x-a). \tag{1.1}$$

Observe, for a test function f,

$$\begin{aligned}
&\lim_{\sigma \searrow 0} \int_{\mathbf{R^n}} f(x) e^{-|x-a|^2/2\sigma^2} \frac{1}{(2\pi\sigma^2)^{n/2}}\,dx \\
&= \lim_{\sigma \searrow 0} \int_{\mathbf{R^n}} f(\sigma x + a) e^{-|x|^2/2} \frac{1}{(2\pi)^{n/2}}\,dx \qquad \text{by a change of variables} \\
&= \int_{\mathbf{R^n}} f(a) e^{-|x|^2/2} \frac{1}{(2\pi)^{n/2}}\,dx \quad \text{by the dominated convergence theorem} \\
&= f(a).
\end{aligned}$$

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Now let V be a subspace of $\mathbf{R^n}$ and let $a \in V^{\perp}$. We can create a delta function for the affine subspace $a+V$. This delta function is denoted by δ_{a+V} and is defined by its action on a test function f as follows

$$\int_{\mathbf{R^n}} f(x)\delta_{a+V}(x)\,dx = \int_{a+V} f(y)\,dy \tag{1.2}$$

when V is a nontrivial subspace. For the special case $V = \{0\}$, the delta function δ_{a+V} reduces to the ordinary delta function δ. That is,

$$\int_{\mathbf{R^n}} f(x)\delta_{a+0}(x)\,dx = f(a).$$

By adjusting equation (1.1) slightly we can realize δ_{a+V} by

$$\delta_{a+V}(x) = \lim_{\sigma\searrow 0} e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}. \tag{1.3}$$

where $x_{V^\perp}$ represents the orthongonal projection of x onto the subspace $V^{\perp}$ and d is the codimension of V. Observe, for a test function f,

$$\begin{aligned}
&\lim_{\sigma\searrow 0}\int_{\mathbf{R^n}} f(x)e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}\,dx\\
&= \lim_{\sigma\searrow 0}\int_{V^\perp}\int_V f(x_V+x_{V^\perp})e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}\,dx_V\,dx_{V^\perp}\\
&= \lim_{\sigma\searrow 0}\int_{V^\perp}\int_V f(x_V+\sigma x_{V^\perp}+a)e^{-|x_{V^\perp}|^2/2}\frac{1}{(2\pi)^{d/2}}\,dx_V\,dx_{V^\perp}\\
&= \int_{V^\perp}\int_V f(x_V+a)e^{-|x_{V^\perp}|^2/2}\frac{1}{(2\pi)^{d/2}}\,dx_V\,dx_{V^\perp}\\
&= \int_V f(x_V+a)\,dx_V\\
&= \int_{a+V} f(y)\,dy.
\end{aligned}$$

It is also easy to see that

$$\delta_{a+V} = \int_{a+V}\delta(x-y)\,dy \tag{1.4}$$

holds in the distributional sense. Observe, for a test function f,

$$\begin{aligned}
&\int_{\mathbf{R^n}}\int_{a+V}\delta(x-y)\,dy\,f(x)\,dx\\
&= \int_{a+V}\int_{\mathbf{R^n}}\delta(x-y)f(x)\,dx\,dy = \int_{a+V} f(y)\,dy.
\end{aligned} \tag{1.5}$$

Replacing the Lebesgue measure dx with the Gaussian measure on $\mathbf{R^n}$, $d\mu(x) = (2\pi)^{-n/2}e^{-|x|^2/2}dx$, we see from (1.2) that

$$\begin{aligned}\int_{\mathbf{R^n}} \delta_{a+V}(x)f(x)\,d\mu(x) &= \int_{\mathbf{R^n}} \delta_{a+V}(x)f(x)e^{-|x|^2/2}\frac{dx}{(2\pi)^{n/2}}\\ &= \int_{a+V} f(y)e^{-|y|^2/2}\frac{dy}{(2\pi)^{n/2}}\end{aligned}$$

Now we use that $a \in V^{\perp}$ and $y = a + v$ for some $v \in V$ to see $\langle y, a\rangle = |a|^2$ and hence $|y-a|^2 = |y|^2 - |a|^2$. This gives us that the above is

$$\begin{aligned}&= (2\pi)^{-d/2}e^{-|a|^2/2}\int_{a+V} f(y)e^{-|y-a|^2/2}\frac{dy}{(2\pi)^{(n-d)/2}}\\ &= (2\pi)^{-d/2}e^{-|a|^2/2}\int_{a+V} f(y)\,d\mu_{a+V}(y)\end{aligned}$$

where μ_{a+V} is the Gaussian measure on $a+V$.

Thus to define a delta function $\tilde{\delta}_{a+V}$ with respect to the Gaussian measure for the affine subspace $a+V$ we need to make a small adjustment to δ_{a+V}:

$$\tilde{\delta}_{a+V} \stackrel{\text{def}}{=} (2\pi)^{d/2}e^{|a|^2/2}\delta_{a+V}. \tag{1.6}$$

With this small modification we see from the above calculations that

$$\int_{\mathbf{R^n}} \tilde{\delta}_{a+V}(x)f(x)\,d\mu(x) = \int_{a+V} f(x)\,d\mu_{a+V}(x).$$

Also equation (1.5) becomes

$$\tilde{\delta}_{a+V} = (2\pi)^{d/2}e^{|a|^2/2}\int_{a+V}\delta(x-y)\,dy.$$

Observe, for a test function f,

$$\begin{aligned}&\int_{\mathbf{R^n}} (2\pi)^{d/2}e^{|a|^2/2}\int_{a+V}\delta(x-y)\,dy\,f(x)\,d\mu(x)\\ &= (2\pi)^{d/2}e^{|a|^2/2}\int_{a+V}\int_{\mathbf{R^n}}\delta(x-y)\,f(x)e^{-|x|^2/2}\,\frac{dx}{(2\pi)^{n/2}}dy\\ &= (2\pi)^{d/2}e^{|a|^2/2}\int_{a+V} f(y)e^{-|y|^2/2}\,\frac{dy}{(2\pi)^{n/2}}\\ &= \int_{a+V} f(y)e^{-|y-a|^2/2}\,\frac{dy}{(2\pi)^{(n-d)/2}} \qquad \text{since } |y-a|^2 = |y|^2 - |a|^2\\ &= \int_{a+V} f(y)\,d\mu_{a+V}(y).\end{aligned}$$

In white noise distribution theory, one has the Kubo–Yokoi delta function, $\tilde{\delta}_a$. This is the delta function at the point a. In this paper we extend this notion to a delta function for subspace V or an affine subspace $a + V$ of a real separable Hilbert space H_0.

Sections 2, 3, and 4 are devoted to an account of the machinery–such as test functions and distributions–necessary to formulate our results. Section 5 gives an overview of the delta functions currently used in white noise distribution theory. Sections 6 and 7 contains the development of the delta function for an affine subspace along with a formulation of its Wiener–Itô decomposition.

2. Test Functions Over Infinite-dimensional Spaces

In this section we summarize the necessary notions concerning test functions on infinite-dimensional linear spaces. We also set up notation to be used in the rest of the paper.

2.1. Test Functions and Distributions

A distributions over a space X is a continuous linear functional on a space $\mathcal{E}$ of appropriately chosen 'test functions' over X. For analysis we would also have some measure μ on X and

$$\mathcal{E} \subset E = L^2(\mu).$$

The classical example uses the Schwartz space $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$. The topology on $\mathcal{E}$ is given by some family of norms. Thus, in abstract, the basic framework is a pair

$$\mathcal{H} \subset H_0. \tag{2.1}$$

where H_0 is a real separable Hilbert space with norm $|\cdot|_0$ and inner-product $\langle \cdot, \cdot \rangle_0$ and $\mathcal{H}$ a nuclear space. To form $\mathcal{H}$ we take an operator A on H_0 such that there exists an orthonormal basis $\{e_k\,;\, k = 1, 2, 3, \dots\}$ for H_0 satisfying

(1) $Ae_k = \lambda_k e_k$, for $k = 1, 2, 3, \dots$
(2) $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$
(3) $\sum_{k=1}^{\infty} \lambda_k^{-2} < \infty$.

Remark 2.1. The condition that $1 < \lambda_1$ is needed when proving continuity of test functions.
Note that A^{-1} is a bounded operator with norm given by

$$\rho = \|A^{-1}\| = \frac{1}{\lambda_1}. \tag{2.2}$$

Now for each $p \geq 0$ we define the norm $|x|_p = |A^p x|_0$ and let

$$H_p = \{x \in H_0\,;\ |x|_p < \infty\}, \qquad p \geq 0.$$

Then we have that $H_p \subset H_q$ for any $p \geq q$ and the inclusion map $I_{p,p-1} : H_p \hookrightarrow H_{p-1}$ is a Hilbert–Schmidt operator. We then define $\mathcal{H}$ to the the projective limit of $\{H_p\,;\ p = 0, 1, 2, \ldots\}$ and this gives us

$$\mathcal{H} = \bigcap_{p \geq 0} H_p \subset \cdots H_2 \subset H_1 \subset H_0.$$

Below we describe in brief how a space of test functions is constructed over the dual space $\mathcal{H}'$ using this framework.

The symmetric Fock space $\mathcal{F}_s(V)$ over a Hilbert space V is the subspace of symmetric tensors in the completion of the tensor algebra $T(V)$ under the inner–product given by

$$\langle a, b\rangle_{T(V)} = \sum_{n=0}^{\infty} n! \langle a_n, b_n\rangle_{V^{\otimes n}}, \tag{2.3}$$

where $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0}$ are elements of $T(V)$ with a_n, b_n in the tensor power $V^{\otimes n}$. Then we have

$$\mathcal{F}_s(\mathcal{H}) \stackrel{\text{def}}{=} \bigcap_{p \geq 0} \mathcal{F}_s(H_p) \subset \cdots \subset \mathcal{F}_s(H_2) \subset \mathcal{F}_s(H_1) \subset \mathcal{F}_s(H_0). \tag{2.4}$$

Thus, the pair (2.1) gives rise to a corresponding pair by taking symmetric Fock spaces:

$$\mathcal{F}_s(\mathcal{H}) \subset \mathcal{F}_s(H_0) \tag{2.5}$$

The dual space $\mathcal{H}'$ of continuous real linear functionals on $\mathcal{H}$ is the union

$$\mathcal{H}' = \bigcup_{p \geq 0} H_{-p}$$

where H_{-p} is the set of real linear functionals on $\mathcal{H}$ which are continuous in the $|\cdot|_p$ norm. Note that H_{-p} is naturally isomorphic to $H_p' \simeq H_p$ and is thus a Hilbert space. We denote the norm on H_{-p} by $|\cdot|_{-p}$. Thus, the inner product on H_0 extends to a bilinear pairing between H_p and H_{-p} with

$$|\langle f, x\rangle| \leq |x|_p |f|_{-p}$$

for all $p \geq 0$, $x \in H_p$ and $f \in H_{-p}$. We have then a chain of inclusions

$$H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'$$

where the inner product on H_0 extends to a bilinear pairing between $\mathcal{H}$ and $\mathcal{H}'$. The dual space $\mathcal{H}'$ may be equipped with the weak or the strong or the inductive limit topologies.

Fact 2.2. The Borel sigma algebras generated by the weak, strong, and inductive topologies on $\mathcal{H}'$ are equivalent.

Although this result is known and has been used implicitly or explicitly in the literature, a complete readily accessible proof can be found in [1].

2.2. The Gaussian measure μ

In infinite dimensions the role of Lebesgue measure is played by Gaussian measure. The standard Gaussian measure μ for the pair (2.1) is a Borel measure on $\mathcal{H}'$, specified uniquely by

$$\int_{\mathcal{H}'} e^{i\hat{x}}\, d\mu = e^{-|x|_0^2/2} \qquad \text{for all } x \in \mathcal{H}, \tag{2.6}$$

where

$$\hat{x} : \mathcal{H}' \to \mathbf{R} : f \mapsto f(x)$$

There is a standard unitary isomorphism, the *Wiener-Itô isomorphism* or wave-particle duality map, which identifies the complexified Fock space $\mathcal{F}_s(H_0)_c$ with $L^2(\mathcal{H}', \mu)$. This is uniquely specified by

$$I : \mathcal{F}_s(H_0)_c \to L^2(\mathcal{H}', \mu) : \mathrm{Exp}(x) \mapsto e^{\hat{x} - \frac{1}{2}x^2} \tag{2.7}$$

where $x^2 = |x|_0^2$ and

$$\mathrm{Exp}(x) = \sum_{n \geq 0} \frac{1}{n!} x^{\otimes n}.$$

Indeed, it is readily checked that I preserves inner–products (the inner–product is as described in (2.3)). Using I, for each $\mathcal{F}_s(H_p)_c$, we have the corresponding space $[H]_p \subset L^2(\mathcal{H}', \mu)$ with with norm $\|\cdot\|_p$ induced by the norm on the space $\mathcal{F}_s(H_p)_c$. From this the chain of spaces (2.4) can be transferred into a chain of function spaces:

$$[\mathcal{H}] = \bigcap_{p \geq 0} [H]_p \subset \cdots \subset [H]_2 \subset [H]_1 \subset [H]_0 = L^2(\mathcal{H}', \mu). \tag{2.8}$$

Observe $[\mathcal{H}]$ is a nuclear space with topology induced by the norms $\{\|\cdot\|_p\, ; p = 0, 1, 2, \ldots\}$.

Thus, starting with the pair

$$\mathcal{H} \subset H_0$$

one obtains a corresponding pair

$$[\mathcal{H}] \subset L^2(\mathcal{H}', \mu).$$

Note that the measure μ uses H_0 as a real separable Hilbert space. In the following section we describe a number of properties about the space $[\mathcal{H}]$, which demonstrate that it is sensible to take $[\mathcal{H}]$ as a test function space over $\mathcal{H}'$.

3. Properties of $[\mathcal{H}]$

The following theorem summarizes the properties of $[\mathcal{H}]$ we need. The results here are standard (see, for instance, Kuo's monograph [5]), and we compile them here for ease of reference.

Theorem 3.1. *Every function in $[\mathcal{H}]$ is μ-almost-everywhere equal to a unique continuous function on $\mathcal{H}'$. Moreover, working with these continuous versions,*

(1) *$[\mathcal{H}]$ is an algebra under pointwise operations;*
(2) *pointwise addition and multiplication are continuous operations $[\mathcal{H}] \times [\mathcal{H}] \to [\mathcal{H}]$;*
(3) *for any $f \in \mathcal{H}'$, the evaluation map*

$$\delta_f : [\mathcal{H}] \to \mathbf{R} : \phi \mapsto \phi(f)$$

is continuous;

(4) *the exponentials $e^{\hat{x} - \frac{1}{2}|x|_0^2}$, with x running over $\mathcal{H}$, span a dense subspace of $[\mathcal{H}]$.*

A complete characterization of the space $[\mathcal{H}]$ was obtained by Y. J. Lee (see [6] or the account in [5]).

Remark 3.2. Note that (4) gives us immediately that the span of $e^{\hat{x}}$ with x running over $\mathcal{H}$ is dense in $[\mathcal{H}]$. It also follows from (4) that the span of $e^{\hat{z}}$ with z running over $\mathcal{H}_c$ is dense in $[\mathcal{H}]$.

Now we would like to formulate a more explicit relationship between $[H]_p$ and $\mathcal{F}_s(H_p)_c$. To do this we use the notion of Wick Tensor.

3.1. Wick Tensors

We begin by introducing the trace operator. The trace operator is in $(\mathcal{H}')^{\widehat{\otimes} 2}$ and is defined by

$$\langle \tau, x \otimes y \rangle = \langle x, y \rangle \qquad x, y \in \mathcal{H}$$

It can be represented by

$$\sum_{k=1}^{\infty} e_k \otimes e_k$$

where $\{e_k\}_{k=1}^{\infty}$ forms a complete orthonormal basis of H_0.

3.1.1. Hermite Polynomials

We now review some concepts and properties concerning Hermite polynomials. The function defined by

$$:x^n:_{\sigma^2} \equiv H_n^\sigma(x) = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}$$

is the *Hermite polynomial* of degree n with parameter σ^2. They can also be defined by the generating function:

$$e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} :x^n:_{\sigma^2} \tag{3.1}$$

Using this one can derive that

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbf{R}} H_n^\sigma(x)\, H_m^\sigma(x)\, e^{-x^2/2\sigma^2} dx = n!\delta_{mn} \tag{3.2}$$

where δ_{mn} is 1 if $m = n$ and 0 otherwise.

We have the following formulas for Hermite polynomials

$$:x^n:_{\sigma^2} = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-\sigma^2)^k x^{n-2k} \tag{3.3}$$

$$x^n = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!\, \sigma^{2k} :x^{n-2k}:_{\sigma^2} \tag{3.4}$$

3.1.2. Definition of Wick Tensor

The formula given in (3.3) provides the motivation for the following definition:

Definition 3.3.

Given an element $f \in \mathcal{H}'$, we define the *Wick tensor for f of order n* to be

$$:f^{\otimes n}: = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-1)^k f^{\otimes(n-2k)} \widehat{\otimes} \tau^{\otimes k}.$$

Remark 3.4. For an element $f \in \mathcal{H}'$, we can also define $:f^{\otimes n}:$ inductively as follows:

$$\begin{cases} :f^{\otimes 0}:= 1 \\ :f^{\otimes 1}:= f \\ :f^{\otimes n}:= f\widehat{\otimes} :f^{\otimes (n-1)}: -(n-1)\tau\widehat{\otimes} :f^{\otimes (n-2)}: \qquad \text{for } n \geq 2. \end{cases}$$

Similar to the formula in (3.4) for the Hermite polynomials, we have the following formula for Wick tensors:

$$f^{\otimes n} = \sum_{k=0}^{[n/2]} \binom{n}{2k}(2k-1)!! :f^{\otimes(n-2k)}: \widehat{\otimes}\tau^{\otimes k}.$$

Proposition 3.5. *For any $f \in \mathcal{H}'$ and $x \in \mathcal{H}$ we have*

$$\langle :f^{\otimes n}:, x^{\otimes n}\rangle =: \langle f, x\rangle^n :_{|x|_0^2} \qquad \textit{and} \qquad \|\langle :f^{\otimes n}:, x^{\otimes n}\rangle\|_0 = \sqrt{n!}|x|_0^n.$$

For a proof see page 33, Lemma 5.2 in [5].

Corollary 3.6. *Let $x_1, x_2, \dots \in \mathcal{H}$ be orthogonal vectors in H_0. Then for any $f \in \mathcal{H}'$ we have*

$$\langle :f^{\otimes n}:, x_1^{\otimes n_1} \widehat{\otimes}\, x_2^{\otimes n_2} \widehat{\otimes} \cdots\rangle =: \langle f, x_1\rangle^{n_1}:_{|x_1|_0^2}: \langle f, x_2\rangle^{n_2}:_{|x_2|_0^2} \cdots$$

where $n_1 + n_2 + \cdots = n$. Moreover, the following holds

$$\|\langle :\cdot^{\otimes n}:, x_1^{\otimes n_1} \widehat{\otimes}\, x_2^{\otimes n_1} \widehat{\otimes} \cdots\rangle\|_0 = \sqrt{n_1! n_2! \cdots}|x_1|_0^{n_1}|x_2|_0^{n_2}\cdots$$

By Corollary 3.6. one can see that if $x \in \mathcal{H}^{\widehat{\otimes} n}$, then

$$\|\langle :\cdot^{\otimes n}:, x\rangle\|_0 = \sqrt{n!}|x|_0$$

Using this, we take an element $y \in {H_0}^{\widehat{\otimes} n}$ and a sequence $\{y_k\}$ in ${H_0}^{\widehat{\otimes} n}$ with $y_k \to y$ in ${H_0}^{\widehat{\otimes} n}$. Then, by the equality above, $\{\langle :\cdot^{\otimes n}:, y_k\rangle\}$ is Cauchy in $L^2(\mathcal{H}', \mu)$. Therefore we can define the function $\langle :\cdot^{\otimes n}:, y\rangle$ μ–almost everywhere as the limit in $L^2(\mathcal{H}', \mu)$ of the functions $\{\langle :\cdot^{\otimes n}:, y_k\rangle\}$. Defined in this way we have

$$\|\langle :\cdot^{\otimes n}:, y\rangle\|_0 = \sqrt{n!}|y|_0 \tag{3.5}$$

Of course, for $z = x + iy \in H_{0,c}^{\widehat{\otimes} n}$ we can define for almost every $f \in \mathcal{H}'$,

$$\langle :\cdot^{\otimes n}:, z\rangle = \langle :\cdot^{\otimes n}:, x\rangle + i\langle :\cdot^{\otimes n}:, y\rangle$$

and equation (3.5) still holds.

Corollary 3.7. *Let $x_1, x_2, \cdots \in H_0$ be orthogonal vectors in H_0. Then for almost every $f \in \mathcal{H}'$ we have*

$$\langle :f^{\otimes n}:, x_1^{\otimes n_1} \widehat{\otimes} x_2^{\otimes n_2} \widehat{\otimes} \cdots \rangle = :\langle f, x_1\rangle^{n_1}:_{|x_1|_0^2} :\langle f, x_2\rangle^{n_2}:_{|x_2|_0^2} \cdots$$

where $n_1 + n_2 + \cdots = n$.

For a proof refer to page 34, Corollary 5.3 in [5].

The importance of the Wick tensor lies in the fact that it allows us to make the Wiener–Itô isomorphism more explicit. Take $x \in H_0$ with $x \neq 0$ and form the function

$$\phi_x = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, x^{\otimes n}\rangle$$

Observe that

$$\phi_x(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :f^{\otimes n}:, x^{\otimes n}\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{|x|_0}{\sqrt{2}}\right)^n :\left(\frac{\langle f, x\rangle}{\sqrt{2}|x|_0}\right)^n:$$

Now the last series is the generating function of the Hermite polynomials. So we obtain

$$\phi_x(f) = \exp\left[2\frac{\langle f, x\rangle}{\sqrt{2}|x|_0}\frac{|x|_0}{\sqrt{2}} - \left(\frac{|x|_0}{\sqrt{2}}\right)^2\right] = \exp(\langle f, x\rangle - \tfrac{1}{2}\langle x, x\rangle_0) = e^{\hat{x}(f) - \frac{1}{2}|x|_0^2}$$

Therefore for every function $\phi \in L^2(\mathcal{H}', \mu)$ the Wiener–Itô ismorphism gives a unique $(a_n)_{n=0}^{\infty} \in \mathcal{F}_s(H_0)_c$ such that

$$\phi(f) = \sum_{n=0}^{\infty} \langle :f^{\otimes n}:, a_n\rangle \tag{3.6}$$

in $L^2(\mathcal{H}', \mu)$. This is called the Wiener–Itô expansion for ϕ.

Moreover, for any positive integer p we have

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!|a_n|_p^2 \tag{3.7}$$

where $|\cdot|_p$ is the canonical extension of the H_p norm to $H_{p,c}^{\widehat{\otimes} n}$. This implies that if $\phi \in [H]_p$, then $a_n \in H_{p,c}^{\widehat{\otimes} n}$ for all n. Conversely, it is easy to see that given any $(a_n)_{n=0}^{\infty} \in \mathcal{F}_s(H_p)_c$, we have that $\phi(f) = \sum_{n=0}^{\infty} \langle :f^{\otimes n}:, a_n\rangle$ defines a unique function ϕ in $[H]_p$.

With the notion of Wiener–Itô expansion the following theorem gives us property (4) of 3.1. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be the inner–product on $L^2(\mathcal{H}', \mu)$.

Proposition 3.8. *Let $\phi, \psi \in [\mathcal{H}]$. Recall that for $x \in \mathcal{H}$ the function $e^{\hat{x}-|x|_0^2/2}$ is a test function in $[\mathcal{H}]$. If $\langle\langle \phi, e^{\hat{x}-|x|_0^2/2}\rangle\rangle = \langle\langle \psi, e^{\hat{x}-|x|_0^2/2}\rangle\rangle$ for all $x \in \mathcal{H}$, then $\phi = \psi$.*

This is a well known result in white noise distribution theory. For instance, see page 39, Proposition 5.10 in [5].

It follows immediately from the above propostion that the exponentials $e^{\hat{x}-|x|_0^2/2}$, with x running over $\mathcal{H}$, span a dense subspace of $[\mathcal{H}]$.

4. Function Spaces and Duals Over Infinite-dimensional Spaces

The identification of H_0' with H_0 leads to a complete chain

$$\mathcal{H} = \bigcap_{p\geq 0} H_p \subset \cdots \subset H_1 \subset H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p\geq 0} H_{-p} = \mathcal{H}'. \tag{4.1}$$

In the same way we have a chain for the 'second quantized' spaces $\mathcal{F}_s(H_q)_c \simeq [H]_q$. The unitary isomorphism I extends to unitary isomorphisms

$$I : \mathcal{F}_s(H_{-p})_c \to [H]_{-p} \stackrel{\text{def}}{=} [H]_p' \subset [\mathcal{H}]', \tag{4.2}$$

for all $p \geq 0$. In more detail, for $a \in \mathcal{F}_s(H_{-p})_c$ the distribution $I(a)$ is specified by

$$\langle I(a), f\rangle = \langle a, I^{-1}(f)\rangle, \tag{4.3}$$

for all $f \in [\mathcal{H}]$. On the right side here we have the pairing of $\mathcal{F}_s(H_{-p})_c$ and $\mathcal{F}_s(H_p)_c$ induced by the duality pairing of H_{-p} and H_p; in particular, the pairings above are complex bilinear (not sesquilinear).

The Wiener–Itô expansion from 3.6 can be extended to functions Φ in $[\mathcal{H}]'$.

Theorem 4.1. *Given a $\Phi \in [H]_{-p}$, there exists a unique element $A = (A_n)_{n=0}^\infty \in \mathcal{F}_s(H_{-p})_c$ such that*

$$\langle\langle \Phi, \phi\rangle\rangle = \sum_{n=0}^{\infty} n!\langle A_n, a_n\rangle \qquad \text{for all } \phi \in [H]_p \tag{4.4}$$

where $\phi(f) = \sum_{n=0}^\infty \langle : f^{\otimes n} :, a_n\rangle$ μ–a.e. Conversely, given a sequence $A = (A_n)_{n=0}^\infty \in \mathcal{F}_s(H_{-p})_c$ we can define a $\Phi \in [H]_{-p}$ by (4.4)*. Moreover, we have that*

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n!|A_n|_{-p}^2 = \|A\|_{\mathcal{F}_s(H_{-p})_c}^2. \tag{4.5}$$

See the account in [10] for a proof (page 36, Theorem 3.1.6).

Using the previous theorem we can adopt a formal expression for $\Phi \in [\mathcal{H}]'$ as follows:

$$\Phi(f) = \sum_{n=0}^{\infty} \langle : f^{\otimes n} :, A_n \rangle. \tag{4.6}$$

Here $\langle : f^{\otimes n} :, A_n \rangle$ is not a function of $f \in \mathcal{H}'$, but a generalized function. It can only be understood through the pairing with a test function in $[\mathcal{H}]$. The expression given by (4.6) is called the *Wiener–Itô expansion* of Φ.

4.1. The Segal-Bargmann Transform

Next we look at the Segal–Bargmann transform. In its simplest form this is defined for any function $\psi \in L^2(\mathcal{H}', \mu)$ to be the function $S\psi$ on the complexified space $\mathcal{H}_c$ given by

$$S\psi(z) = \int_{\mathcal{H}'} e^{\tilde{z} - z^2/2} \psi \, d\mu \tag{4.7}$$

with notation as follows: if $z = a + ib$, with $a, b \in \mathcal{H}$ then

$$\tilde{z}(x) \stackrel{\text{def}}{=} zx \stackrel{\text{def}}{=} \langle x, a \rangle + i \langle x, b \rangle, \tag{4.8}$$

and $z^2 = zz$, where the product zu is specified through

$$zu \stackrel{\text{def}}{=} \langle a, s \rangle_0 - \langle b, t \rangle_0 + i \left(\langle a, t \rangle_0 + \langle b, s \rangle_0 \right) \tag{4.9}$$

if $z = a + ib$ and $u = s + it$, where $a, b, s, t \in \mathcal{H}$.

Let μ_c be the Gaussian measure $\mathcal{H}_c'$ specified by the requirement that

$$\int_{\mathcal{H}_c'} e^{ax+by} \, d\mu_c(x + iy) = e^{(a^2+b^2)/4} \tag{4.10}$$

for every $a, b \in \mathcal{H}$. For convenience, let us introduce the renormalized exponential function

$$c_w = e^{\tilde{w} - w^2/2} \tag{4.11}$$

for all $w \in \mathcal{H}_c$. Thankfully, c_w lies in $L^2(\mathcal{H}', \mu)$, which means the integrand in (4.7) exists for all $z \in \mathcal{H}_c$. It is readily checked that

$$[Sc_w](z) = e^{wz} \tag{4.12}$$

for all $w, z \in \mathcal{H}_c$. Thus we may take Sc_w as a function on $\mathcal{H}'_c$ given by

$$Sc_w = e^{\tilde{w}}, \tag{4.13}$$

where now $\tilde{w}$ is a function on $\mathcal{H}'_c$ in the natural way. Then $Sc_w \in L^2(\mathcal{H}'_c, \mu_c)$ and one has

$$\langle Sc_w, Sc_u \rangle_{L^2(\mu_c)} = \langle c_w, c_u \rangle_{L^2(\mu_c)} = e^{w\overline{u}}.$$

This shows that S provides an isometry from the linear span of the exponentials c_w in $L^2(\mathcal{H}', \mu)$ onto the linear span of the complex exponentials $e^{\tilde{w}}$ in $L^2(\mathcal{H}'_c, \mu_c)$. Passing to the closure one obtains the **Segal-Bargmann** unitary isomorphism

$$S : L^2(\mathcal{H}', \mu) \to Hol^2(\mathcal{H}'_c, \mu_c)$$

where $Hol^2(\mathcal{H}'_c, \mu_c)$ is the closed linear span of the complex exponential functions $e^{\tilde{w}}$ in $L^2(\mathcal{H}'_c, \mu_c)$.

An explicit expression for $SF(z)$ is suggested by (4.7). For any $F \in [\mathcal{H}]$ and $z \in \mathcal{H}'_c$, we have

$$(SF)(z) = \langle\langle I(\mathrm{Exp}(z)) \, , F \rangle\rangle \tag{4.14}$$

where the right side is the evaluation of the distribution $I(\mathrm{Exp}(z))$ on the test function F. Indeed it may be readily checked that if $SF(z)$ is defined in this way then $[Sc_w](z) = e^{wz}$.

In view of (4.14), it natural to extend the Segal-Bargmann transform to distributions: for $\Phi \in [\mathcal{H}]'$, define $S\Phi$ to be the function on $\mathcal{H}_c$ given by

$$S\Phi(z) \stackrel{\mathrm{def}}{=} \langle\langle \Phi, I(\mathrm{Exp}(z)) \rangle\rangle \qquad z \in \mathcal{H}_c. \tag{4.15}$$

One of the many applications of the the S–transform includes its usefulness in characterizing generalized functions in $[\mathcal{H}]'$.

Theorem 4.2. (Potthoff-Streit) *Suppose a function F on $\mathcal{H}_c$ satisfies:*

(1) *For any $z, w \in \mathcal{H}_c$, the function $F(\alpha z + w)$ is an entire function of $\alpha \in \mathbf{C}$.*

(2) *There exists nonnegative constants A, p, and C such that*

$$|F(z)| \leq Ce^{A|z|_p^2} \qquad \text{for all } z \in \mathcal{H}_c.$$

Then there is a unique generalized function $\Phi \in [\mathcal{H}]'$ such that $F = S\Phi$. Conversely, given such a $\Phi \in [\mathcal{H}]'$, then $S\Phi$ satisfies (1) and (2) above.

The proof of this theorem can be found in Theorem 8.2 on page 79 in [5].

5. White Noise Delta Functions

We now review some of the popular delta function type distributions commonly used in white noise distribution theory.

5.1. Kubo–Yokoi Delta Function

The White Noise analogue of the finite dimensional Dirac's delta function is the Kubo–Yokoi delta function, $\tilde{\delta}_f$. The distribution $\tilde{\delta}_f$ has the following effect on a test function $\phi \in [\mathcal{H}]$:

$$\langle\langle \tilde{\delta}_f, \phi \rangle\rangle = \phi(f).$$

Note that continuous versions of test functions are assumed as per Theorem 3.1. Also, Theorem 3.1 gives us that the Kubo–Yokoi delta function is continuous.

It is easy enough to compute the *S*–transform of the Kubo–Yokoi delta function

$$S(\tilde{\delta}_f)(z) = \langle\langle \tilde{\delta}_f, e^{\tilde{z}-z^2/2} \rangle\rangle = e^{\tilde{z}(f)-z^2/2} = e^{\langle f,z\rangle - \langle z,z\rangle/2}. \tag{5.1}$$

Lastly, the next theorem gives us the Wiener–Itô expansion for the Kubo–Yokoi delta function.

Theorem 5.1. *The Wiener–Itô expansion of the Kubo–Yokoi delta function at* $f \in \mathcal{H}'$ *is given by*

$$\tilde{\delta}_f = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, : f^{\otimes n} : \rangle.$$

Proof. By Theorem 4.1., $\tilde{\delta}_f$ has Wiener–Itô expansion given by

$$\tilde{\delta}_f = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, A_n \rangle \tag{5.2}$$

where $A_n \in (\mathcal{H}')^{\widehat{\otimes} n}$. For each n consider the function given by $\phi_a = \langle : \cdot^{\otimes n} :, a \rangle$ where $a \in H_{0,c}^{\widehat{\otimes} n}$. By the definition of $\tilde{\delta}_f$ we have that

$$\langle\langle \tilde{\delta}_f, \phi_a \rangle\rangle = \phi_a(f) = \langle : f^{\otimes n} :, a \rangle. \tag{5.3}$$

But, by using 5.2, we have

$$\langle\langle \tilde{\delta}_f, \phi_a \rangle\rangle = n! \langle A_n, a \rangle. \tag{5.4}$$

Combining equations (5.3) and (5.4) we see that $n!\langle A_n, a\rangle = \langle : f^{\otimes n} :, a \rangle$ for all $a \in H_{0,c}^{\widehat{\otimes} n}$. Therefore $A_n = \frac{1}{n!} : f^{\otimes n} :$ and we have the Wiener–Itô expansion for $\tilde{\delta}_f$ promised by the theorem. ∎

5.2. Donsker's Delta Function

Another delta function often used in White Noise Analysis is Donsker's delta function. It is defined using the classic Gel'fand triple $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$.

Let δ_s be the Dirac delta function at s and note that $\langle \cdot, 1_{[0,t]} \rangle = B(t)$, $t \geq 0$, is a Brownian motion. The generalized function $\delta_s(B(t))$ is *Donsker's delta function*. To see that it is in fact a generalized function we need the following theorem:

Theorem 5.2. *Let $F \in S_c'(\mathbf{R})$ and $f \in L^2(\mathbf{R})$ with $f \neq 0$. Then $F(\langle \cdot, f \rangle)$ is a generalized function and has S–transform given by*

$$SF(\langle \cdot, f \rangle)(z) = \frac{1}{\sqrt{2\pi}|f|_0} \int_{\mathbf{R}} F(y) \exp\left[-\frac{1}{2|f|_0^2}(y - \langle f, z \rangle)^2 \right] dy, \qquad z \in S_c(\mathbf{R})$$

where the integral is understood to be the bilinear pairing of $S_c'(\mathbf{R})$ and $S_c(\mathbf{R})$. For a proof refer to [4] or page 63 in [5].

Using this theorem, we can see that $\delta_s(B(t))$ is in fact a generalized function. Moreover, we have the *S*–transform of $\delta_s(B(t))$ is given by

$$\begin{aligned} S[\delta_s(B(t))](z) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbf{R}} \delta_s(y) e^{-\frac{1}{2t}(y - \langle 1_{[0,t]}, z \rangle)^2} \, dy \\ &= \frac{1}{\sqrt{2\pi t}} \exp\Big[-\frac{1}{2t}\Big(s - \int_0^t z(u)\, du\Big)^2 \Big]. \end{aligned}$$

6. The Delta Function for an Affine Subspace

Let H_0 be a real separable Hilbert space with V a subspace of H_0. In the paper [8], Lomonaco and Kauffman present the formal integral

$$\int_V \delta(x - v)\, Dv \tag{6.1}$$

where Dv is the (nonexistent) Lebesgue measure on V. This integral is meant to represent a delta function of sorts on the subspace V. In this section we make this idea rigorous through the framework of White Noise Distribution Theory. We also extend this idea slightly to include not only subspaces, but affine subspaces $a + V$, where $a \in V^{\perp}$. Here,

$$\delta_{a+V} = \int_{a+V} \delta(x - y)\, Dy. \tag{6.2}$$

6.1. Formal Calculations

Let f be a function on H_0. We will show that when the terms in (6.2) are integrated against a function f, we arrive at the expected result, $\int_{a+V} \delta(x-y)\, Dy$. Thus we say the equalities in (6.2) hold in the distributional sense. Observe:

(1) Essentially by the "definition" of δ_{a+V} we have

$$\int_{H_0} \delta_{a+V}(x) f(x)\, Dx = \int_{a+V} f(y)\, Dy.$$

(2) Now for $\int_{a+V} \delta(x-y) Dy$ we calculate

$$\begin{aligned}\textstyle\int_{H_0}\int_{a+V} \delta(x-y) Dy\, f(x)\, Dx &= \int_{a+V}\int_{H_0} \delta(x-y) f(x) Dx\, Dy \\ &= \int_{a+V} f(v) Dv.\end{aligned}$$

This formally verifies the equation in (6.2).

Of course, in infinite dimensions there is no notion of Lebesgue measure, however there is the notion of Gaussian measure. So just as was done in equation (1.6) we will "tweek" the delta function δ_{a+V} in order to form a delta function with respect to μ:

$$\tilde{\delta}_{a+V} = (2\pi)^{\dim V^{\perp}/2} e^{|a|_0^2/2} \delta_{a+V}..$$

This is just a formal definition as $\dim V^{\perp}$ could very well be infinite. However, this gives us at a very formal level

$$\int_{H_0} \tilde{\delta}_{a+V}(x) f(x)\, d\mu(x) = \int_{a+V} f(y)\, d\mu_{a+V}(y)$$

where μ and μ_{a+V} are Gaussian type measures on H_0 and $a+V$.

Now the Gaussian measure cannot live on H_0 or $a+V$. However, just as we used the Minlos theorem to form the Gaussian measure μ on $\mathcal{H}'$ (which we think of as the Gaussian measure on H_0), we can again use the Minlos theorem to form the Gaussian measure for the affine subspace $a+V$. For a subspace W of $\mathbf{R^n}$ and $a \in W^{\perp}$ we have the Gaussian measure on $a+W$ given by:

$$\int_{a+W} e^{i\langle x,y\rangle}\, d\mu_{a+W}(x) = \int_{a+W} e^{i\langle x,y\rangle} e^{-\frac{1}{2}|x-a|^2} \frac{dx}{(2\pi)^{\dim W/2}} = e^{i\langle a,y\rangle - \frac{1}{2}\langle y_W, y_W\rangle}$$

where $y \in \mathbf{R^n}$ and y_W is the projection of y onto W.

6.2. Gaussian Measure on $a+V$

Using the Minlos theorem one can find that there is a measure μ_{a+V} on $\mathcal{H}'$ with

$$\int_{\mathcal{H}'} e^{i\langle x,y\rangle}\, d\mu_{a+V}(x) = e^{i\langle a,y\rangle - \frac{1}{2}\langle y_V, y_V\rangle}$$

for any $y \in \mathcal{H}$. This measure μ_{a+V} is the Gaussian measure for the affine subspace $a+V$. For an alternate (and more explicit) construction of this measure see [9].

6.3. Hida Measure

The Gaussian measure μ_{a+V} is a special type of measure known as a Hida measure. In this section we define the notion of Hida measure and give an overview of some its properties.

Definition 6.1. A measure ν on $\mathcal{H}'$ is called a *Hida measure* if $\phi \in L^1(\nu)$ for all $\phi \in [\mathcal{H}]$ and the linear functional

$$\phi \mapsto \int_{\mathcal{H}'} \phi(x)\, d\nu(x)$$

is continuous on $[\mathcal{H}]$.

We say that a generalized function $\Phi \in [\mathcal{H}]'$ is *induced* by a Hida measure ν if for any $\phi \in [\mathcal{H}]$ we have

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi(x)\, d\nu(x).$$

The following theorem characterizes those generalized functions which are induced by a Hida measure.

Theorem 6.2. *Let $\Phi \in [\mathcal{H}]'$. Then the following are equivalent:*

(1) *For any nonnegative $\phi \in [\mathcal{H}]$, $\langle\langle \Phi, \phi \rangle\rangle \geq 0$*
(2) *$\mathcal{T}(\Phi)(x) = \langle\langle \Phi, e^{i\langle \cdot, x\rangle} \rangle\rangle$ is positive definite on $\mathcal{H}$*
(3) *Φ is induced by a Hida measure*

A proof of this theorem can be found in [5] (page 320, Theorem 15.3).

Corollary 6.3. *Let ν be a finite measure on $\mathcal{H}'$ such that for any $x \in \mathcal{H}$*

$$\langle\langle \Phi, e^{i\langle \cdot, x\rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x\rangle}\, d\nu(y)$$

for some $\Phi \in [\mathcal{H}]'$. Then Φ is induced by ν.

Proof. Since $\langle\langle \Phi, e^{i\langle \cdot, x\rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x\rangle}\, d\nu(y)$ it is clear that $\langle\langle \Phi, e^{i\langle \cdot, x\rangle} \rangle\rangle$ is positive definite. So we can apply Theorem 6.2. to get a finite measure m which is induced by Φ. Hence for all $\phi \in [\mathcal{H}]$,

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi\, dm.$$

Letting $\phi = e^{i\langle\cdot,x\rangle}$ in the above equation, we see that the characteristic functions for m and ν are identical. Therefore $m = \nu$ and we have that Φ is induced by ν. ∎

6.4. Definition of $\tilde{\delta}_{a+V}$

Observe the effect of μ_{a+V} on the renormalized exponential $e^{\langle\cdot,z\rangle-\frac{1}{2}\langle z,z\rangle}$,

$$\begin{aligned}\int_{\mathcal{H}'} e^{\langle x,z\rangle-\frac{1}{2}\langle z,z\rangle}\, d\mu_{a+V}(x) &= e^{-\langle z,z\rangle}\int_{\mathcal{H}'} e^{\langle x,z\rangle}\, d\mu_{a+V}(x)\\ &= e^{-\langle z,z\rangle} e^{\langle a,z\rangle+\frac{1}{2}\langle z_V,z_V\rangle}\\ &= e^{\langle a,z\rangle-\frac{1}{2}\langle z_{V^\perp},z_{V^\perp}\rangle}.\end{aligned}$$

Let the function $F(z)$ denote the result from the calculations above. That is,

$$F(z) = e^{\langle a,z\rangle-\frac{1}{2}\langle z_{V^\perp},z_{V^\perp}\rangle} \tag{6.3}$$

We will show that $F(z)$ satisfies properties (1) and (2) of Theorem 4.2.

For property (1) consider $F(\alpha z+w)$ where $z, w \in \mathcal{H}_c$ and $\alpha \in \mathbf{C}$. Then notice that

$$\begin{aligned}F(\alpha z+w) &= e^{\langle a,\alpha z+w\rangle-\frac{1}{2}\langle \alpha z_{V^\perp}+w_{V^\perp},\alpha z_{V^\perp}+w_{V^\perp}\rangle}\\ &= \exp[\alpha\langle a,z\rangle+\langle a,w\rangle-\tfrac{1}{2}(\alpha^2\langle z_{V^\perp},z_{V^\perp}\rangle\\ &\quad +2\alpha\langle z_{V^\perp},w_{V^\perp}\rangle+\langle w_{V^\perp},w_{V^\perp}\rangle)]\\ &= e^{-\frac{\alpha^2}{2}\langle z_{V^\perp},z_{V^\perp}\rangle} e^{\alpha(\langle a,z\rangle-\langle z_{V^\perp},w_{V^\perp}\rangle)} e^{\langle a,w\rangle-\frac{1}{2}\langle w_{V^\perp},w_{V^\perp}\rangle}\end{aligned}$$

which is an entire function of $\alpha \in \mathbf{C}$.

Now for property (2) of Theorem 4.2 we write z as $z = x+iy$ with $x, y \in \mathcal{H}$ and observe that

$$\begin{aligned}|F(z)| &= |e^{\langle a,z\rangle-\frac{1}{2}\langle z_{V^\perp},z_{V^\perp}\rangle}|\\ &= |e^{\langle a,x+iy\rangle-\frac{1}{2}\langle x_{V^\perp}+iy_{V^\perp},x_{V^\perp}+iy_{V^\perp}\rangle}|\\ &= e^{\langle a,x\rangle} e^{-\frac{1}{2}|x_{V^\perp}|_0^2+\frac{1}{2}|y_{V^\perp}|_0^2}\\ &\le e^{\langle a,x\rangle} e^{\frac{1}{2}|z_{V^\perp}|_0^2}\\ &\le e^{\frac{1}{2}|a|_0^2+\frac{1}{2}|z|_0^2} e^{\frac{1}{2}|z|_0^2}\\ &\le e^{\frac{1}{2}|a|_0^2} e^{\frac{3}{2}|z|_0^2}.\end{aligned}$$

So property (2) of Theorem 4.2 is satisfied.

Therefore by Theorem 4.2 there exist some $\Phi \in [\mathcal{H}]'$ such that $S(\Phi)(z) = F(z)$. Then by Corollary 6.3 we have that Φ is induced by μ_{a+V}. We simply denote this Φ by $\tilde{\delta}_{a+V}$. This leads us to the following definition:

Definition 6.4. The *delta function for the affine subspace* $a + V$ is the distribution in $[\mathcal{H}]'$ induced by the measure μ_{a+V}. We denote this generalized function by $\tilde{\delta}_{a+V}$.

Thus for any test function $\phi \in [\mathcal{H}]$ we have

$$\langle\langle \tilde{\delta}_{a+V}, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi \, d\mu_{a+V}.$$

Note that this is analogous to the effect of the corresponding distribution in $\mathbf{R^n}$.

6.5. *S*–transform

Using the definition of the distribution $\tilde{\delta}_{a+V}$ we can directly compute its *S*–transform. By the calculations directly preceding 6.3 we have

$$S(\tilde{\delta}_{a+V})(z) = e^{\langle a,z\rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle} \qquad \text{for} \quad z \in \mathcal{H}_c. \tag{6.4}$$

6.5.1. Relationship with the Kubo–Yokoi Delta Function

As the notation indicates, the delta function for the affine subspace $a + V$, $\tilde{\delta}_{a+V}$, is related to the Kubo–Yokoi delta function from Section 5. If we interpret the Kubo–Yokoi delta function at a, $\tilde{\delta}_a$, as the delta function for the affine subspace $a + 0$, then observing that $V^\perp = 0^\perp = H_0$, we see from the definition above that the *S*–transform of $\tilde{\delta}_{a+0}$ is given by $e^{\langle a,z\rangle - \frac{1}{2}\langle z,z\rangle}$, which is the *S*–transform of the Kubo–Yokoi delta function $\tilde{\delta}_a$ (see equation 5.1).

6.5.2. Relationship with Donsker's delta function

What is not so apparent is that the delta function for an affine subspace, $\tilde{\delta}_{a+V}$, is related to Donsker's delta function. We saw in Section 5.2 that Donsker's delta function is defined for the Gel'fand triple $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$ and is usually given by $\delta_s(B(t))$ where $B(t) = \langle \cdot, 1_{[0,t]} \rangle$, $t \geq 0$, and δ_s is the Dirac delta function at s. However, this definition can be extended slightly. In fact, by Theorem 5.2, we have for any $f \in L^2(\mathbf{R})$, $\delta_s(\langle \cdot, f \rangle)$ is in $\mathcal{S}'(\mathbf{R})$.

Now given a unit vector $f \in L^2(\mathbf{R})$ consider the generalized function $\delta_s(\langle \cdot, f \rangle)$. Intuitively, this is a function that gives enormous weight to vectors $g \in L^2(\mathbf{R})$ with $\langle f, g \rangle = s$. (Considering the case where $s = 0$ we can see that this distribution should have density at $V = \{f\}^\perp$.) So taking $V = \{f\}^\perp$ and $a = sf$, the distribution $\delta_s(\langle \cdot, f \rangle)$ should be related to $\tilde{\delta}_{a+V}$.

Using Theorem 5.2. to find the *S*–transform of $\delta_s(\langle\cdot, f\rangle)$ we see that

$$\begin{aligned} S[\delta_s(\langle\cdot, f\rangle)](z) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \delta(y-s)e^{-\frac{1}{2}(y-\langle f,z\rangle)^2}dy \\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(s-\langle f,z\rangle)^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-s^2/2+s\langle f,z\rangle-\langle f,z\rangle^2/2}. \end{aligned}$$

Now since $V=\{f\}^\perp$, we have $V^\perp = \{\mathbf{R}f\}$. Thus the $V^\perp$ component of z, $z_{V^\perp}$, is given by $\langle f,z\rangle f$. Observe

$$\langle f,z\rangle^2 = \langle\langle f,z\rangle f, \langle f,z\rangle f\rangle = \langle z_{V^\perp}, z_{V^\perp}\rangle$$

and

$$s\langle f,z\rangle = \langle sf, z\rangle = \langle a, z\rangle.$$

Therefore, we have

$$S(\delta_s(\langle\cdot, f\rangle))(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}|a|_0^2} e^{\langle a,z\rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle}.$$

Thus, Proposition 3.8 tells us that $\sqrt{2\pi}\, e^{\frac{1}{2}|a|_0^2}\,\delta_s(\langle\cdot, f\rangle) = \tilde{\delta}_{a+V}$.

6.5.3. Realizing $\tilde{\delta}_{a+V}$ through the limit

In finite dimension for V a subspace of $\mathbf{R^n}$ with $a\in V^\perp$ and d the codimension of V we have that

$$\tilde{\delta}_{a+V} = \lim_{\sigma\searrow 0} e^{-|x_V|^2/2}\frac{1}{(2\pi)^{(n-d)/2}} e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}. \tag{6.5}$$

Observe

$$\begin{aligned} &\lim_{\sigma\searrow 0}\int_{\mathbf{R^n}} f(x)e^{-|x_V|^2/2}\frac{1}{(2\pi)^{(n-d)/2}} e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}\,dx \\ &= \lim_{\sigma\searrow 0}\int_V\int_{V^\perp} f(x_V+x_{V^\perp})e^{-|x_V|^2/2}\frac{1}{(2\pi)^{(n-d)/2}} \\ &\times e^{-|x_{V^\perp}-a|^2/2\sigma^2}\frac{1}{(2\pi\sigma^2)^{d/2}}\,dx_{V^\perp}dx_V \\ &= \lim_{\sigma\searrow 0}\int_V\int_{V^\perp} f(x_V+\sigma x_{V^\perp}+a)e^{-|x_V|^2/2}\frac{1}{(2\pi)^{(n-d)/2}} \\ &\times e^{-|x_{V^\perp}|^2/2}\frac{1}{(2\pi)^{d/2}}dx_{V^\perp}\,dx_V \\ &= \int_V\int_{V^\perp} f(x_V+a)e^{-|x_V|^2/2}\frac{1}{(2\pi)^{(n-d)/2}} e^{-|x_{V^\perp}|^2/2}\frac{1}{(2\pi)^{d/2}}\,dx_{V^\perp}dx_V \end{aligned}$$

$$= \int_V f(x_V + a) e^{-|x_V|^2/2} \frac{1}{(2\pi)^{(n-d)/2}}\, dx_V$$
$$= \int_{a+V} f(y) e^{-|y-a|^2/2} \frac{1}{(2\pi)^{(n-d)/2}}\, dy$$
$$= \int_{a+V} f(y)\, d\mu_{a+V}(y).$$

Using the Minlos theorem one can find that there is a measure ρ_σ with

$$\int_{\mathcal{H}'} e^{i\langle x,z\rangle}\, d\rho_\sigma(x) = e^{i\langle a,z\rangle - \frac{1}{2}\langle z_V, z_V\rangle - \frac{\sigma^2}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle}$$

for any $z \in \mathcal{H}$, which is analogous to the characteristic function for the measure found inside the limit of (6.5). Thus

$$S(\tilde{\rho}_\sigma)(z) = e^{\langle a,z\rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle + \frac{\sigma^2}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle}.$$

Moreover,

$$\lim_{\sigma \searrow 0} S(\tilde{\rho}_\sigma)(z) = S(\tilde{\delta}_{a+V})(z) \tag{6.6}$$

for all $z \in \mathcal{H}_c$.

Now we use the following theorem about convergence in $[\mathcal{H}]'$:

Theorem 6.5. *Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence in $[\mathcal{H}]'$ and let $F_n = S\Phi_n$. Then Φ_n converges strongly to Φ in $[\mathcal{H}]'$ if and only if*

(1) *For each $z \in \mathcal{H}_c$, $\lim_{n\to\infty} F_n(z) = F(z)$, where $F(z) = S(\Phi)(z)$.*

(2) *There exists nonnegative constants K, a, and p (independent of n) such that*

$$|F_n(z)| \le K e^{a|z|_p^2}, \qquad \text{for all } z \in \mathcal{H}_c \text{ and } n \in \{1,2,3,\ldots\}.$$

For a proof refer to [11] or page 86, Theorem 8.6 in [5].

We would like to use this theorem to see that $\lim_{\sigma\searrow 0} \tilde{\rho}_\sigma = \tilde{\delta}_{a+V}$ in $[\mathcal{H}]'$. We have already seen that condition (1) of Theorem 6.5 is satisfied. We now show that condition (2) is satisfied. Observe that writing $z = x + iy$ with $x, y \in \mathcal{H}$ we have

$$\begin{aligned}
|S(\tilde{\rho}_\sigma)(z)| &= |e^{\langle a,z\rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle + \frac{\sigma^2}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle}| \\
&= |e^{\langle a,x+iy\rangle - \frac{1}{2}\langle x_{V^\perp}+iy_{V^\perp}, x_{V^\perp}+iy_{V^\perp}\rangle + \frac{\sigma^2}{2}\langle x_{V^\perp}+iy_{V^\perp}, x_{V^\perp}+iy_{V^\perp}\rangle}| \\
&= e^{\langle a,x\rangle} e^{-\frac{1}{2}|x_{V^\perp}|_0^2 + \frac{1}{2}|y_{V^\perp}|_0^2} e^{\frac{\sigma^2}{2}|x_{V^\perp}|_0^2 - \frac{\sigma^2}{2}|y_{V^\perp}|_0^2} \\
&\le e^{\langle a,x\rangle} e^{\frac{1}{2}|z_{V^\perp}|_0^2} e^{\frac{\sigma^2}{2}|x_{V^\perp}|_0^2 + \frac{\sigma^2}{2}|y_{V^\perp}|_0^2} \\
&\le e^{\frac{1}{2}|a|_0^2 + \frac{1}{2}|z|_0^2} e^{\frac{1}{2}|z|_0^2} e^{\frac{\sigma^2}{2}|z|_0^2}.
\end{aligned}$$

Since we are letting $\sigma \searrow 0$ there is no harm in assuming $\sigma \leq 1$ to get

$$\leq e^{\frac{1}{2}|a|_0^2} e^{\frac{3}{2}|z|_0^2}.$$

Therefore condition (2) is satisfied and we get

$$\lim_{\sigma \searrow 0} \tilde{\rho}_\sigma = \tilde{\delta}_{a+V}$$

in $[\mathcal{H}]'$. This is the infinite dimensional analog of (6.5).

7. The Wiener-Itô Decomposition of $\tilde{\delta}_{a+V}$

In this section we find the Wiener–Itô decomposition of $\tilde{\delta}_{a+V}$. We begin by generalizing the definition of the trace operator and Wick tensor found in Section 3.1.

7.1. Subspace Trace Operator

As usual, let V be a closed subspace of our Hilbert space H_0.

Definition 7.1. The *V–trace operator*, which we denote by τ_V is the element in $(\mathcal{H}')^{\widehat{\otimes}2}$ given by

$$\langle \tau_V, z \otimes w \rangle = \langle z_V, w_V \rangle \qquad z, w \in \mathcal{H}_c.$$

The V–trace operator can be represented as

$$\tau_V = \sum_{k=1}^{\infty} e_k \otimes P_V e_k$$

where P_V is the orthogonal projection onto the subspace V. Observe

$$\begin{aligned}
\langle \tau_V, z \otimes w \rangle &= \Big\langle \sum_{k=1}^{\infty} e_k \otimes P_V e_k, z \otimes w \Big\rangle \\
&= \sum_{k=1}^{\infty} \langle e_k, z \rangle \langle P_V e_k, w \rangle \\
&= \sum_{k=1}^{\infty} \langle e_k, z \rangle \langle e_k, w_V \rangle \quad \text{ where } w_V = P_V w \\
&= \Big\langle \sum_{k=1}^{\infty} \langle e_k, z \rangle e_k, \sum_{k=1}^{\infty} \langle e_k, w_V \rangle e_k \Big\rangle \\
&= \langle z, w_V \rangle = \langle z_V, w_V \rangle.
\end{aligned}$$

Remark 7.2. We can also represent τ_V as $\sum_{k=1}^{\infty} P_V e_k \otimes P_V e_k$ or $\sum_{k=1}^{\dim V} v_k \otimes v_k$ where $\{v_k\}_1^{\dim V}$ is an orthonormal basis for V. However, we find that the representation of τ_V given above is more suitable for our computations.

7.2. Subspace Wick Tensor

With the notion of a subspace trace operator securely behind us, we can now define the subspace Wick tensor. Again we let V be a closed subspace of our Hilbert space H_0.

Definition 7.3. For $f \in \mathcal{H}'$ the *V-Wick tensor for f of order n* is defined to be

$$:f^{\otimes n}:_V = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-1)^k f^{\otimes(n-2k)} \widehat{\otimes} \tau_V^{\otimes k}.$$

Proposition 7.4. *For any $f \in \mathcal{H}'$ and $x \in \mathcal{H}$ we have*

$$\langle :f^{\otimes n}:_V, x^{\otimes n}\rangle = :\langle f, x\rangle^n:_{|x_V|_0^2}.$$

Proof. From the definition we have

$$\langle :f^{\otimes n}:_V, x^{\otimes n}\rangle = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-|x_V|_0^2)^k \langle f, x\rangle^{n-2k}.$$

Comparing this with (3.3) we see that $\langle :f^{\otimes n}:_V, x^{\otimes n}\rangle = :\langle f, x\rangle^n:_{|x_V|_0^2}$. ■

7.3. Wiener–Itô Expansion Consider the following function

$$\Phi_f^V = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, :f^{\otimes n}:_V\rangle.$$

We would like to see that Φ_f^V is in $[\mathcal{H}]'$.

Lemma 7.5. *For any $n \geq 1$ and $f \in H_{-p}$ we have*

$$|:f^{\otimes n}:_V|_{-p} \leq \sqrt{n!}(|f|_{-p} + |\tau_V|_{-p}^{1/2})^n.$$

Proof. From the definition of $:f^{\otimes n}:_V$ we see that

$$|:f^{\otimes n}:_V|_{-p} \leq \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!\, |f|_{-p}^{n-2k} |\tau_V|_{-p}^k.$$

Now for $k \le [n/2]$ we use that $(2k-1)!! \le (n-1)!! \le \sqrt{n!}$ to get

$$\begin{aligned}|:f^{\otimes n}:_V|_{-p} &\le \sqrt{n!}\sum_{k=0}^{[n/2]}\binom{n}{2k}|f|_{-p}^{n-2k}(|\tau_V|_{-p}^{1/2})^{2k}\\ &\le \sqrt{n!}\sum_{k=0}^{n}\binom{n}{k}|f|_{-p}^{n-k}(|\tau_V|_{-p}^{1/2})^{k}\\ &\le \sqrt{n!}(|f|_{-p}+|\tau_V|_{-p}^{1/2})^n.\end{aligned}$$

■

Proposition 7.6. *For $f \in \mathcal{H}'$, $\Phi_f^V = \sum_{n=0}^\infty \frac{1}{n!}\langle :\cdot^{\otimes n}:, :f^{\otimes n}:_V\rangle$ is a generalized function (i.e. Φ_f^V is in $[\mathcal{H}]'$).*

Proof. Since $f \in \mathcal{H}'$, f is in H_{-q} for $q \ge 0$. Thus for any $p \ge q$ we have

$$\begin{aligned}\|\Phi_f^V\|_{-p}^2 &= \sum_{n=0}^\infty n!\frac{1}{(n!)^2}|:f^{\otimes n}:_V|_{-p}^2\\ &\le \sum_{n=0}^\infty \frac{1}{n!}\left(\sqrt{n!}\right)^2\left(|x|_{-p}+|\tau_V|_{-p}^{1/2}\right)^{2n} \qquad \text{by Lemma 7.5}\\ &= \sum_{n=0}^\infty \left(|f|_{-p}+|\tau_V|_{-p}^{1/2}\right)^{2n}.\end{aligned}$$

Since $\tau_V \in (\mathcal{H}')^{\widehat{\otimes}2}$, we know $|\tau_V|_{-p} \to 0$ as $p \to \infty$. Also for $f \in \mathcal{H}'$, we have $|f|_{-p} \to 0$ as $p \to \infty$. Therefore we can take p so that $|f|_{-p} + |\tau_V|_{-p}^{1/2} < 1$. From (7.1) above this gives us

$$\|\Phi_f^V\|_{-p}^2 \le \frac{1}{1-\left(|f|_{-p}+|\tau_V|_{-p}^{1/2}\right)^2}.$$

Thus $\Phi_f^V \in [\mathcal{H}]'$. ■

Theorem 7.7. *The Wiener–Itô decomposition of $\tilde{\delta}_{a+V}$ is given by*

$$\sum_{n=0}^\infty \frac{1}{n!}\langle :\cdot^{\otimes n}:, :a^{\otimes n}:_{V^\perp}\rangle.$$

Proof. From Proposition 7.6 we know that $\Phi_a^{V^\perp} = \sum_{n=0}^\infty \frac{1}{n!}\langle :\cdot^{\otimes n}:, :a^{\otimes n}:_{V^\perp}\rangle$ is in $[\mathcal{H}]'$. Taking the *S*–transform of $\Phi_a^{V^\perp}$ with $z \in \mathcal{H}_c$ we get

$$\begin{aligned}S(\Phi_a^{V^\perp})(z) &= \sum_{n=0}^\infty n!\frac{1}{(n!)^2}\langle :a^{\otimes n}:_{V^\perp}, z^{\otimes n}\rangle\\ &= \sum_{n=0}^\infty \frac{1}{n!} :\langle a, z\rangle:_{\langle z_{V^\perp}, z_{V^\perp}\rangle} \qquad \text{by Proposition 7.4.}\end{aligned}$$

Noting that the above is the generating function for the Hermite polynomials gives us

$$S(\Phi_a^{V^\perp})(z) = e^{\langle a,z\rangle - \frac{1}{2}\langle z_{V^\perp}, z_{V^\perp}\rangle}.$$

Comparing this with the *S*–transform of $\tilde{\delta}_{a+V}$ in Definition 6.4 we see that

$$\langle\langle \Phi_a^{V^\perp}, :e^{\langle\cdot,z\rangle}:\rangle\rangle = S(\Phi_a^{V^\perp})(z) = S(\tilde{\delta}_{a+V})(z) = \langle\langle \tilde{\delta}_{a+V}, :e^{\langle\cdot,z\rangle}:\rangle\rangle$$

for all $z \in \mathcal{H}_c$. Thus by Corollary 3.1 we have $\tilde{\delta}_{a+V} = \Phi_a^{V^\perp}$ μ–almost everywhere. Therefore $\tilde{\delta}_{a+V} = \sum_{n=0}^{\infty} \frac{1}{n!}\langle :\cdot^{\otimes n}:, :a^{\otimes n}:_{V^\perp}\rangle$. ■

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