TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 8, pp. 2009-2029, November 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON HYBRID PROXIMAL-TYPE ALGORITHMS IN BANACH SPACES

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Abstract. In this paper, we propose new hybrid proximal-type algorithms for a maximal monotone operator in a Banach spaces and establish some strong convergence results. An application to the problem of finding a minimizer of a convex function is given.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E. A mapping $S : C \to C$ is called nonexpansive if $\|Sx-Sy\| \leq \|x-y\|$ for all $x, y \in C$. Denote by $F(S)$ the set of fixed points of S; that is, $F(S) = \{x \in C : Sx = x\}$. Whenever E is a Hilbert space, Nakajo and Takahashi [16] proposed the following iterative algorithm for a single nonexpansive mapping $S : C \to C$

$$(1.1) \qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where P_K denotes the metric projection from E onto a nonempty closed convex subset K of E and proved that the sequence $\{x_n\}$ converges strongly to $P_{F(S)}x_0$.

Received June 25, 2008.
2000 *Mathematics Subject Classification*: 47H05, 47J25, 47H09.
Key words and phrases: Hybrid proximal-type algorithm, Maximal monotone operator, Relatively nonexpansive mapping, Banach space, Strong convergence, Generalized projection.
[1]This research was partially supported by the National Science Foundation of China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), and Science and Technology Commission of Shanghai Municipality grant (075105118).
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[3]This research was partially supported by a grant from the National Science Council of Taiwan.

In 2006, Martinez-Yanes and Xu [14] introduced one iterative algorithm for a nonexpansive mapping $S : C \to C$, with C a bounded closed convex subset of a real Hilbert space H

$$(1.2)\qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen}, \\ z_n = \beta_n x_n + (1-\beta_n) S x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) S z_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ \qquad +(1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

and also defined another iterative algorithm

$$(1.3)\qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen}, \\ y_n = \alpha_n x_0 + (1-\alpha_n) S x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 \\ \qquad +\alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in the interval $[0,1]$. They proved that both the sequence $\{x_n\}$ generated by algorithm (1.2) and the sequence $\{x_n\}$ generated by algorithm (1.3), converge strongly to the same point $P_{F(S)} x_0$.

Very recently, utilizing Nakajo and Takahashi's idea [16], Qin and Su [20] modified algorithms (1.2) and (1.3) for relatively nonexpansive mappings in a Banach space E. They first introduced one iterative algorithm for a relatively nonexpansive mapping $S : C \to C$, with C a closed convex subset of a uniformly convex and uniformly smooth Banach space E

$$(1.4)\qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen}, \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J S x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J S z_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, J x_0 - J x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where J is the single-valued normalized duality mapping on E, $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y,x)$. Second, they also defined another iterative algorithm

$$\text{(1.5)} \qquad \begin{cases} x_0 \in C \text{ arbitrarily chosen,} \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JSx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases}$$

They proved that under appropriate conditions both the sequence $\{x_n\}$ generated by algorithm (1.4) and the sequence $\{x_n\}$ generated by algorithm (1.5), converge strongly to the same point $\Pi_{F(S)}x_0$.

On the other hand, let $T : H \to 2^H$ be a maximal monotone operator in a real Hilbert space H. The problem of finding an element $x \in H$ such that $0 \in Tx$ is very important in the area of optimization and related fields.

Example 1.1. If $T = \partial f$ the subdifferential of a proper lower semicontinuous convex function $f : H \to (-\infty, \infty]$, then T is a maximal monotone operator and the inclusion $0 \in \partial f(x)$ is equivalent to $f(x) = \min\{f(z) : z \in H\}$.

Example 1.2. Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be a monotone and Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - y, w\rangle \ge 0, \ \forall y \in C\}$. Consider the following variational inequality problem (for short, VI(A, C)): find a $\overline{x} \in C$ such that

$$\langle A\overline{x}, y - \overline{x}\rangle \ge 0 \quad \text{for all } y \in C.$$

Define $T : H \to 2^H$ as follows:

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if v is a solution of the VI(A, C); see [23].

A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. Denote by I the identity operator on H. The proximal point algorithm generates, for any initial point $x_0 = x \in H$, a sequence $\{x_n\}$ in H, by the iterative scheme

$$\text{(1.6)} \qquad x_{n+1} = (I + r_n T)^{-1} x_n, \quad n = 0, 1, 2, ...,$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$. Note that (1.6) is equivalent to

$$0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1} - x_n), \quad n = 0, 1, 2, ... \tag{1.7}$$

This algorithm was first introduced by Martinet [18] and generally studied by Rockafellar [24] in the framework of a Hilbert space. Later many authors studied the convergence of (1.6) in a Hilbert space or a Banach space. See for instance, [7, 9, 10, 13, 21, 25] and the references therein. Rockafellar [24] proved that if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence generated by (1.6) converges weakly to an element of $T^{-1}0$. Further, Rockafellar [24] posed an open question of whether or not the sequence generated by (1.6) converges strongly to an element of $T^{-1}0$. This question was solved by Güler [10], who introduced an example for which the sequence generated by (1.6) converges weakly but not strongly. On the other hand, Kamimura and Takahashi [11] and Solodov and Svaiter [26] recently modified the proximal point algorithm to generate a strongly convergent sequence. Solodov and Svaiter [26] introduced the following algorithm:

$$\begin{cases} x_0 \in H \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(y_n - x_n), \quad v_n \in Ty_n, \\ H_n = \{v \in H : \langle v - y_n, v_n\rangle \leq 0\}, \\ W_n = \{v \in H : \langle v - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases} \tag{1.8}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. They proved that if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence generated by algorithm (1.8) converges strongly to $P_{T^{-1}0}x_0$.

Let E be a real Banach space with the dual E^*. A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i,\ i = 1, 2$. A monotone operator T is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. Recently, Kamimura and Takahashi [12] introduced and studied the following proximal-type algorithm in a uniformly convex and uniformly smooth Banach space E, which is an extension of (1.8):

$$\begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \quad v_n \in Ty_n, \\ H_n = \{v \in E : \langle v - y_n, v_n\rangle \leq 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases} \tag{1.9}$$

where $\{r_n\}$ is a sequence in the interval $(0, \infty)$ and J is the normalized duality mapping on E. They derived a strong convergence theorem which extends and improves Solodov and Svaiter's result [26].

Let E be a real Banach space with the dual E^*. Assume that $T : E \to 2^{E^*}$ is a maximal monotone operator and $S : E \to E$ is a relatively nonexpansive mapping. The purpose of this paper is to introduce and study two new hybrid proximal-type algorithms (1.10) and (1.11) in a uniformly convex and uniformly smooth Banach space E, which combine (1.4) with (1.9) and (1.5) with (1.9), respectively.

Algorithm I.

$$(1.10)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen,} \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ H_n = \{v \in E : \phi(v, y_n) \le \alpha_n \phi(v, \widetilde{x}_n) \\ \qquad +(1-\alpha_n)\phi(v, z_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

Algorithm II.

$$(1.11)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen,} \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n), \\ H_n = \{v \in E : \phi(v, y_n) \le \alpha_n \phi(v, x_0) \\ \qquad +(1-\alpha_n)\phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$.

In this paper, strong convergence results on these two hybrid proximal-type algorithms are established; that is, under appropriate conditions, both the sequence

$\{x_n\}$ generated by algorithm (1.10) and the sequence $\{x_n\}$ generated by algorithm (1.11), converge strongly to the same point $\Pi_{T^{-1}0\cap F(S)}x_0$. Moreover, these new results are applied to the problem of finding a minimizer of a convex function on a uniformly convex and uniformly smooth Banach space. Our results represent the improvement, generalization and development of the previously known results in the literature including Solodov and Svaiter [12], Kamimura and Takahashi [12] and Qin and Su [20].

Throughout this paper the symbol $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.

2. Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E is smooth then J is single-valued and if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We shall still denote the single-valued duality mapping by J.

Recall that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and hence, it is not available in more general Banach spaces. Nevertheless , Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as in [1,2] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for all } x, y \in E. \tag{2.1}$$

It is clear that in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x-y\|^2$, $\forall x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [3]). In a Hilbert space, $\Pi_C = P_C$. From [2], in uniformly convex and uniformly smooth Banach spaces, we have

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2 \quad \text{for all } x, y \in E. \tag{2.3}$$

Let C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is called an asymptotically fixed point of S [17] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $Sx_n - x_n \to 0$. The set of asymptotical fixed points of S will be denoted by $\widehat{F}(S)$. A mapping S from C into itself is called relatively nonexpansive [4-6] if $\widehat{F}(S) = F(S)$ and $\phi(p, Sx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(S)$.

A Banach space E is called strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $x_n - y_n \to 0$ for any two sequences $\{x_n\}, \{y_n\} \subset E$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. Then the Banach space E is called smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. Recall also that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. A Banach space is said to have the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, whenever $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [8,19] for more details.

Remark 2.1. [20]. If E is a reflexive, strictly convex and smooth Banach space, then for any $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|y\|^2$. From the definition of J, we have $Jx = Jy$. Therefore, we have $x = y$; see [8,19] for more details.

We need the following lemmas for the proof of our main results.

Lemma 2.1. (Kamimura and Takahashi [12]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.2. (Alber [2]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx\rangle \ge 0 \quad \text{for all } z \in C.$$

Lemma 2.3. (Alber [2]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \text{for all } y \in C.$$

Lemma 2.4. (Matsushita and Takahashi [15]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let S be a relatively nonexpansive mapping from C into itself. Then $F(S)$ is closed and convex.*

3. Main Results

Throughout this section, unless otherwise stated, we assume that $T : E \to 2^{E^*}$ is a maximal monotone operator and $S : E \to E$ is a relatively nonexpansive mapping. In this section, we study the following algorithm in a smooth Banach space E, which is a combination of (1.4) with (1.9).

$$(3.1) \qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ H_n = \{v \in E : \phi(v, y_n) \leq \alpha_n \phi(v, \widetilde{x}_n) \\ \qquad +(1-\alpha_n)\phi(v, z_n) \text{ and } \langle v - \widetilde{x}_n, v_n \rangle \leq 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

First we investigate the condition under which the algorithm (3.1) is well defined. Rockafellar [23] proved the following result.

Lemma 3.1. [23]. *Let E be a reflexive, strictly convex, and smooth Banach space, and let $T : E \to 2^{E^*}$ be a monotone operator. Then T is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$.*

Utilizing this theorem, we can show the following result.

Lemma 3.2. *Let E be a reflexive, strictly convex, and smooth Banach space. If $T^{-1}0 \cap F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (3.1) is well defined.*

Proof. For each $n \geq 0$, define two sets C_n and D_n as follows:

$$C_n = \{v \in E : \phi(v, y_n) \leq \alpha_n \phi(v, \widetilde{x}_n) + (1 - \alpha_n)\phi(v, z_n)\}$$
$$\text{and} \quad D_n = \{v \in E : \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}.$$

It is obvious that C_n is closed and D_n, W_n are closed convex sets for each $n \geq 0$. Let us show that C_n is convex. For $v_1, v_2 \in C_n$ and $t \in (0, 1)$, put $v = tv_1 + (1-t)v_2$. It is sufficient to show that $v \in C_n$. Indeed, observe that

$$\phi(v, y_n) \leq \alpha_n \phi(v, \widetilde{x}_n) + (1 - \alpha_n)\phi(v, z_n)$$

is equivalent to

$$2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1-\alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \leq \alpha_n\|\widetilde{x}_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|y_n\|^2.$$

Note that there hold the following

$$\phi(v, y_n) = \|v\|^2 - 2\langle v, Jy_n\rangle + \|y_n\|^2, \quad \phi(v, \widetilde{x}_n) = \|v\|^2 - 2\langle v, J\widetilde{x}_n\rangle + \|\widetilde{x}_n\|^2$$

and $\phi(v, z_n) = \|v\|^2 - 2\langle v, Jz_n\rangle + \|z_n\|^2$. Thus we have

$$\begin{aligned}
&2\alpha_n\langle v, J\widetilde{x}_n\rangle + 2(1 - \alpha_n)\langle v, Jz_n\rangle - 2\langle v, Jy_n\rangle \\
&= 2\alpha_n\langle tv_1 + (1 - t)v_2, J\widetilde{x}_n\rangle \\
&\quad +2(1 - \alpha_n)\langle tv_1 + (1 - t)v_2, Jz_n\rangle - 2\langle tv_1 + (1 - t)v_2, Jy_n\rangle \\
&= 2t\alpha_n\langle v_1, J\widetilde{x}_n\rangle + 2(1 - t)\alpha_n\langle v_2, J\widetilde{x}_n\rangle + 2(1 - \alpha_n)t\langle v_1, Jz_n\rangle \\
&\quad +2(1 - \alpha_n)(1 - t)\langle v_2, Jz_n\rangle - 2t\langle v_1, Jy_n\rangle - 2(1 - t)\langle v_2, Jy_n\rangle \\
&\leq \alpha_n\|\widetilde{x}_n\|^2 + (1 - \alpha_n)\|z_n\|^2 - \|y_n\|^2.
\end{aligned}$$

This implies that $v \in C_n$. Therefore, C_n is convex and hence $H_n = C_n \cap D_n$ is closed and convex.

On the other hand, let $w \in T^{-1}0 \cap F(S)$ be arbitrarily chosen. Then $w \in T^{-1}0$ and $w \in F(S)$. From (3.1), we have for $w \in F(S)$

$$\begin{aligned}
\phi(w, y_n) &= \phi(w, J^{-1}(\alpha_n J\widetilde{x}_n + (1 - \alpha_n)JSz_n)) \\
&= \|w\|^2 - 2\langle w, \alpha_n J\widetilde{x}_n + (1 - \alpha_n)JSz_n\rangle + \|\alpha_n J\widetilde{x}_n + (1 - \alpha_n)JSz_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n\langle w, J\widetilde{x}_n\rangle - 2(1 - \alpha_n)\langle w, JSz_n\rangle + \alpha_n\|\widetilde{x}_n\|^2 + (1 - \alpha_n)\|Sz_n\|^2 \\
&\leq \alpha_n\phi(w, \widetilde{x}_n) + (1 - \alpha_n)\phi(w, Sz_n) \\
&\leq \alpha_n\phi(w, \widetilde{x}_n) + (1 - \alpha_n)\phi(w, z_n).
\end{aligned}$$

So $w \in C_n$ for all $n \geq 0$. Now, from Lemma 3.1 it follows that there exists $(\widetilde{x}_0, v_0) \in E \times E^*$ such that $0 = v_0 + \frac{1}{r_0}(J\widetilde{x}_0 - Jx_0)$ and $v_0 \in T\widetilde{x}_0$. Since T is monotone, it follows that

$$\langle \widetilde{x}_0 - w, v_0 \rangle \geq 0,$$

which implies that $w \in D_0$ and hence $w \in H_0$. Furthermore, it is clear that $w \in W_0 = E$. Then $w \in H_0 \cap W_0$, and therefore $x_1 = \Pi_{H_0 \cap W_0} x_0$ is well defined. Suppose that $w \in H_{n-1} \cap W_{n-1}$ and x_n is well defined for some $n \geq 1$. Again by Lemma 3.1, we deduce that $(\widetilde{x}_n, v_n) \in E \times E^*$ such that $0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n)$ and $v_n \in T\widetilde{x}_n$. Then from the monotonicity of T we conclude that

$$\langle \widetilde{x}_n - w, v_n \rangle \geq 0$$

which implies that $w \in D_n$ and hence $w \in H_n$. It follows from Lemma 2.4 that

$$\langle w - x_n, Jx_0 - Jx_n \rangle = \langle w - \Pi_{H_{n-1} \cap W_{n-1}} x_0, Jx_0 - J\Pi_{H_{n-1} \cap W_{n-1}} x_0 \rangle \leq 0,$$

which implies that $w \in W_n$. Consequently, $w \in H_n \cap W_n$ and so $T^{-1}0 \cap F(S) \subset H_n \cap W_n$. Therefore $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ is well defined. Then, by induction, the sequence $\{x_n\}$ generated by (3.1) is well defined for each nonnegative integer n.■

Remark 3.1. From the above proof, we obtain

$$T^{-1}0 \cap F(S) \subset H_n \cap W_n$$

for each nonnegative integer n.

Now we are in a position to prove the main theorems.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let $\{r_n\}_{n=0}^{\infty}$ be a sequence in $(0, \infty)$ and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ be sequences in $[0, 1]$ such that*

$$\liminf_{n\to\infty} r_n > 0, \quad \limsup_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 1.$$

Let $T^{-1}0 \cap F(S) \neq \emptyset$. If S is uniformly continuous, then the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to $\Pi_{T^{-1}0 \cap F(S)} x_0$.

Proof. First of all, it follows from the definition of W_n that $x_n = \Pi_{W_n} x_0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in W_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \text{for all } n \geq 0.$$

Thus $\{\phi(x_n, x_0)\}$ is nondecreasing. Also from $x_n = \Pi_{W_n} x_0$ and Lemma 2.3, we have that

$$\phi(x_n, x_0) = \phi(\Pi_{W_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0)$$

for each $w \in T^{-1}0 \cap F(S) \subset W_n$ and for each $n \geq 0$. Consequently, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, according to the inequality

$$(\|x_n\| - \|x_0\|)^2 \leq \phi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2,$$

we conclude that $\{x_n\}$ is bounded. Thus, we have that $\lim_{n\to\infty} \phi(x_n, x_0)$ exists. From Lemma 2.3, we derive

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n} x_0) \\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \end{aligned}$$

for all $n \geq 0$. This implies that $\phi(x_{n+1}, x_n) \to 0$. So it follows from Lemma 2.1 that $x_{n+1} - x_n \to 0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, from the definition of H_n, we also have

$$\begin{aligned} &\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, \widetilde{x}_n) \\ &+(1-\alpha_n)\phi(x_{n+1}, z_n) \quad \text{and} \quad \langle x_{n+1} - \widetilde{x}_n, v_n \rangle \leq 0. \end{aligned} \tag{3.2}$$

Observe that

$$\begin{aligned} \phi(x_{n+1}, z_n) &= \phi(x_{n+1}, J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n) JS\widetilde{x}_n)) \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n J\widetilde{x}_n + (1-\beta_n) JS\widetilde{x}_n \rangle \\ &\quad + \|\beta_n J\widetilde{x}_n + (1-\beta_n) JS\widetilde{x}_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\beta_n \langle x_{n+1}, J\widetilde{x}_n \rangle - 2(1-\beta_n)\langle x_{n+1}, JS\widetilde{x}_n \rangle \\ &+\beta_n \|\widetilde{x}_n\|^2 + (1-\beta_n)\|S\widetilde{x}_n\|^2 \\ &= \beta_n \phi(x_{n+1}, \widetilde{x}_n) + (1-\beta_n)\phi(x_{n+1}, S\widetilde{x}_n). \end{aligned} \tag{3.3}$$

At the same time,

$$\begin{aligned} \phi(\Pi_{H_n} x_n, x_n) - \phi(\widetilde{x}_n, x_n) &= \|\Pi_{H_n} x_n\|^2 - \|\widetilde{x}_n\|^2 + 2\langle \widetilde{x}_n - \Pi_{H_n} x_n, Jx_n \rangle \\ &\geq 2\langle \Pi_{H_n} x_n - \widetilde{x}_n, J\widetilde{x}_n \rangle + 2\langle \widetilde{x}_n - \Pi_{H_n} x_n, Jx_n \rangle \\ &= 2\langle \widetilde{x}_n - \Pi_{H_n} x_n, Jx_n - J\widetilde{x}_n \rangle. \end{aligned}$$

Since $\Pi_{H_n} x_n \in H_n$ and $v_n = \frac{1}{r_n}(Jx_n - J\widetilde{x}_n)$, it follows that

$$\langle \widetilde{x}_n - \Pi_{H_n} x_n, Jx_n - J\widetilde{x}_n \rangle = r_n \langle \widetilde{x}_n - \Pi_{H_n} x_n, v_n \rangle \geq 0$$

and hence that $\phi(\Pi_{H_n}x_n, x_n) \geq \phi(\widetilde{x}_n, x_n)$. Further, from $x_{n+1} \in H_n$, we have $\phi(x_{n+1}, x_n) \geq \phi(\Pi_{H_n}x_n, x_n)$, which yields

$$\phi(x_{n+1}, x_n) \geq \phi(\Pi_{H_n}x_n, x_n) \geq \phi(\widetilde{x}_n, x_n).$$

Then it follows from $\phi(x_{n+1}, x_n) \to 0$ that $\phi(\widetilde{x}_n, x_n) \to 0$. Hence it follows from Lemma 2.1 that $\widetilde{x}_n - x_n \to 0$. Since from (3.2) we derive

$$\begin{aligned}
&\phi(x_{n+1}, \widetilde{x}_n) - \phi(\widetilde{x}_n, x_n) \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J\widetilde{x}_n\rangle + \|\widetilde{x}_n\|^2 - (\|\widetilde{x}_n\|^2 - 2\langle \widetilde{x}_n, Jx_n\rangle + \|x_n\|^2) \\
&= \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1}, J\widetilde{x}_n\rangle + 2\langle \widetilde{x}_n, Jx_n\rangle \\
&= \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1} - \widetilde{x}_n, J\widetilde{x}_n - Jx_n\rangle \\
&\quad -2\langle x_{n+1} - \widetilde{x}_n, Jx_n\rangle + 2\langle \widetilde{x}_n, Jx_n - J\widetilde{x}_n\rangle \\
&= \|x_{n+1}\|^2 - \|x_n\|^2 + 2r_n\langle x_{n+1} - \widetilde{x}_n, v_n\rangle - 2\langle x_{n+1} - \widetilde{x}_n, Jx_n\rangle \\
&\quad +2\langle \widetilde{x}_n, Jx_n - J\widetilde{x}_n\rangle \\
&\leq \|x_{n+1}\|^2 - \|x_n\|^2 - 2\langle x_{n+1} - \widetilde{x}_n, Jx_n\rangle + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\| \\
&\leq (\|x_{n+1}\| - \|x_n\|)(\|x_{n+1}\| + \|x_n\|) + 2\|x_{n+1} - \widetilde{x}_n\|\|x_n\| + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\| \\
&\leq \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) + 2(\|x_{n+1} - x_n\| + \|x_n - \widetilde{x}_n\|)\|x_n\|,
\end{aligned}$$

we have

$$\begin{aligned}
\phi(x_{n+1}, \widetilde{x}_n) &\leq \phi(\widetilde{x}_n, x_n) + \|x_{n+1} - x_n\|(\|x_{n+1}\| + \|x_n\|) \\
&\quad +2(\|x_{n+1} - x_n\| + \|x_n - \widetilde{x}_n\|)\|x_n\| + 2\|\widetilde{x}_n\|\|Jx_n - J\widetilde{x}_n\|.
\end{aligned}$$

Thus from $\phi(\widetilde{x}_n, x_n) \to 0$, $x_n - \widetilde{x}_n \to 0$ and $x_{n+1} - x_n \to 0$, we know that $\phi(x_{n+1}, \widetilde{x}_n) \to 0$. Consequently from (3.3), $\phi(\widetilde{x}_n, x_n) \to 0$ and $\beta_n \to 1$ it follows that

$$\phi(x_{n+1}, z_n) \to 0. \tag{3.4}$$

So it follows from (3.2), $\phi(x_{n+1}, \widetilde{x}_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$ that

$$\phi(x_{n+1}, y_n) \to 0. \tag{3.5}$$

Utilizing Lemma 2.1 we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - \widetilde{x}_n\| = \lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{3.6}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\widetilde{x}_n\| = 0. \tag{3.7a}$$

On the other hand, we have

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|.$$

It follows from $x_{n+1} - x_n \to 0$ and $x_{n+1} - z_n \to 0$ that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.7b}$$

Noticing that

$$\begin{aligned}\|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n)\| \\ &= \|\alpha_n(Jx_{n+1} - J\widetilde{x}_n) + (1-\alpha_n)(Jx_{n+1} - JSz_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JSz_n) - \alpha_n(J\widetilde{x}_n - Jx_{n+1})\| \\ &\ge (1-\alpha_n)\|Jx_{n+1} - JSz_n\| - \alpha_n\|J\widetilde{x}_n - Jx_{n+1}\|,\end{aligned}$$

we have

$$\|Jx_{n+1} - JSz_n\| \le \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|J\widetilde{x}_n - Jx_{n+1}\|).$$

From (3.7) and $\limsup_{n\to\infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JSz_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - Sz_n\| = 0. \tag{3.7c}$$

Observe that

$$\|x_n - Sx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Sz_n\| + \|Sz_n - Sx_n\|.$$

Since S is uniformly continuous, it follows from (3.7b), (3.7c) and $x_{n+1} - x_n \to 0$ that $x_n - Sx_n \to 0$.

Finally we prove that $x_n \to \Pi_{T^{-1}0\cap F(S)}x_0$. Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widetilde{x} \in E$, then $\widetilde{x} \in F(S)$. Now let us show that $\widetilde{x} \in T^{-1}0$. Since $x_n - \widetilde{x}_n \to 0$, we have that $\widetilde{x}_{n_i} \rightharpoonup \widetilde{x}$. Moreover, since J is uniformly norm-to-norm continuous on bounded subsets of E and $\liminf_{n\to\infty} r_n > 0$, we obtain

$$v_n = \frac{1}{r_n}(Jx_n - J\widetilde{x}_n) \to 0.$$

It follows from $v_n \in T\widetilde{x}_n$ and the monotonicity of T that

$$\langle z - \widetilde{x}_n, z' - v_n \rangle \geq 0$$

for all $z \in D(T)$ and $z' \in Tz$. This implies that

$$\langle z - \widetilde{x}, z' \rangle \geq 0$$

for all $z \in D(T)$ and $z' \in Tz$. Thus from the maximality of T, we infer that $\widetilde{x} \in T^{-1}0$. Therefore $\widetilde{x} \in T^{-1}0 \cap F(S)$.

Next let us show that $\widetilde{x} = \Pi_{T^{-1}0 \cap F(S)} x_0$ and convergence is strong. Put $\overline{x} = \Pi_{T^{-1}0 \cap F(S)} x_0$. From $x_{n+1} = \Pi_{H_n \cap W_n} x_0$ and $\overline{x} \in T^{-1}0 \cap F(S) \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\overline{x}, x_0)$. Now from weakly lower semicontinuity of the norm, we derive

$$\begin{aligned}
\phi(\widetilde{x}, x_0) &= \|\widetilde{x}\|^2 - 2\langle \widetilde{x}, Jx_0 \rangle + \|x_0\|^2 \\
&\leq \liminf_{i\to\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2) \\
&= \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \phi(\overline{x}, x_0).
\end{aligned}$$

It follows from the definition of $\Pi_{T^{-1}0 \cap F(S)} x_0$ that $\widetilde{x} = \overline{x}$ and hence

$$\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0).$$

So we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we conclude that $\{x_{n_i}\}$ converges strongly to $\Pi_{T^{-1}0 \cap F(S)} x_0$. Since $\{x_{n_i}\}$ is an arbitrarily weakly convergent sequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0 \cap F(S)} x_0$. This completes the proof. ∎

Corollary 3.1. (Kamimura and Takahashi [12, Theorem 8]). *Let E be a uniformly convex and uniformly smooth Banach space. If $T^{-1}0 \neq \emptyset$ and $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by the following algorithm*

$$\left\{\begin{aligned}
x_0 &\in E \text{ arbitrarily chosen,} \\
0 &= v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\
H_n &= \{v \in E : \langle v - \widetilde{x}_n, v_n \rangle \leq 0\}, \\
W_n &= \{v \in E : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\
x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ...,
\end{aligned}\right. \tag{3.8}$$

converges strongly to $\Pi_{T^{-1}0}x_0$.

Proof. In Theorem 3.1, we take $\alpha_n = 0$ and $\beta_n = 1$ for all n, and $S = I$ the identity mapping of E. Then $\widetilde{x}_n = z_n = y_n$ for all n, and hence $H_n = \{v \in E : \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}$. Thus algorithm (3.1) reduces to algorithm (3.8). By Theorem 3.1 we obtain the desired result. ∎

We remark that Theorem 3.1 covers [20, Theorem 2.1] as a special case.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and $S : E \to E$ be a relatively nonexpansive mapping. Assume that $\{r_n\}_{n=0}^{\infty}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$ and that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ by the following algorithm*

$$(3.10)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen,} \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in T\widetilde{x}_n, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n), \\ H_n = \{v \in E : \phi(v, y_n) \leq \alpha_n\phi(v, x_0) \\ \qquad +(1-\alpha_n)\phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n}x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where J is the single-valued duality mapping on E. Let $T^{-1}0 \cap F(S) \neq \emptyset$. If S is uniformly continuous, then $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap F(S)}x_0$.

Proof. For each $n \geq 0$, define two sets C_n and D_n as follows:

$$C_n = \{v \in E : \phi(v, y_n) \leq \alpha_n\phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)\}$$
$$\text{and} \quad D_n = \{v \in E : \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}.$$

It is obvious that C_n is closed and D_n, W_n are closed convex sets for each $n \geq 0$. Let us show that C_n is convex and so $H_n = C_n \cap D_n$ is closed and convex. Similarly to the proof of Lemma 3.2, since

$$\phi(v, y_n) \leq \alpha_n\phi(v, x_0) + (1-\alpha_n)\phi(v, \widetilde{x}_n)$$

is equivalent to

$$2\alpha_n\langle v, Jx_0\rangle + 2(1-\alpha_n)\langle v, J\widetilde{x}_n\rangle - 2\langle v, Jy_n\rangle \leq \alpha_n\|x_0\|^2 + (1-\alpha_n)\|\widetilde{x}_n\|^2 - \|y_n\|^2,$$

we know that C_n is convex and so is $H_n = C_n \cap D_n$. Next, let us show that $T^{-1}0 \cap F(S) \subset C_n$ for each $n \geq 0$. Indeed, we have, for each $w \in F(S)$

$$\begin{aligned} \phi(w, y_n) &= \phi(w, J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n)) \\ &= \|w\|^2 - 2\langle w, \alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n\|^2 \\ &\leq \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2(1-\alpha_n)\langle w, JS\widetilde{x}_n\rangle \\ &\quad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|S\widetilde{x}_n\|^2 \\ &\leq \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, S\widetilde{x}_n) \\ &\leq \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, \widetilde{x}_n). \end{aligned}$$

So $w \in C_n$ for all $n \geq 0$ and $F(S) \subset C_n$. As in the proof of Lemma 3.2, we can obtain $w \in D_n$ and hence $w \in H_n$. It follows from Lemma 2.4 that

$$\langle w - x_n, Jx_0 - Jx_n\rangle = \langle w - \Pi_{H_{n-1}\cap W_{n-1}}x_0, Jx_0 - J\Pi_{H_{n-1}\cap W_{n-1}}x_0\rangle \leq 0,$$

which implies that $w \in W_n$. Consequently, $w \in H_n \cap W_n$ and so $T^{-1}0 \cap F(S) \subset H_n \cap W_n$ for all $n \geq 0$. Therefore, the sequence $\{x_n\}$ generated by (3.10) is well defined. As in the proof of Theorem 3.1, we can obtain $\phi(x_{n+1}, x_n) \to 0$. Since $x_{n+1} = \Pi_{H_n\cap W_n}x_0 \in H_n$, from the definition of H_n we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n\phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \widetilde{x}_n) \quad \text{and} \quad \langle x_{n+1} - \widetilde{x}_n, v_n\rangle \leq 0.$$

As in the proof of Theorem 3.1, we can deduce not only from $\phi(x_{n+1}, x_n) \to 0$ that $\phi(\widetilde{x}_n, x_n) \to 0$ but also from $\phi(\widetilde{x}_n, x_n) \to 0$, $x_n - \widetilde{x}_n \to 0$ and $x_{n+1} - x_n \to 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, \widetilde{x}_n) = 0. \tag{3.11}$$

Since $x_{n+1} = \Pi_{H_n\cap W_n}x_0 \in H_n$, from the definition of H_n, we also have

$$\phi(x_{n+1}, y_n) \leq \alpha_n\phi(x_{n+1}, x_0) + (1-\alpha_n)\phi(x_{n+1}, \widetilde{x}_n).$$

It follows from (3.11) and $\alpha_n \to 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{3.12}$$

Utilizing Lemma 2.1 we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - \widetilde{x}_n\| = 0. \tag{3.13}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - J\widetilde{x}_n\| = 0. \tag{3.14}$$

Note that

$$\begin{aligned}\|JS\widetilde{x}_n - Jy_n\| &= \|JS\widetilde{x}_n - (\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n)\| \\ &= \alpha_n\|Jx_0 - JS\widetilde{x}_n\|.\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \|JS\widetilde{x}_n - Jy_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} \|S\widetilde{x}_n - y_n\| = 0. \tag{3.15}$$

It follows that

$$\|x_n - Sx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - S\widetilde{x}_n\| + \|S\widetilde{x}_n - Sx_n\|. \tag{3.16}$$

Since S is uniformly continuous, it follows from (3.13) and (3.15) that $x_n - Sx_n \to 0$.

Finally, we prove that $x_n \to \Pi_{T^{-1}0\cap F(S)}x_0$. Indeed, assume that $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \widetilde{x} \in E$, then $\widetilde{x} \in F(S)$. Now let us show that $\widetilde{x} \in T^{-1}0$. Since $x_n - \widetilde{x}_n \to 0$, we have that $\widetilde{x}_{n_i} \rightharpoonup \widetilde{x}$. Moreover, since J is uniformly norm-to-norm continuous on bounded subsets of E and $\liminf_{n\to\infty} r_n > 0$, we obtain that $v_n = \frac{1}{r_n}(Jx_n - J\widetilde{x}_n) \to 0$. It follows from $v_n \in T\widetilde{x}_n$ and the monotonicity of T that $\langle z - \widetilde{x}_n, z' - v_n\rangle \ge 0$ for all $z \in D(T)$ and $z' \in Tz$. This implies that $\langle z - \widetilde{x}, z'\rangle \ge 0$ for all $z \in D(T)$ and $z' \in Tz$. Thus from the maximality of T, we infer that $\widetilde{x} \in T^{-1}0$. Therefore $\widetilde{x} \in T^{-1}0 \cap F(S)$. Now, put $\overline{x} = \Pi_{T^{-1}0\cap F(S)}x_0$. From $x_{n+1} = \Pi_{H_n\cap W_n}x_0$ and $\overline{x} \in T^{-1}0 \cap F(S) \subset H_n \cap W_n$, we have $\phi(x_{n+1}, x_0) \le \phi(\overline{x}, x_0)$. On the other hand, from weak lower semicontinuity of the norm, we obtain

$$\begin{aligned}\phi(\widetilde{x}, x_0) &= \|\widetilde{x}\|^2 - 2\langle \widetilde{x}, Jx_0\rangle + \|x_0\|^2 \\ &\le \liminf_{i\to\infty}(\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_)\|^2) \\ &= \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\le \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\ &\le \phi(\overline{x}, x_0).\end{aligned}$$

It follows from the definition of $\Pi_{T^{-1}0\cap F(S)}x_0$ that $\widetilde{x} = \overline{x}$ and hence $\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\overline{x}, x_0)$. So, we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\overline{x}\|$. Utilizing the Kadec-Klee property of E, we know that $\{x_{n_i}\}$ converges strongly to $\Pi_{T^{-1}0\cap F(S)}x_0$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent sequence of $\{x_n\}$, we know that $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0\cap F(S)}x_0$. This completes the proof. ■

We remark that Theorem 3.2 covers [20, Theorem 2.2] as a special case.

4. Application

Let $f : E \mapsto (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined by

$$\partial f(z) = \{v \in E^* : f(y) \geq f(z) + \langle y - z, v\rangle,\ \forall y \in E\} \quad \text{for all } z \in E.$$

Using Theorems 3.1 and 3.2, we consider the problem of finding a minimizer of the function f.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space. Let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function and $S : E \to E$ be a relatively nonexpansive mapping. Assume that $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and that $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \beta_n = 1$. Let $\{x_n\}$ be the sequence generated by*

$$(4.1)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ \widetilde{x}_n = \text{argmin}_{z\in E}\{f(z) + \dfrac{1}{2r_n}\|z\|^2 - \dfrac{1}{r_n}\langle z, Jx_n\rangle\}, \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in \partial f(\widetilde{x}_n), \\ z_n = J^{-1}(\beta_n J\widetilde{x}_n + (1-\beta_n)JS\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n J\widetilde{x}_n + (1-\alpha_n)JSz_n), \\ H_n = \{v \in E : \phi(v, y_n) \leq \alpha_n\phi(v, \widetilde{x}_n) \\ \qquad +(1-\alpha_n)\phi(v, z_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \leq 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where J is the single-valued duality mapping on E. Let $(\partial f)^{-1}0 \cap F(S) \neq \emptyset$. If S is uniformly continuous, then $\{x_n\}$ converges strongly to the minimizer of f.

Proof. Since $f : E \mapsto (-\infty, \infty]$ is a proper convex lower semicontinuous function, by Rockafellar [22], the subdifferential ∂f of f is a maximal monotone operator. We also know that

$$\widetilde{x}_n = \text{argmin}_{z\in E}\{f(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle z, Jx_n\rangle\}$$

is equivalent to

$$0 \in \partial f(\widetilde{x}_n) + \frac{1}{r_n}J\widetilde{x}_n - \frac{1}{r_n}Jx_n.$$

Thus, we have $v_n \in \partial f(\widetilde{x}_n)$ such that $0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n)$. By Theorem 3.1 we obtain the desired result.

We remark that Theorem 4.1 covers [12, Theorem 9] as a special case.

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space. Let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function and $S : E \to E$ be a relatively nonexpansive mapping. Assume that $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ and that $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$. Let $\{x_n\}$ be the sequence generated by*

$$(4.3)\qquad \begin{cases} x_0 \in E \text{ arbitrarily chosen}, \\ \widetilde{x}_n = \operatorname{argmin}_{z\in E}\{f(z) + \dfrac{1}{2r_n}\|z\|^2 - \dfrac{1}{r_n}\langle z, Jx_n\rangle\}, \\ 0 = v_n + \dfrac{1}{r_n}(J\widetilde{x}_n - Jx_n), \quad v_n \in \partial f(\widetilde{x}_n), \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JS\widetilde{x}_n), \\ H_n = \{v \in E : \phi(v, y_n) \le \alpha_n \phi(v, x_0) \\ \qquad +(1-\alpha_n)\phi(v, \widetilde{x}_n) \text{ and } \langle v - \widetilde{x}_n, v_n\rangle \le 0\}, \\ W_n = \{v \in E : \langle v - x_n, Jx_0 - Jx_n\rangle \le 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, ..., \end{cases}$$

where J is the single-valued duality mapping on E. Let $(\partial f)^{-1}0 \cap F(S) \neq \emptyset$. If S is uniformly continuous, then $\{x_n\}$ converges strongly to the minimizer of f.

Proof. As in the proof of Theorem 4.1, we know that

$$\widetilde{x}_n = \operatorname{argmin}_{z\in E}\{f(z) + \frac{1}{2r_n}\|z\|^2 - \frac{1}{r_n}\langle z, Jx_n\rangle\}$$

is equivalent to

$$0 \in \partial f(\widetilde{x}_n) + \frac{1}{r_n}J\widetilde{x}_n - \frac{1}{r_n}Jx_n.$$

Thus, we have $v_n \in \partial f(\widetilde{x}_n)$ such that $0 = v_n + \frac{1}{r_n}(J\widetilde{x}_n - Jx_n)$. By Theorem 3.2 we obtain the desired result. ∎

References

1. Ya. I. Alber And S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, Panamer. *Math. J.*, **4(2)** (1994), 39-54.

2. Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type, Marcel Dekker, New York, 1996, pp. 15-50.

3. Ya. I. Alber and S. Guerre-Delabriere, On the projection methods for fixed point problems, *Analysis (Munich)*, **21** (2001), 17-39.

4. D. Butnariu, S. Reich and A. J. Zaslavski, Asymptotic behavior of relatively nonexpansive operators in Banach spaces, *J. Appl. Anal.*, **7** (2001), 151-174.

5. D. Butnariu, S. Reich and A. J. Zaslavski, Weak convergence of orbits of nonlinear operators in reflexive Banach spaces, *Numer. Funct. Anal. Optim.*, **24** (2003), 489-508.

6. Y. Censor and S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, *Optimization*, **37** (1996), 323-339.

7. L. C. Ceng, T. C. Lai and J. C. Yao, Approximate proximal algorithms for generalized variational inequalities with paramonotonicity and pseudomonotonicity, *Comput. Math. Appl.*, **55(6)** (2008), 1262-1269.

8. I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

9. L. C. Ceng, S. Y. Wu and J. C. Yao, New accuracy criteria for modified approximate proximal point algorithms in Hilbert spaces, *Taiwan. J. Math.*, **12(6)** (2008), 1691-1705.

10. O. Güler, On the convergence of the proximal point algorithm for convex minimization, *SIAM J. Control Optim.*, **29** (1991), 403-419.

11. S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, *J. Approx. Theory*, **106** (2000), 226-240.

12. S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, **13** (2003), 938-945.

13. L. C. Ceng and J. C. Yao, Generalized implicit hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *Taiwan. J. Math.*, **12(3)** (2008), 753-766.

14. C. Martinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.*, **64** (2006), 2400-2411.

15. S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory*, **134** (2005), 257-266.

16. K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372-379.

17. S. Reich, *A weak convergence theorem for the alternating method with Bergman distance*, in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313-318.

18. B. Martinet, Regularisation d'inequations variationnelles par approximations successives, *Rev. Franc. Inform. Rech. Oper.*, **4** (1970), 154-159.

19. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

20. X. L. Qin and Y. F. Su, Strong convergence theorems for relative nonexpansive mappings in a Banach space, *Nonlinear Anal.*, **67** (2007), 1958-1965.

21. L. C. Zeng and J. C. Yao, An inexact proximal-type algorithm in Banach spaces, *J. Optim. Theory Appl.*, **135(1)** (2007), 145-161.

22. R. T. Rockafellar, Characterization of the subdifferentials of convex functions, *Pacific J. Math.*, **17** (1966), 497-510.

23. R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, **149** (1970), 75-88.

24. R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.*, **14** (1976), 877-898.

25. L. C. Ceng and J. C. Yao, Approximate proximal algorithms for generalized variational inequalities with pseudomonotone multifunctions, *J. Comput. Appl. Math.*, **213(2)** (2008), 423-438.

26. M. V. Solodov and B. F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Program.*, **87** (2000), 189-202.

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