TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 8, pp. 1883-1910, November 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

PROXIMAL POINT ALGORITHMS AND FOUR RESOLVENTS OF NONLINEAR OPERATORS OF MONOTONE TYPE IN BANACH SPACES

Wataru Takahashi

Abstract. In this article, motivated by Rockafellar's proximal point algorithm in Hilbert spaces, we discuss various weak and strong convergence theorems for resolvents of accretive operators and maximal monotone operators which are connected with the proximal point algorithm. We first deal with proximal point algorithms in Hilbert spaces. Then, we consider weak and strong convergence theorems for resolvents of accretive operators in Banach spaces which generalize the results in Hilbert spaces. Further, we deal with weak and strong convergence theorems for three types of resolvents of maximal monotone operators in Banach spaces which are related to proximal point algorithms. Finally, in Section 7, we apply some results obtained in Banach spaces to the problem of finding minimizers of convex functions in Banach spaces.

1. Introduction

Let H be a real Hilbert space. We know many problems in nonlinear analysis and optimization which are formulated as follows: Find

$$u \in H \text{ such that } 0 \in Au, \tag{1}$$

where A is a maximal monotone operator from H to H. Such $u \in H$ is called a *zero point* (or a *zero*) of A. A well-known method for solving (1) in a Hilbert space H is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{2}$$

where $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$. This algorithm was first introduced by Martinet [26]. In [39], Rockafellar proved that if $\liminf_{n\to\infty} r_n > 0$ and

Received and accepted October 2, 2007.
2000 *Mathematics Subject Classification*: 47H05, 47J25.
Key words and phrases: Banach space, Proximal point algorithm, Resolvent, Nonexpansive mapping, Maximal monotone operator, Retraction, Projection, Convex optimization.

$A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (2) converges weakly to a solution of (1); see also Brézis and Lions [3] and Lions [24]. It was shown by Güler [8] that the sequence $\{x_n\}$ generated by this algorithm does not converge strongly in general. On the other hand, we know three iterative methods for approximation of fixed points of nonexpansive mappings in a Hilbert space.

Halpern [9] introduced the following iterative scheme to approximate a fixed point of a nonexpansive mapping in a Hilbert space. For the proof, see Wittmann [55] and Takahashi [45].

Theorem 1.1. ([55]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that the set $F(T)$ of fixed points of T is nonempty. Let P be the metric prjection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then, $\{x_n\}$ converges strongly to $Px \in F(T)$.

Mann [25] also introduced the iterative scheme for finding a fixed point of a nonexpansive mapping. For the proof, see Takahashi [45].

Theorem 1.2. ([33]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x \in C$ and let $\{x_n\}$ be a sequence defined by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \ldots,$$

where $\{x_n\} \subset [0, 1]$ satisfies

$$0 \leq \alpha_n < 1 \text{ and } \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, $\{x_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} Px_n$.

Nakajo and Takahashi [30] proved the following strong convergence theorem for nonexpansive mappings in a Hilbert space by using the hybrid method in mathematical programming.

Theorem 1.3. ([30]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let P be the metric projection of H onto $F(T)$. Let $x_1 = x \in C$ and*

$$\begin{cases} y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_1), \quad n = 1, 2, \ldots, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $\limsup_{n\to\infty} \alpha_n < 1$ and $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$. Then, $\{x_n\}$ converges strongly to $Px_1 \in F(T)$.

In this article, motivated by Rockafellar's proximal point algorithm and three iterative methods for approximation of fixed points of nonexpansive mappings, we discuss various weak and strong convergence theorems for resolvents of accretive operators and maximal monotone operators which are connected with Rockafellar's proximal point algorithm. In Section 3, we first discuss two proximal point algorithms in Hilbert spaces modified by Kamimura and Takahashi. Further, we deal with Solodov and Svaiter's strong convergence theorem which was proved by using the hybrid method in mathematical programming. Then, we try to extend such proximal point algorithms in Hilbert spaces to Banach spaces. These algorithms in Banach spaces are connected with four resolvents for accretive operators and maximal monotone operators in Banach spaces. In Section 4, we prove weak and strong convergence theorems for resolvents of accretive operators in Banach spaces which generalize the results in Hilbert spaces. In Section 5, we deal with weak and strong convergence theorems for two types of resolvents of maximal monotone operators which are called metric resolvents and relative resolvents. In Section 6, we introduce a new notion of resolvents of maximal monotone operators called generalized resolvents which is different from metric resolvents and relative resolvents. Then, we obtain weak and strong convergence theorems for such resolvents. Finally, in Section 7, we apply some results obtained in Sections 5 and 6 to the problem of finding minimizers of convex functions in Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C(x)$, we call P_C the *metric projection* of E onto C. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists. In the case, E is called *smooth*. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (3) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (3) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (3) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. We know the following result; see, for instance, [44].

Theorem 2.1. *Let E be a smooth Banach space. Let C be a nonempty closed convex subset of E, $x_1 \in E$ and $x_0 \in C$. Then $x_0 = P_C x_1$ if and only if*

$$\langle x_0 - y, J(x_1 - x_0) \rangle \ge 0$$

for all $y \in C$, where J is the duality mapping of E.

Let E be a smooth Banach space. Following Alber [1] and Kamimura and Takahashi [18], we denote by $\phi : E \times E \to [0, \infty)$ the mapping defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $(x, y) \in E \times E$. It is easy to see that $(\|x\| - \|y\|)^2 \le \phi(x, y)$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \ge 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{4}$$

for all $x, y, z \in E$. It is also easy to see that if E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{5}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for all $x \in E$, there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x). \tag{6}$$

The mapping Π_C is called the *generalized projection* from E onto C. We know the following theorem:

Theorem 2.2. ([1, 18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then the following hold:*

(a) $z = \Pi_C x$ *if and only if* $\langle y - z, Jx - Jz \rangle \leq 0$ *for all* $y \in C$;

(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(z, x)$.

A Banach space E is said to satisfy *Opial's condition* [32] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup y$ implies

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|$$

for all $z \in E$ with $z \neq y$. A Hilbert space satisfies Opial's condition.

Let C be a closed convex subset of E. A mapping $T \colon C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of all fixed points of T by $F(T)$. A closed convex subset C of E is said to have *normal structure* if for each bounded closed convex subset K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, i.e.,

$$\sup\{\|x - y\| : y \in K\} < \delta(K),$$

where $\delta(K)$ is the diameter of K. We know that a closed convex subset of a uniformly convex Banach space has normal structure. We know Kirk's fixed point theorem [20] for nonexpansive mappings.

Theorem 2.3. *Let E be a reflexive Banach space and let C be a bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of C into itself. Then, T has a fixed point in C.*

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$. If A is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all $r > 0$. An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset \bigcap_{r>0} R(I + rA)$. If A is accretive, then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$. It is called the *accretive resolvent* of A. We also define the *Yosida approximation* A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know that for an accretive operator A satisfying the range condition, $A^{-1}0 = F(J_r)$ for all $r > 0$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. Let C be a closed convex subset of E. Let D be a subset of C and let P be a mapping of C into D. Then P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a *retraction* if $P^2 = P$. We denote the closure of the convex hull of D by $\overline{\mathrm{co}}D$.

Theorem 2.4. *Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. Suppose that $D \subset C$ and P is a retraction of C onto D. Then P is sunny and nonexpansive if and only if*

$$\langle x - Px, J(Px - y)\rangle \geq 0$$

for all $x \in C$ and $y \in D$, where J is the duality mapping of E.

A multi-valued operator $A : E \to 2^{E^*}$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$. A monotone operator A is said to be *maximal* if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. The following theorems are well known; for instance, see [45].

Theorem 2.5. *Let E be a reflexive, strictly convex and smooth Banach space and let $A : E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Theorem 2.6 *Let E be a strictly convex and smooth Banach space and let $x, y \in E$. If $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

A duality mapping J of a smooth Banach space is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ implies that $Jx_n \overset{*}{\rightharpoonup} Jx$, where $\overset{*}{\rightharpoonup}$ means the weak* convergence.

3. Proximal Point Algorithms in Hilbert Spaces

Let H be a Hilbert space. Then, from Theorem 2.5 we know that a monotone operator $A \subset H \times H$ is maximal if and only if A is m-accretive. Motivated by Rockafellar's result [39], Kamimura and Takahashi [16] proved the following two convergence theorems.

Theorem 3.1. ([16]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Let $J_r = (I + rA)^{-1}$ for all $r > 0$ and let $x_0 = x \in H$ and let $\{x_n\}$ be a sequence generated by*

$$y_n \approx J_{\lambda_n} x_n, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n) y_n, \quad n = 1, 2, \dots,$$

under criterion $\|y_n - J_{\lambda_n} x_n\| \leq \delta_n$, where $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} \lambda_n = \infty.$$

Then $\{x_n\}$ converges strongly to Px, where P is the metric projection of H onto $A^{-1}0$.

Theorem 3.2. ([16]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Let $J_r = (I + rA)^{-1}$ for all $r > 0$ and let $x_0 = x \in H$ and let $\{x_n\}$ be a sequence generated by*

$$y_n \approx J_{\lambda_n} x_n, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad n = 1, 2, \dots,$$

under criterion $\|y_n - J_{\lambda_n} x_n\| \leq \delta_n$, where $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} \lambda_n > 0.$$

Then $\{x_n\}$ converges weakly to $A^{-1}0$, where $v = \lim_{n\to\infty} Px_n$.

Solodov and Svaiter [41] also proved the following strong convergence theorem by the hybrid method in mathematical programming.

Theorem 3.3. ([41]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Let $x \in H$ and let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_1 &= x \in H, \\ y_n &= J_{r_n} x_n, \\ H_n &= \{z \in H : \langle z - y_n, x_n - y_n \rangle \leq 0\}, \\ W_n &= \{z \in H : \langle z - x_n, x_1 - x_n \rangle \leq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n} x_1, \quad n = 1, 2, \dots, \end{cases}$$

where $\{r_n\}$ *is a sequence of positive numbers. If* $A^{-1}0 \neq \emptyset$ *and* $\liminf_{n\to\infty} r_n > 0$*, then* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x_1$.

Motivated by Kamimura and Takahashi [16], Iemoto and Takahashi [13] obtained the following theorem which generalizes Theorems 3.1 and 3.2, simultaneously. This theorem is a complete generalization of the theorems in [16].

Theorem 3.4. *Let* $A : H \to 2^H$ *be a maximal monotone operator with* $A^{-1}0 \neq \emptyset$*. Let* $x_0 = x \in H$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\text{(7)} \qquad \begin{cases} x_0 &= x \in H, \\ y_n &\approx J_{\lambda_n} x_n, \\ x_{n+1} &= \alpha_n x + \beta_n x_n + \gamma_n y_n, \quad n = 0, 1, 2, \ldots, \end{cases}$$

under criterion $\|y_n - J_{\lambda_n}x_n\| \leq \delta_n$*. Then the following hold:*

(1) *Suppose that*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \ \lim_{n\to\infty} \beta_n = 0 \text{ and } \lim_{n\to\infty} \lambda_n = \infty.$$

Then $\{x_n\}$ *converges strongly to* Px*, where* P *is the metric projection of* H *onto* $A^{-1}0$.

(2) *Suppose that*

$$\sum_{n=0}^{\infty} \alpha_n < \infty, \ \limsup_{n\to\infty} \beta_n < 1 \text{ and } \liminf_{n\to\infty} \lambda_n > 0.$$

Then $\{x_n\}$ *converges weakly to* $v \in A^{-1}0$*, where* $v = \lim_{n\to\infty} Px_n$.

We also know that Xu [56, Theorems 5.1 and 5.2] proved strong and weak convergence theorems which are related to Theorems 3.1 and 3.2.

4. Proximal Point Algorithms for Accretive Operators

Let E be a Banach space and let $A \subset E \times E$ be an accretive operator with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$. For $r > 0$ and $x \in R(I + rA)$, define the resolvent of A as follows:

$$J_r x = \{z \in E : x \in z + rAz\}.$$

Then, as in Preliminaries, a single valued nonexpansive mapping $J_r \colon R(I+rA) \to D(A)$ denoted by $J_r = (I + rA)^{-1}$ is called the *accretive resolvent* of A. We can also define the *Yosida approximation* A_r by $A_r = (I - J_r)/r$. We know that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. An accretive operator A is said to be *m-accretive* if $R(I + rA) = E$ for all $r > 0$. Reich [35] proved the following result:

Theorem 4.1. ([35]). *Let E be a uniformly convex and uniformly smooth Banach space and let $A \subset E \times E$ be an m-accretive operator such that $A^{-1}0$ is nonempty. Then, for any $x \in E$, the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$. Define $Qx = \lim_{t\to\infty} J_t x$. Then Q is a sunny nonexpansive retraction of E onto $A^{-1}0$.*

We first obtain the following strong convergence theorem by the viscosity approximation method which generalizes Theorem 4.1.

Theorem 4.2. ([52]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E which has normal structure. Let $A \subset E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying*

$$D(A) \subset C \subset \bigcap_{t>0} R(I + tA)$$

and let J_t be the resolvent of A for $t > 0$. Let f be a contractive mapping of C into itself. Then the following hold:

(1) *For $t > 0$, $J_t f$ has a unique fixed point u_t in C;*

(2) *if $t \to \infty$, then the net $\{u_t\}$ converges strongly to $u \in A^{-1}0$, where $u = Q_{A^{-1}0} f u$ and $Q_{A^{-1}0}$ is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

Using Theorem 4.2, we deal with the proximal point algorithm by the viscosity approximation method for resolvents of accretive operators in Banach spaces.

Theorem 4.3. ([52]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E which has normal structure. Let $A \subset E \times E$ be an accretive operator with $A^{-1}0 \neq \emptyset$ satisfying*

$$D(A) \subset C \subset \bigcap_{t>0} R(I + tA)$$

and let J_t be the resolvent of A for $t > 0$. Let f be a contractive mapping of C into itself. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{t_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{t_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} t_n = \infty.$$

Then $\{x_n\}$ *converges strongly to* $u \in A^{-1}0$*, where* $u = Q_{A^{-1}0} f(u)$ *and* $Q_{A^{-1}0}$ *is a sunny nonexpansive retraction of* C *onto* $A^{-1}0$*.*

As a direct consequence of Theorem 4.3, we have the following:

Theorem 4.4. *Let* E *be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let* $A \subset E \times E$ *be an m-accretive operator. Let* $x_1 = x \in E$ *and let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to an element of* $A^{-1}0$*.*

Next, we prove a weak convergence theorem of Mann's type for accretive operators in a Banach space. Before stating the theorem, we need the following two lemmas.

Lemma 4.5. ([5]). *Let* C *be a closed bounded convex subset of a uniformly convex Banach space* E *and let* T *be a nonexpansive mapping of* C *into itself. If* $\{x_n\}$ *converges weakly to* $z \in C$ *and* $\{x_n - Tx_n\}$ *converges strongly to* 0*, then* $Tz = z$*.*

Lemma 4.6. ([33]). *Let* E *be a uniformly convex Banach space whose norm is Fréchet differentiable, let* C *be a closed convex subset of* E *and let* $\{T_0, T_1, T_2, \ldots\}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\bigcap_{n=0}^{\infty} F(T_n)$ *is nonempty. Let* $x \in C$ *and* $S_n = T_n T_{n-1} \cdots T_0$ *for all* $n = 1, 2, \ldots$*. Then the set* $\bigcap_{n=0}^{\infty} \overline{\mathrm{co}}\{S_m x : m \geq n\} \cap U$ *consists of at most one point, where* $U = \bigcap_{n=0}^{\infty} F(T_n)$*.*

For the proof of Lemma 4.6, see Takahashi and Kim [53]. Now we can prove the following weak convergence theorem.

Theorem 4.7. ([17]). *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition, let $A \subset E \times E$ be an accretive operator which satisfies the range condition, and let C be a nonempty closed convex subset of E such that $D(A) \subset C \subset \bigcap_{r>0} R(I+rA)$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

As a direct consequence of Theorem 4.7, we have the following:

Theorem 4.8. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable or which satisfies Opial's condition and let $A \subset E \times E$ be an m-accretive operator. Let $x_1 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.

Recently, Iemoto and Takahashi [14] proved the following theorem. The proof is mainly due to Kamimura and Takahashi [17].

Theorem 4.9. *Let E be a uniformly convex Banach space whose norm is uniformly smooth and let $A \subset E \times E$ be an m-accretive operator. Let $x_0 = x \in E$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n x + \beta_n x_n + \gamma_n J_{\lambda_n} x_n + e_n, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{8}$$

Assume that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $A^{-1}0 \neq \emptyset$. Then the following hold:

(1) *Suppose that*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \ \lim_{n\to\infty} \alpha_n = 0, \ \lim_{n\to\infty} \beta_n = 0, \ \textit{and} \ \lim_{n\to\infty} \lambda_n = \infty.$$

Then $\{x_n\}$ *converges strongly to an element of* $A^{-1}0$. *Further, if* $Px = \lim_{n\to\infty} x_n$ *for each* $x \in E$, *then* P *is a sunny nonexpansive retraction of* E *onto* $A^{-1}0$.

(2) *Suppose that*

$$\sum_{n=0}^{\infty} \alpha_n < \infty, \ \limsup_{n\to\infty} \beta_n < 1, \ \textit{and} \ \liminf_{n\to\infty} \lambda_n > 0.$$

Then $\{x_n\}$ *converges weakly to* $v \in A^{-1}0$.

Probelm. Can we prove a theorem of Solodov and Svaiter's type for resolvents of accretive operators in Banach spaces?

5. Proximal Point Algorithms for Monotone Operators

Let E be a smooth Banach space. Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an *asymptotic fixed point* of T [36] if C contains a sequence $\{x_n\}$ which converges weakly to p and the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* [28] if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorems 2.5 and 2.6, we obtain that for every $r > 0$ and $x \in E$, there exists a unique $x_r \in D(A)$ such that

$$Jx \in Jx_r + rAx_r.$$

If $Q_r x = x_r$, then we can define a single valued mapping $Q_r : E \to D(A)$ by $Q_r = (J + rA)^{-1}J$ and such a Q_r is called the *relative resolvent* of A. We know that $A^{-1}0 = F(Q_r)$ for all $r > 0$; see [44, 45] for more details. For such resolvents, we know the following convergence theorem.

Theorem 5.1. ([34]). *Let* E *be a Banach space and let* $A \subset E \times E^*$ *be a maximal monotone operator with* $A^{-1}0 \neq \emptyset$. *If* E^* *is strictly convex and has a Fréchet differentiable norm. Then, for each* $x \in E$, $\lim_{\lambda\to\infty} Q_\lambda x$ *exists and belongs to* $A^{-1}0$.

For $x \in E$ and $r > 0$, we also consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

By Theorems 2.5 and 2.6, this equation has a unique solution x_r. We denote J_r by $x_r = J_r x$ and such J_r is also called the *metric resolvent* of A. We do not know useful properties for metric resolvents; see [45]. On the other hand, we know the following theorem [28] for relative resolvents of maximal monotone operators.

Theorem 5.2. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, let Q_r be the relative resolvent of A, where $r > 0$. If $A^{-1}0$ is nonempty, then Q_r is a relatively nonexpansive mapping on E.*

Kohsaka and Takahashi [21] also proved a strong convergence theorem of Halpen's type for relative resolvents of maximal monotone operators in a Banach space.

Theorem 5.3. ([21]). *Let E be a smooth and uniformly convex Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence difined as follows:*

$$\begin{aligned} x_1 &= x \in E, \\ x_{n+1} &= J^{-1}(\alpha_n J(x) + (1 - \alpha_n)J(Q_{r_n} x_n)), \quad n = 1, 2, \dots, \end{aligned}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \textit{and} \ \lim_{n\to\infty} r_n = \infty.$$

If $A^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}x$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

For the sake of getting a weak convergence theorem of Mann's type for relative resolvents of maximal monotone operators in a Banach space, we need the following strong convergence theorem.

Theorem 5.4. ([15]). *Let E be a smooth and uniformly convex Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0$ is nonempty, let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\Pi_{A^{-1}0}$ be the generalized projection of E onto $A^{-1}0$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n)J(Q_{r_n} x_n)), \quad n = 1, 2, \dots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$. *Then, the sequence* $\{\Pi_{A^{-1}0}(x_n)\}$ *converges strongly to an element of* $A^{-1}0$*, which is a unique element* $v \in A^{-1}0$ *such that*

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in A^{-1}0} \lim_{n\to\infty} \phi(y, x_n).$$

Using Theorem 5.4, we can prove the following theorem in a Banach space which generalizes the results of Rockafellar [39] and Kamimura and Takahashi [16] in a Hilbert space.

Theorem 5.5. ([15]). *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $A \subset E \times E^*$ *be a maximal monotone operator such that* $A^{-1}0$ *is nonempty, let* $Q_r = (J + rA)^{-1}J$ *for all* $r > 0$ *and let* $\Pi_{A^{-1}0}$ *be the generalized projection of* E *onto* $A^{-1}0$*. Let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$x_{n+1} = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n)J(Q_{r_n}x_n)), \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \textit{ and } \liminf_{n\to\infty} r_n > 0.$$

Then, the sequence $\{x_n\}$ *converges weakly to an element v of* $A^{-1}0$*, where* $v = \lim_{n\to\infty} \Pi_{A^{-1}0}(x_n)$.

As a direct consequence of Theorem 5.5, we obtain the following:

Theorem 5.6. *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $A \subset E \times E^*$ *be a maximal monotone operator such that* $A^{-1}0$ *is nonempty, let* $Q_r = (J + rA)^{-1}J$ *for all* $r > 0$ *and let* $\Pi_{A^{-1}0}$ *be the generalized projection of* E *onto* $A^{-1}0$*. Let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$x_{n+1} = Q_{r_n}x_n, \quad n = 1, 2, \ldots,$$

where $\{r_n\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$*. Then, the sequence* $\{x_n\}$ *converges weakly to an element v of* $A^{-1}0$*, where* $v = \lim_{n\to\infty} \Pi_{A^{-1}0}(x_n)$.

Problem. If E and E^* are uniformly convex Banach spaces, does Theorem 5.6 hold without assumming that J is weakly sequentially continuous?

Now, we extend Solodov and Svaiter's result [41] to that of a Banach space. Kamimura and Takahashi [18] obtained the following strong convergence theorem by using Theorem 2.2.

Theorem 5.7. ([18]). *Let E be a uniformly convex and uniformly smooth Banach space and let A be a maximal monotone operator from E into E^* such that $A^{-1}0 \neq \emptyset$. Let $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = Q_{r_n} x_n, \\ H_n = \{z \in E : \langle z - y_n, Jx_n - Jy_n \rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots, \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers such that $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0} x_1$, where $\Pi_{A^{-1}0}$ is the generalized projection of E onto $A^{-1}0$.

Matsushita and Takahashi [28] also proved the following strong convergence theorem of Solodov and Svaiter's type for relative resolvents of maximal monotone operators by using an idea of Nakajo and Takahashi's hybrid method.

Theorem 5.8. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, let Q_r be the relative resolvent of A, where $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_1 = x \in E, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JQ_r x_n), \\ H_n = \{z \in E : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in E : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} = Q_{H_n \cap W_n} x, \quad n = 1, 2, \ldots, \end{cases}$$

where J is the duality mapping on E. If $A^{-1}0$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0} x$, where $\Pi_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

Ohsawa and Takahashi [31] proved another extension of Solodov and Svaiter's result [41] by using the metric resolvents of maximal monotone operators and the metric projection in Banach spaces.

Theorem 5.9. ([31]). *Let E be a uniformly convex and smooth Banach space and let A be a maximal monotone operator from E into E^*. Suppose $\{x_n\}$ is the* *sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = J_{r_n} x_n, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_1, \quad n = 1, 2, \ldots, \end{cases}$$

where $\{r_n\}$ is a sequence of positive numbers. If $A^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{A^{-1}0} x_1$, where $P_{A^{-1}0}$ is the metric projection of E onto $A^{-1}0$.

Problem. Can we prove theorems of Halpern's type and Mann's type for metric resolvents of maximal monotone operators?

6. Proximal Point Algorithms for New Resolvents

In this section, we introduce a new notion of resolvents of maximal monotone operators in a Banach space. Then, we discuss weak and strong convergence theorems for the resolvents. Let E be a smooth Banach space and let D be a nonempty closed convex subset of E. A mapping $R : D \to D$ is called *generalized nonexpansive* [10] if $F(R) \neq \emptyset$ and

$$\phi(Rx, y) \leq \phi(x, y), \quad \forall x \in D, \forall y \in F(R),$$

where $F(R)$ is the set of fixed points of R. We can first get the following theorem for generalized nonexpansive mappings.

Theorem 6.1. *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_C be a retraction of E onto C. Then R_C is sunny and generalized nonexpansive if and only if*

$$\langle x - R_C x, J(R_C x) - J(y) \rangle \geq 0$$

for each $x \in E$ and $y \in C$.

Compare this theorem with Theorems 2.1, 2.2 and 2.4. A point p in C is said to be a *generalized asymptotic fixed point* of T [12] if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and the strong $\lim_{n\to\infty}(Jx_n - JTx_n) = 0$. The set of generalized asymptotic fixed points of T will be denoted by $\check{F}(T)$.

Let E be a reflexive, strictly convex and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, consider the set

$$R_\lambda x := \{z \in E : x \in z + \lambda BJ(z)\}.$$

Then $R_\lambda x$ consists of one point. We also denote the domain and the range of R_λ by $D(R_\lambda) = R(I + \lambda BJ)$ and $R(R_\lambda) = D(BJ)$, respectively. Such R_λ is called the *generalized resolvent* of B and is denoted by

$$R_\lambda = (I + \lambda BJ)^{-1}.$$

We get some properties of R_λ and $(BJ)^{-1}0$.

Proposition 6.2. *Let E be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) $D(R_\lambda) = E$ *for each* $\lambda > 0$*;*

(2) $(BJ)^{-1}0 = F(R_\lambda)$ *for each* $\lambda > 0$*, where* $F(R_\lambda)$ *is the set of fixed points of* R_λ*;*

(3) $(BJ)^{-1}0$ *is closed;*

(4) R_λ *is generalized nonexpansive for each* $\lambda > 0$.

Using Theorem 5.1, we get the following result.

Theorem 6.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) *For each* $x \in E$, $\lim_{\lambda\to\infty} R_\lambda x$ *exists and belongs to* $(BJ)^{-1}0$*;*

(2) *if* $Rx := \lim_{\lambda\to\infty} R_\lambda x$ *for each* $x \in E$*, then* R *is a sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0$.

Next, we discuss proximal point algorithms for new resolvents of a maximal monotone operator $B \subset E^* \times E$. We start with the following lemma. Compare this lemma with the results in Kamimura and Takahashi [18], and Kohsaka and Takahashi [21].

Lemma 6.4. *Let E be a reflexive, strictly convex, and smooth Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and $R_r = (I + rBJ)^{-1}$ for all $r > 0$. Then*

$$\phi(x, J_r x) + \phi(J_r x, u) \leq \phi(x, u)$$

for all $r > 0$*,* $u \in (BJ)^{-1}0$*, and* $x \in E$*.*

Now, we can prove the following strong convergence theorem, which is a generalization of Kamimura and Takahashi's strong convergence theorem (Theorem 3.1).

Theorem 6.5. ([11]). *Let* E *be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let* $B \subset E^* \times E$ *be a maximal monotone operator. Let* $R_r = (I + rBJ)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) R_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy* $\lim_{n\to\infty} \alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$*, and* $\lim_{n\to\infty} r_n = \infty$*. If* $B^{-1}0 \neq \emptyset$*, then the sequence* $\{x_n\}$ *converges strongly to* $R_{(BJ)^{-1}0}(x)$*, where* $R_{(BJ)^{-1}0}$ *is a sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0$*.*

Next, we can prove the following weak convergence theorem, which is a generalization of Kamimura and Takahashi's weak convergence theorem (Theorem 3.2).

Theorem 6.6. ([11]). *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $B \subset E^* \times E$ *be a maximal monotone operator, let* $R_r = (I + rBJ)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) R_{r_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \liminf_{n\to\infty} r_n > 0.$$

If $B^{-1}0 \neq \emptyset$*, then the sequence* $\{x_n\}$ *converges weakly to an element of* $(BJ)^{-1}0$*.*

Further, we can prove strong convergence theorems by hybrid methods for generalized nonexpansive mappings in Banach spaces. Before discussing them, we need the following definitions and theorems. Let E be a smooth and strictly convex Banach space and let C be a nonempty closed subset of E. Then, a sunny generalized nonexpansive retraction of E onto C is unique. Let C is a nonempty closed subset of E. Then, C is said to be a *sunny generalized nonexpansive retract* (resp. *generalized nonexpansive retract*) if there exists a sunny generalized nonexpansive retraction (resp. generalized nonexpansive retraction) of E onto C. Kohsaka and

Takahashi [23] obtained the following results for sunny generalized nonexpansive retractions in a Banach space.

Theorem 6.7. ([23]). *Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed subset of E. Then, the following conditions are equivalent:*

(1) *C is a sunny generalized nonexpansive retract of E;*

(2) *C is a generalized nonexpansive retract of E;*

(3) *JC is closed and convex.*

Theorem 6.8. ([23]). *Let E be a reflexive, strictly convex and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, $B^{-1}0$ is a sunny generalized nonexpansive retract of E.*

Ibaraki and Takahashi [12] also obtained the following result concerning the set of fixed points of a generalized nonexpansiv mapping.

Theorem 6.9. ([12]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansiv mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is colsed and convex.*

As a direct consequence of Theorems 6.7 and 6.9, we obtain the following result.

Theorem 6.10. ([12]). *Let E be a reflexive, strictly convex and smooth Banach space and let T be a generalized nonexpansiv mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Further, Ibaraki and Takahashi [12] obtained the following results.

Lemma 6.11. *Let E be a smooth and uniformly convex Banach space E, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, and let R_r be the generalized resolvent of B for some $r > 0$. Then $\check{F}(R_r) = F(R_r)$.*

Lemma 6.12. *Let D be a nonempty closed subset of a reflexive, strictly convex and smooth Banach space E, let R be a sunny generalized nonexpansive retraction of E onto D. Then $\check{F}(R) = F(R)$.*

Using these theorems and lemmas, Kohsaka and Takahashi [23] and Ibaraki and Takahashi [12] proved the following strong convergence theorems.

Theorem 6.13. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator* *such that* $B^{-1}0$ *is nonempty. Let* $R_r = (I + rBJ)^{-1}$ *for all* $r > 0$ *and let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in E$ *and*

$$\begin{cases} y_n &= R_{r_n} x_n; \\ H_n &= \{z \in E : \langle x_n - y_n, Jz - Jy_n\rangle \le 0\}; \\ W_n &= \{z \in E : \langle x - x_n, Jz - Jx_n\rangle \le 0\}; \\ x_{n+1} &= R_{H_n \cap W_n}(x), \quad n = 1, 2, \dots, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_n r_n > 0$ *and* $R_{H_n \cap W_n}$ *denotes the sunny generalized nonexpansive retraction from E onto* $H_n \cap W_n$ *for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $R(x)$, *where* R *denotes the sunny generalized nonexpansive retraction from* E *onto* $J^{-1}B^{-1}0$.

Theorem 6.14. (Ibaraki and Takahashi [12]). *Let* E *be a uniformly convex and uniformly smooth Banach space, let* $B \subset E^* \times E$ *be a maximal monotone operator with* $B^{-1}0 \ne \emptyset$, *let* R_r *the generalized resolvent of* B *for some* $r > 0$, *and let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \le \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Suppose that* $\{x_n\}$ *is given by*

$$\begin{cases} x_1 &= x \in E, \\ y_n &= \alpha_n x_n + (1 - \alpha_n) R_r x_n), \\ H_n &= \{z \in E : \phi(y_n, z) \le \phi(x_n, z)\}, \\ W_n &= \{z \in E : \langle x - x_n, Jx_n - Jz\rangle \ge 0\}, \\ x_{n+1} &= R_{H_n \cap W_n} x, \quad n = 1, 2, \dots, \end{cases}$$

where J *is the duality mapping on* E. *Then* $\{x_n\}$ *converges strongly to* $R_{(BJ)^{-1}0}x$, *where* $R_{(BJ)^{-1}0}$ *is a sunny generalized nonexpansive retraction from* E *onto* $(BJ)^{-1}0$.

7. Applications

In this section, we apply some results obtained in Sections 5 and 6 to find minimizers of convex functions in Banach spaces. Let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the *subdifferential* ∂f of f is as follows:

$$\partial f(z) = \{v^* \in E^* : f(y) \ge f(z) + \langle y - z, v^*\rangle, \forall y \in E\}, \quad \forall z \in E.$$

Then, we know the following theorem which was proved by Rockafellar [41].

Theorem 7.1. ([41]). *Let E be a real Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then, the subdifferential ∂f of f is maximal monotone.* ■

Using Theorems 7.1 and 5.7, we obtain the following theorem.

Theorem 7.2. ([18]). *Let E be a uniformly convex and uniformly smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = \arg\min_{z\in E}\{f(z) + \dfrac{1}{2r_n}\|z\|^2 - \dfrac{1}{r_n}\langle z, Jx_n\rangle\}, \\ 0 = v_n + \dfrac{1}{r_n}(Jy_n - Jx_n),\ v_n \in \partial f(y_n), \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \le 0\}, \\ W_n = \{z \in E : \langle z - x_n, Jx_1 - Jx_n\rangle \le 0\}, \\ x_{n+1} = Q_{H_n\cap W_n} x_1, \quad n = 1, 2, \ldots. \end{cases}$$

If $(\partial f)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to $Q_{(\partial f)^{-1}0}(x_1)$, where $Q_{(\partial f)^{-1}0}$ is the generalized projection of E onto $(\partial f)^{-1}0$.

Proof. Since $f : E \to (-\infty, \infty]$ is a proper convex lower semicontinuous function, by Rockafellar's Theorem 7.1, the subdifferential ∂f of f is a maximal monotone operator. We also know that

$$y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle z, Jx_n\rangle\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n}Jy_n - \frac{1}{r_n}Jx_n.$$

So, we have $v_n \in \partial f(y_n)$ such that $0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n)$. Using Theorem 5.7, we get the conclusion. ■

Using Theorems 7.1 and 5.9, we also obtain the following convergence theorem.

Theorem 7.3. ([31]). *Let E be a uniformly convex and smooth Banach space and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function.*

Assume that $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$ *and let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in E, \\ y_n = \arg\min_{z\in E}\{f(z) + \dfrac{1}{2r_n}\|z - x_n\|^2\}, \\ H_n = \{z \in E : \langle y_n - z, J(x_n - y_n\rangle \geq 0\}, \\ W_n = \{z \in E : \langle x_n - z, J(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{H_n\cap W_n} x_1, \quad n = 1, 2, \ldots. \end{cases}$$

If $(\partial f)^{-1}0 \neq \emptyset$*, then* $\{x_n\}$ *converges strongly to* $P_{(\partial f)^{-1}0}(x_1)$*, where* $P_{(\partial f)^{-1}0}$ *is the metric projection of* E *onto* $(\partial f)^{-1}0$.

Proof. As in the proof, we know that

$$y_n = \arg\min_{z\in E}\{f(z) + \frac{1}{2r_n}\|z - x_n\|^2\}$$

is equivalent to

$$0 \in \partial f(y_n) + \frac{1}{r_n} J(y_n - x_n).$$

So, we have

$$0 \in J(y_n - x_n) + r_n \partial f(y_n).$$

Using Theorem 5.9, we get the conclusion. ∎

Further, using Theorems 5.3 and 5.5, we have the following theorems of Halpern' type and Mann's type.

Theorem 7.4. ([21]). *Let* E *be a smooth and uniformly convex Banach space and let* $f : E \to (-\infty, \infty]$ *be a proper lower semicontinuous convex function such that* $(\partial f)^{-1}0$ *is nonempty. Let* $\{x_n\}$ *be a sequence defined as follows:*

$$\begin{aligned} x_1 &= x \in E, \\ y_n &= \arg\min_{y\in E}\{f(y) + \frac{1}{2r_n}\|y\|^2 - \frac{1}{r_n}\langle y, Jx_n\rangle\}, \\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n), \quad n = 1, 2, \ldots, \end{aligned}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \textit{and} \ \lim_{n\to\infty} r_n = \infty.$$

Then, $\{x_n\}$ *converges strongly to* $Q_{(\partial f)^{-1}0}x$.

Theorem 7.5. ([15]). *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $f : E \to (-\infty, \infty]$ *be a proper lower semicontinuous convex function such that* $(\partial f)^{-1}0$ *is nonempty. Let* $\{x_n\}$ *be a sequence defined as follows:*

$$\begin{aligned} x_1 &= x \in E, \\ y_n &= \arg\min_{y \in E}\{f(y) + \frac{1}{2r_n}\|y\|^2 - \frac{1}{r_n}\langle y, Jx_n\rangle\}, \\ x_{n+1} &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jy_n), \quad n = 1, 2, \ldots, \end{aligned}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \textit{ and } \liminf_{n\to\infty} r_n > 0.$$

Then, $\{x_n\}$ *converges weakly to* $v \in (\partial f)^{-1}0$. *Further* $v = \lim Q_{(\partial f)^{-1}0}(x_n)$, *where* $Q_{(\partial f)^{-1}0}$ *is the generalized projection of* E *onto* $(\partial f)^{-1}0$.

On the other hand, using the theorem (Theorem 6.6) for new resolvents of maximal monotone operators, we obtain the following result.

Theorem 7.6. *Let* E *be a smooth and uniformly convex Banach space whose duality mapping* J *is weakly sequentially continuous. Let* $f : E^* \to (-\infty, \infty]$ *be a proper lower semicontinuous convex function such that* $(\partial f)^{-1}(0) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined as follows:* $x_1 = x \in E$ *and*

$$\begin{aligned} y_n^* &= \arg\min_{y^* \in E^*}\left\{f(y^*) + \frac{1}{2r_n}\|y^*\|^2 - \frac{1}{r_n}\langle x_n, y^*\rangle\right\}, \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)J^{-1}y_n^*, \quad n = 1, 2, \ldots, \end{aligned} \tag{9}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy*

$$\limsup_{n\to\infty} \alpha_n < 1 \textit{ and } \liminf_{n\to\infty} r_n > 0.$$

Then the sequence $\{x_n\}$ *converges weakly to an element of* $(\partial f J)^{-1}0$.

Proof. By Rockafellar's Theorem 7.1, the subdifferential mapping $\partial f \subset E^* \times E$ is maximal monotone. If J_r is the generalized resolvent of ∂f for $r > 0$, then we have, for $z \in E$,

$$z \in J_r z + r\partial f J J_r z$$

and hence

$$0 \in \partial f JJ_r z + \frac{1}{r} J^{-1} JJ_r z - \frac{1}{r} z = \partial \left(f + \frac{1}{2r} \| \cdot \|^2 - \frac{1}{r} \langle z, \cdot \rangle \right) JJ_r z.$$

Thus, we have

$$JJ_r z = \arg \min_{y^* \in E^*} \left\{ f(y^*) + \frac{1}{2r} \|y^*\|^2 - \frac{1}{r} \langle z, y^* \rangle \right\}.$$

Therefore, $J^{-1} y_n^* = J^{-1} JJ_{r_n} x_n = J_{r_n} x_n$ for all $n \in \mathbb{N}$. By Theorem 6.6, $\{x_n\}$ converges weakly to an element of $(\partial f J)^{-1} 0$. ■

Finally, using Theorem 6.13, we have the following theorem:

Theorem 7.7. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $f : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$\begin{cases} y_n &= J^{-1} \Big(\arg \min_{y^* \in E^*} \Big\{ f(y^*) + \dfrac{1}{2r_n} \|y^*\|^2 - \dfrac{1}{r_n} \langle x_n, y^* \rangle \Big\} \Big), \\ H_n &= \{ z \in E : \langle x_n - y_n, Jz - Jy_n \rangle \le 0 \}, \\ W_n &= \{ z \in E : \langle x - x_n, Jz - Jx_n \rangle \le 0 \}, \\ x_{n+1} &= R_{H_n \cap W_n}(x), \quad n = 1, 2, \dots, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_n r_n > 0$ and $R_{H_n \cap W_n}$ denotes the sunny generalized nonexpansive retraction from E onto $H_n \cap W_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $R(x)$, where R denotes the sunny generalized nonexpansive retraction from E onto $J^{-1}(\partial f)^{-1}(0)$.

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Wataru Takahashi
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
2-12-1, Ohokayama,
Meguro-ku, Tokyo 152-8552,
Japan
E-mail: wataru@is.titech.ac.jp