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NONDIFFERENTIABLE MULTIOBJECTIVE SECOND ORDER SYMMETRIC DUALITY WITH CONE CONSTRAINTS

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Abstract. We introduce two pairs of nondifferentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones, which is different from the one proposed by Mishra and Lai [12]. Under suitable second order pseudo-invexity assumptions we establish weak, strong and converse duality theorems as well as self-duality relations. Our symmetric duality results include an extension of the symmetric duality results for the first order case obtained by Kim and Kim [7] to the second order case. Several known results are abtained as special cases.

1. Introduction

Symmetric duality for quadratic programming was introduced by Dorn [5], who defined symmetric duality in mathematical programming if the dual of the dual is the primal problem. Applying these results to nonlinear programming, Dantzig et al. [4] formulated a symmetric dual and established symmetric duality relations. The notion of symmetric duality was developed significantly by Mond and Weir [14], Chandra and Husain [3] and Mond and Weir [15]. Also Mond and Weir [15] presented two pairs of symmetric dual multiobjective programming problems for efficient solutions and obtained symmetric duality results concerning pseudo-convex or convex functions. Later, Mond and Schechter [13] first introduced a symmetric dual programs where the objective function contains a support function.

On the other hand, Bector and Chandra [2] studied Mond-Weir type second order primal and dual nonlinear programs and established second order symmetric duality results. Mishra [11] considered second order symmetric duality under second order F-convexity, F-pseudo-convexity for second order Wolfe and Mond-Weir models, respectively. Recently, Yang et al. [19] introduced a symmetric dual for

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a class of multiobjective programs, which is Mond-Weir type. Then in Yang et al. [20] formulated a pair of Wolfe type second order symmetric dual programs in nondifferentiable multiobjective nonlinear prigramming and presented duality results for these programs. Very recently, Kim et al. [8] gave a pair of nonfifferentiable multiobjective generalized second order symmetric dual programs as unified models and established duality relations.

In this paper we focus on symmetric duality with cone constraints. Bazaraa and Goode [1] established symmetric duality results for convex function with arbitrary cones. Nanda and Das [17] formulated a pair of symmetric dual nonlinear programming problems for pseudo-invex functions and arbitrary cones. In the multiobjective case, Kim et al. [9] formulated a pair of multiobjective symmetric dual programs for pseudo-invex functions and arbitrary cones and established duality results. Subsequently, Suneja et al. [18] formulated a pair of symmetric dual multiobjective programs of Wolfe type over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the function involved to be cone-convex. Recently, Khurana [6] introduced cone-pseudo-invex and strongly cone-pseudo-invex functions and established duality theorems for a pair of Mond-Weir type multiobjective symmetric dual over arbitrary cones. Very recently, Kim and Kim [7] studied two pairs of non-differentiable multiobjective symmetric dual problems with cone constraints over arbitrary closed convex cones, which are Wolfe type and Mond-Weir type.

In the second order case, Mishra [10] formulated a pair of multiobjective second order symmetric dual nonlinear programming problems under second order pseudo-invexity assumptions on the functions involved over arbitrary cones and established duality results. The concept of cone-second order pseudo-invex and strongly cone-second order pseudo-invex functions was introduced by Mishra and Lai [12]. They formulated a pair of Mond-Weir type multiobjective second order symmetric dual programs over arbitrary cones.

In this paper, we consider two pairs of nondifferentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type. These are slightly different from Mishra and Lai ([10], [12]). Weak, strong, converse and self-duality theorems are established under the assumptions of second order pseudo-invex functions. Our results extend the results in Kim and Kim [7] to the second order case. Moreover, we give some special cases of our symmetric duality results.

2. Preliminaries

Definition 2.1. A nonempty set K in $\mathbb{R}^k$ is said to be a cone with vertex zero, if $x \in K$ implies that $\lambda x \in K$ for all $\lambda \geqq 0$. If, in addition, K is convex, then K

is called a convex cone.

Consider the following multiobjective programming problem:

$$(KP) \quad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad -g(x) \in Q, x \in C,$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $C \subset \mathbb{R}^n$, Q is a closed convex cone with nonempty interior in $\mathbb{R}^m$.

We shall denote the feasible set of $(\mathbf{KP})$ by $X = \{x| - g(x) \in Q, x \in C\}$.

Definition 2.2. A feasible point $\overline{x}$ is a K-weakly efficient solution of $(\mathbf{KP})$, if there exists no other $x \in X$ such that $f(\overline{x}) - f(x) \in intK$.

Definition 2.3. The positive polar cone K^* of K is defined by

$$K^* = \{z \in \mathbb{R}^k \quad | \quad x^T z \geqq 0 \quad \text{for all} \quad x \in K\}.$$

Definition 2.4. ([10]). Let $f : X(\subset \mathbb{R}^n) \times Y(\subset \mathbb{R}^m) \to \mathbb{R}$ be a twice differentiable function.

(i) f is said to be second order invex in the first variable at u for fixed v, if there exists a function $\eta_1 : X \times X \to X$ such that for $r \in \mathbb{R}^n$,

$$f(x,v) - f(u,v) \geqq \eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)r] - \frac{1}{2} r^T \nabla_{xx} f(u,v)r.$$

(ii) f is said to be second order pseudo-invex in the first variable at u for fixed v, if there exists a function $\eta_1 : X \times X \to X$ such that for $r \in \mathbb{R}^n$,

$$\eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)r] \geqq 0 \Rightarrow f(x,v) - f(u,v) + \frac{1}{2} r^T \nabla_{xx} f(u,v)r \geqq 0.$$

Definition 2.5. ([13]). The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \text{ for all } y \in B.$$

Equivalently,

$$z^T x = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z-x) \leq 0 \ \text{ for all } \ z \in S\}.$$

It is readily verified that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y.

Definition 2.6. ([14]). A function $f(x,y), x \in \mathbb{R}^n, y \in \mathbb{R}^n$ is said to be skew-symmetric if

$$f(x,y) = -f(y,x)$$

for all x and y in the domain of f.

3. Mond-weir Type Symmetric Duality

We consider the following pair of second order Mond-Weir type non-differentiable multiobjective programming problem with k-objectives :

$(\mathbf{MP})$ Minimize

$$P(x,y,\lambda,w,p)$$
$$= \Big(f_1(x,y) + s(x|B_1) - y^T w_1 - \frac{1}{2}\sum_{i=1}^{k}\lambda_i p_i^T \nabla_{yy} f_i(x,y) p_i, \cdots,$$
$$f_k(x,y) + s(x|B_k) - y^T w_k - \frac{1}{2}\sum_{i=1}^{k}\lambda_i p_i^T \nabla_{yy} f_i(x,y) p_i\Big)$$

$$\text{subject to } \ -\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y) - w_i + \nabla_{yy} f_i(x,y)p_i] \in C_2^*, \tag{1}$$

$$y^T\sum_{i=1}^{k}\lambda_i[\nabla_y f_i(x,y) - w_i + \nabla_{yy} f_i(x,y)p_i] \geqq 0, \tag{2}$$

$$x \in C_1, \quad w_i \in D_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,$$

$(\mathbf{MD})$ Maximize

$$D(u,v,\lambda,z,r)$$
$$= \Big(f_1(u,v) - s(v|D_1) + u^T z_1 - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(u,v) r_i, \cdots,$$
$$f_k(u,v) - s(v|D_k) + u^T z_k - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(u,v) r_i\Big)$$

$$\text{subject to } \quad \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v) + z_i + \nabla_{xx} f_i(u,v)r_i] \in C_1^*, \tag{3}$$

$$(4) \qquad u^T \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + z_i + \nabla_{xx} f_i(u,v) r_i] \leqq 0,$$
$$v \in C_2, \quad z_i \in B_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,$$

where

(i) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,

(ii) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(iii) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,

(iv) K is a closed convex cone in $\mathbb{R}^k$ with $intK \neq \emptyset$ and $\mathbb{R}_+^k \subset K$,

(v) $r_i, z_i (i = 1, \cdots, k)$ are vectors in $\mathbb{R}^n$, $p_i, w_i (i = 1, \cdots, k)$ are vectors in $\mathbb{R}^m$,

(vi) $e = (1, \cdots, 1)^T$ is a vector in $\mathbb{R}^k$,

(vii) B_i and $D_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Now we establish the symmetric duality theorems of $(\mathbf{MP})$ and $(\mathbf{MD})$.

Theorem 3.1. (Weak Duality). *Let (x, y, λ, w, p) and (u, v, λ, z, r) be feasible solutions of $(\mathbf{MP})$ and $(\mathbf{MD})$, respectively. Assume that,*

(i) $\sum_{i=1}^{k} \lambda_i [f_i(\cdot, y) + (\cdot)^T z_i]$ *is second order pseudo-invex in the first variable for fixed y with respect to* η_1,

(ii) $-\sum_{i=1}^{k} \lambda_i [f_i(x, \cdot) - (\cdot)^T w_i]$ *is second order pseudo-invex in the second variable for fixed x with respect to* η_2,

(iii) $\eta_1(x, u) + u \in C_1$,

(iv) $\eta_2(v, y) + y \in C_2$. *Then*

$$D(u, v, \lambda, z, r) - P(x, y, \lambda, w, p) \notin intK.$$

Proof. From (3) and (iii), we obtain

$$[\eta_1(x,u)+u]^T\sum_{i=1}^k \lambda_i[\nabla_x f_i(u,v)+z_i+\nabla_{xx}f_i(u,v)r_i] \geqq 0.$$

From (4), it implies

$$\eta_1(x,u)^T\sum_{i=1}^k \lambda_i[\nabla_x f_i(u,v)+z_i+\nabla_{xx}f_i(u,v)r_i] \geqq 0.$$

By the second order pseudo-invexity of $\sum_{i=1}^k \lambda_i[f_i(\cdot,y)+(\cdot)^T z_i]$, we have

$$\sum_{i=1}^k \lambda_i[f_i(x,v)+x^T z_i - f_i(u,v) - u^T z_i + \frac{1}{2}r_i^T\nabla_{xx}f_i(u,v)r_i] \geqq 0. \tag{5}$$

Similarly, using (1), (2), (ii) and (iv), we have

$$\sum_{i=1}^k \lambda_i[f_i(x,v)-v^T w_i - f_i(x,y) + y^T w_i + \frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i] \leqq 0. \tag{6}$$

From the inequality (5) and the inequality (6), we get

$$\begin{aligned}&\sum_{i=1}^k \lambda_i[f_i(u,v)-v^T w_i + y^T w_i - \frac{1}{2}r_i^T\nabla_{xx}f_i(u,v)r_i]\\ &\quad -\sum_{i=1}^k \lambda_i[f_i(x,y)+x^T z_i - u^T z_i - \frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i] \leqq 0.\end{aligned}$$

Using the fact that $x^T z_i \leqq s(x|B_i)$ and $v^T w_i \leqq s(v|D_i)$ for $i=1,\cdots,k$, we obtain

$$\begin{aligned}&\sum_{i=1}^k \lambda_i[f_i(u,v)-s(v|D_i) + u^T z_i - \frac{1}{2}r_i^T\nabla_{xx}f_i(u,v)r_i]\\ &\quad -\sum_{i=1}^k \lambda_i[f_i(x,y)+s(x|B_i) - y^T w_i - \frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i] \leqq 0,\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{i=1}^k \lambda_i[f_i(u,v)-s(v|D_i) + u^T z_i - \frac{1}{2}\sum_{i=1}^k \lambda_i r_i^T\nabla_{xx}f_i(u,v)r_i]\\ &\quad -\sum_{i=1}^k \lambda_i[f_i(x,y)+s(x|B_i) - y^T w_i - \frac{1}{2}\sum_{i=1}^k \lambda_i p_i^T\nabla_{yy}f_i(x,y)p_i] \leqq 0.\end{aligned} \tag{7a}$$

But suppose that

$$D(u, v, \lambda, z, r) - P(x, y, \lambda, w, p) \in intK.$$

Since $\lambda \in intK^*$, it yields

$$\sum_{i=1}^{k} \lambda_i [f_i(u, v) - s(v|D_i) + u^T z_i - \frac{1}{2} \sum_{i=1}^{k} \lambda_i r_i^T \nabla_{xx} f_i(u, v) r_i]$$
$$- \sum_{i=1}^{k} \lambda_i [f_i(x, y) + s(x|B_i) - y^T w_i - \frac{1}{2} \sum_{i=1}^{k} \lambda_i p_i^T \nabla_{yy} f_i(x, y) p_i] > 0,$$

which is a contradiction to the inequality (7a). ■

Remark 3.1. If we replace (i) and (ii) of Theorem 3.1 by

(i) $[f_i(\cdot, y) + (\cdot)^T z_i]$, $i = 1, \cdots, k$, is second order invex in the first variable for fixed y with respect to η_1,

(ii) $-[f_i(x, \cdot) - (\cdot)^T w_i]$, $i = 1, \cdots, k$, is second order invex in the second variable for fixed x with respect to η_2,

then the same conclusion of Theorem 3.1 also holds.

Lemma 3.1. ([7]). *If $\overline{x}$ is a K-weakly efficient solution of* $(\mathbf{KP})$*, then there exist $\alpha \in K^*$ and $\beta \in Q^*$ not both zero such that*

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))(x - \overline{x}) \geqq 0, \quad \textit{for all} \quad x \in C,$$
$$\beta^T g(\overline{x}) = 0.$$

Equivalently, there exist $\alpha \in K^*$, $\beta \in Q^*$, $\beta_1 \in C^*$ and $(\alpha, \beta, \beta_1) \neq 0$ such that

$$\begin{aligned} \alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) - \beta_1^T I &= 0, \\ \beta^T g(\overline{x}) &= 0, \\ \beta_1^T \overline{x} &= 0. \end{aligned}$$

Proof. We can check that the first part of Lemma 3.1[1]. Now we prove the latter part of Lemma 3.1. (Sufficiency) Substituting $x = 0$ and $x = 2\overline{x}$, we get

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))\overline{x} = 0.$$

Since $\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) \in C^*$, let $\beta_1 = \alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x})$. Then

$$\begin{aligned} \alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) - \beta_1^T I &= 0, \\ \beta^T g(\overline{x}) &= 0, \\ \beta_1^T \overline{x} &= 0. \end{aligned}$$

(Necessity) Since $\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}) = \beta_1 \in C^*$, we get

$$(\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))x \geqq 0, \quad \text{for all} \quad x \in C$$

and

$$\beta_1^T\overline{x} = (\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))\overline{x} = 0.$$

Therefore,

$$(\alpha^T\nabla f(\overline{x}) + \beta^T\nabla g(\overline{x}))(x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C,$$

$$\beta^T g(\overline{x}) = 0.$$

■

Theorem 3.2. (Strong Duality). *Let* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ *be a* K*-weakly efficient solution of* $(\mathbf{MP})$. *Fix* $\lambda = \overline{\lambda}$ *in* $(\mathbf{MD})$. *Assume that*

(i) $\nabla_{yy}f_i$ *is positive definite for* $i = 1, \cdots, k$ *and* $\sum_{i=1}^{k}\overline{\lambda}_i\overline{p}_i^T[\nabla_y f_i - \overline{w}_i] \geqq 0$;*or*
$\nabla_{yy}f_i$ *is negative definite for* $i = 1, \cdots, k$ *and* $\sum_{i=1}^{k}\overline{\lambda}_i\overline{p}_i^T[\nabla_y f_i - \overline{w}_i] \leqq 0$,

(ii) *the set* $\{\nabla_y f_i - \overline{w}_i + \nabla_{yy}f_i\overline{p}_i,\ i = 1, \cdots, k\}$ *is linearly independent, where* $f_i = f_i(\overline{x}, \overline{y})$ *for* $i = 1, \cdots, k$.

Then there exists $\overline{z}_i \in B_i (i = 1, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a feasible solution of* $(\mathbf{MD})$ *and objective values of* $(\mathbf{MP})$ *and* $(\mathbf{MD})$ *are equal. Furthermore, under the assumptions of Theorem 3.1,* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a* K*-weakly efficient solution of* $(\mathbf{MD})$.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ is a K-weakly efficient solution of $(\mathbf{MP})$, by Lemma 3.1, there exist $\alpha \in K^*, \beta \in C_2, \mu \in \mathbb{R}_+, \delta \in C_1^*$ and $\rho \in K$ such that

$$\begin{aligned}&\sum_{i=1}^{k}\alpha_i(\nabla_x f_i + z_i) + (\beta - \mu\overline{y})^T\sum_{i=1}^{k}\overline{\lambda}_i\nabla_{yx}f_i \\ &\quad + \sum_{i=1}^{k}(\beta - \frac{1}{2}(\alpha^T e)\overline{p}_i - \mu\overline{y})^T\overline{\lambda}_i\nabla_x(\nabla_{yy}f_i\overline{p}_i) - \delta = 0,\end{aligned} \tag{7b}$$

$$\begin{aligned}&\sum_{i=1}^{k}(\alpha_i - \mu\overline{\lambda}_i)(\nabla_y f_i - \overline{w}_i) + \sum_{i=1}^{k}(\beta - \mu\overline{p}_i - \mu\overline{y})^T\overline{\lambda}_i\nabla_{yy}f_i \\ &\quad + \sum_{i=1}^{k}(\beta - \frac{1}{2}(\alpha^T e)\overline{p}_i - \mu\overline{y})^T\overline{\lambda}_i\nabla_y(\nabla_{yy}f_i\overline{p}_i) = 0,\end{aligned} \tag{8}$$

$$(\beta-\mu\overline{y})^T(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)-\frac{1}{2}(\alpha^Te)\overline{p}_i^T\nabla_{yy}f_i\overline{p}_i-\rho_i=0, \quad i=1,\cdots,k, \tag{9}$$

$$\alpha_i\overline{y}+(\beta-\mu\overline{y})\overline{\lambda}_i\in N_{D_i}(\overline{w}_i),\quad i=1,\cdots,k, \tag{10}$$

$$(\beta-(\alpha^Te)\overline{p}_i-\mu\overline{y})^T\overline{\lambda}_i\nabla_{yy}f_i=0,\quad i=1,\cdots,k, \tag{11}$$

$$\beta^T\sum_{i=1}^k\overline{\lambda}_i(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)=0, \tag{12}$$

$$\mu\overline{y}^T\sum_{i=1}^k\overline{\lambda}_i(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)=0, \tag{13}$$

$$\delta^T\overline{x}=0, \tag{14}$$

$$\rho^T\overline{\lambda}=0, \tag{15}$$

$$z_i\in B_i,\quad z_i^T\overline{x}=s(\overline{x}|B_i),\quad i=1,\cdots,k, \tag{16}$$

$$(\alpha,\beta,\mu,\delta,\rho)\neq 0. \tag{17}$$

Since $\overline{\lambda}>0$, it follows from (15), that $\rho=0$. As $\nabla_{yy}f_i$ is positive or negative definite for $i=1,\cdots,k$, (11) yields

$$\beta=(\alpha^Te)\overline{p}_i+\mu\overline{y},\quad i=1,\cdots,k. \tag{18}$$

If $\alpha_i=0$ for $i=1,\cdots,k$, then the above equality becomes

$$\beta=\mu\overline{y}. \tag{19}$$

From (8), we obtain

$$\mu\sum_{i=1}^k\overline{\lambda}_i(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)=0. \tag{20}$$

By the assumption (ii), we have $\mu=0$. Also, from (7b) and (19), we get $\delta=0$ and $\beta=0$, respectively. This contradicts (17). So, $\alpha_i>0$ for $i=1,\cdots,k$. From (12) and (13), we obtain

$$\sum_{i=1}^k(\beta-\mu\overline{y})^T\overline{\lambda}_i(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)=0.$$

Using (18) and $\alpha^T e > 0$, it follows that

$$\sum_{i=1}^{k} \overline{p}_i^T \overline{\lambda}_i (\nabla_y f_i - \overline{w}_i + \nabla_{yy} f_i \overline{p}_i) = 0.$$

So,

$$\sum_{i=1}^{k} \overline{p}_i^T \overline{\lambda}_i (\nabla_y f_i - \overline{w}_i) + \sum_{i=1}^{k} \overline{p}_i^T \overline{\lambda}_i \nabla_{yy} f_i \overline{p}_i = 0. \tag{21}$$

We now prove that $\overline{p}_i = 0$ for $i = 1, \cdots, k$. Otherwise, the assumption (i) implies that

$$\sum_{i=1}^{k} \overline{\lambda}_i \overline{p}_i^T (\nabla_y f_i - \overline{w}_i) + \sum_{i=1}^{k} \overline{\lambda}_i (\overline{p}_i^T \nabla_{yy} f_i \overline{p}_i) \neq 0,$$

which contradicts (21). Hence $\overline{p}_i = 0$ for $i = 1, \cdots, k$. From (18), we have

$$\beta = \mu \overline{y}. \tag{22}$$

Using (22) and $\overline{p}_i = 0$, $i = 1, \cdots, k$, in (8), we obtain

$$\sum_{i=1}^{k} (\alpha_i - \mu \overline{\lambda}_i)(\nabla_y f_i - \overline{w}_i) = 0.$$

By the assumption (ii), we get

$$\alpha_i = \mu \overline{\lambda}_i, \quad i = 1, \cdots, k. \tag{23}$$

Therefore, $\mu > 0$ and $\overline{y} \in C_2$ by (22). Using (19) and (23) in (7b), we have

$$\mu \sum_{i=1}^{k} \overline{\lambda}_i (\nabla_x f_i + z_i) = \delta \in C_1^*. \tag{24}$$

Also, since $\mu > 0$, it follows that

$$\sum_{i=1}^{k} \overline{\lambda}_i (\nabla_x f_i + z_i) \in C_1^*.$$

Multiplying (24) by $\overline{x}$ and using equation (14), we get

$$\overline{x}^T \sum_{i=1}^{k} \overline{\lambda}_i (\nabla_x f_i + z_i) = 0.$$

Taking $\overline{z}_i := z_i \in B_i$ for $i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for $(\mathbf{MD})$. Moreover, from (10), we get $\overline{y} \in N_{D_i}(\overline{w}_i)$ for $i = 1, \cdots, k$, so that $\overline{y}^T \overline{w}_i = s(\overline{y}|D_i)$ for $i = 1, \cdots, k$. Consequently, using (16)

$$
\begin{aligned}
& f_i + s(\overline{x}|B_i) - \overline{y}^T \overline{w}_i - \frac{1}{2} \sum_{i=1}^{k} \overline{\lambda}_i \overline{p}_i^T \nabla_{yy} f_i \overline{p}_i \\
&= f_i + \overline{x}^T \overline{z}_i - s(\overline{y}|D_i) \\
&= f_i - s(\overline{y}|D_i) + \overline{x}^T \overline{z}_i - \frac{1}{2} \sum_{i=1}^{k} \overline{\lambda}_i \overline{r}_i^T \nabla_{xx} f_i \overline{r}_i, \quad i = 1, \cdots, k.
\end{aligned}
$$

Thus objective values of $(\mathbf{MP})$ and $(\mathbf{MD})$ are equal.

We will now show that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution of $(\mathbf{MD})$, otherwise, there exists a feasible solution $(u, v, \overline{\lambda}, z, r = 0)$ of $(\mathbf{MD})$ such that

$$D(u, v, \overline{\lambda}, z, r = 0) - D(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0) \in intK.$$

Since objective values of $(\mathbf{MP})$ and $(\mathbf{MD})$ are equal, it follows that

$$D(u, v, \overline{\lambda}, z, r = 0) - P(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) \in intK,$$

which contradicts weak duality. Hence the results hold. ■

Theorem 3.3. (Converse Duality) *Let* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be a* K*-weakly efficient solution of* $(\mathbf{MD})$. *Fix* $\lambda = \overline{\lambda}$ *in* $(\mathbf{MP})$. *Assume that*

(i) $\nabla_{xx} f_i$ *is positive definite for* $i = 1, \cdots, k$ *and* $\sum_{i=1}^{k} \overline{\lambda}_i \overline{r}_i^T [\nabla_x f_i + \overline{z}_i] \geqq 0$; *or*

$\nabla_{xx} f_i$ *is negative definite for* $i = 1, \cdots, k$ *and* $\sum_{i=1}^{k} \overline{\lambda}_i \overline{r}_i^T [\nabla_x f_i + \overline{z}_i] \leqq 0$,

(ii) *the set* $\{\nabla_x f_i + \overline{z}_i + \nabla_{xx} f_i \overline{r}_i, \ i = 1, \cdots, k\}$ *is linearly independent, where* $f_i = f_i(\overline{u}, \overline{v})$ *for* $i = 1, \cdots, k$.

Then there exists $\overline{w}_i \in D_i (i = 1, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution of* $(\mathbf{MP})$ *and objective values of* $(\mathbf{MP})$ *and* $(\mathbf{MD})$ *are equal. Furthermore, under the assumptions of Theorem 3.1,* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a* K*-weakly efficient solution of* $(\mathbf{MP})$.

Proof. It follows on the lines of Theorem 3.2. ■

A mathematical programming problem is said to be self dual if, when the dual is recast in the form of the primal, the new program constructed is the same as the primal problem.

We now prove the following self duality theorem for the primal $(\mathbf{MP})$ and the dual $(\mathbf{MD})$ on the lines of Mond and Weir [14]. We describe $(\mathbf{MP})$ and $(\mathbf{MD})$ as dual programs, if the conclusions of Theorem 3.2. hold.

Theorem 3.4. (Self Duality) *Assume that $m = n$, $C_1 = C_2$ and $B = D$. If f is skew-symmetric, then the program $(\mathbf{MP})$ is self dual. Furthermore, if $(\mathbf{MP})$ and $(\mathbf{MD})$ are dual programs with K-weakly efficient solutions as $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ and $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$, respecively, then $(\overline{y}, \overline{x}, \overline{\lambda}, \overline{z}, \overline{p})$ and $(\overline{y}, \overline{x}, \overline{\lambda}, \overline{w}, \overline{r})$ are K-weakly efficient solutions of $(\mathbf{MP})$ and $(\mathbf{MD})$, respectively. Also the common objective value of the objective functions is* 0.

Proof. Rewriting the dual as in [14], we have

$(\mathbf{MD}')$ Minimize

$$-\Big(f_1(u,v) - s(v|D_1) + u^T z_1 - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(u,v) r_i, \cdots,$$

$$f_k(u,v) - s(v|D_k) + u^T z_k - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(u,v) r_i\Big)$$

$$\text{subject to} \quad \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v) + z_i + \nabla_{xx} f_i(u,v) r_i] \in C_1^*,$$

$$u^T \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(u,v) + z_i + \nabla_{xx} f_i(u,v) r_i] \leqq 0,$$

$$v \in C_2, \quad z_i \in B_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1.$$

Since f is skew-symmetric, therefore, as in [14], $f(u,v) = -f(v,u)$, $\nabla_x f(u,v) = -\nabla_x f(v,u)$ and $\nabla_{xx} f(u,v) = -\nabla_{xx} f(v,u)$. Hence $(\mathbf{MD}')$ becomes

$(\mathbf{MD}')$ Minimize

$$\Big(f_1(v,u) + s(v|D_1) - u^T z_1 - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(v,u) r_i, \cdots,$$

$$f_k(v,u) + s(v|D_k) - u^T z_k - \frac{1}{2}\sum_{i=1}^{k}\lambda_i r_i^T \nabla_{xx} f_i(v,u) r_i\Big)$$

$$\text{subject to} \quad -\sum_{i=1}^{k}\lambda_i[\nabla_x f_i(v,u) - z_i + \nabla_{xx} f_i(v,u) r_i] \in C_1^*,$$

$$u^T \sum_{i=1}^{k}\lambda_i[\nabla_x f_i(v,u) - z_i + \nabla_{xx} f_i(v,u) r_i] \geqq 0,$$

$$v \in C_2, \quad z_i \in B_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,$$

which is just $(\mathbf{MP})$. Thus, if $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ is K-weakly efficient solution of $(\mathbf{MD})$, then $(\overline{y}, \overline{x}, \overline{\lambda}, \overline{z}, \overline{p})$ is K-weakly efficient solution of $(\mathbf{MP})$, and hence by symmetric duality, also $(\overline{y}, \overline{x}, \overline{\lambda}, \overline{w}, \overline{r})$ is K-weakly efficient solution of $(\mathbf{MD})$.

Therefore,

$$\begin{aligned}
&P(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0)\\
&= (f_1(\overline{x}, \overline{y}) + s(\overline{x}|B_1) - \overline{y}^T \overline{w}_1, \cdots, f_k(\overline{x}, \overline{y}) + s(\overline{x}|B_k) - \overline{y}^T \overline{w}_k)\\
&= (f_1(\overline{y}, \overline{x}) + s(\overline{y}|D_1) - \overline{x}^T \overline{z}_1, \cdots, f_k(\overline{y}, \overline{x}) + s(\overline{y}|D_k) - \overline{x}^T \overline{z}_k)\\
&= (-f_1(\overline{x}, \overline{y}) - s(\overline{x}|B_1) + \overline{y}^T \overline{w}_1, \cdots, -f_k(\overline{x}, \overline{y}) - s(\overline{x}|B_k) + \overline{y}^T \overline{w}_k)\\
&= D(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0).
\end{aligned}$$

This implies

$$P(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) = 0 = D(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r}). \qquad \blacksquare$$

4. Wolfe Type Symmetric Duality

We consider the following pair of second order Wolfe type non-differentiable multiobjective programming problem with k-objectives :

$(\mathbf{WP})$ Minimize

$$\begin{aligned}
&P(x, y, \lambda, w, p)\\
&= \Big(f_1(x, y) + s(x|B_1) - y^T w_1 - \sum_{i=1}^{k} \lambda_i [y^T (\nabla_y f_i(x, y)\\
&-w_i + \nabla_{yy} f_i(x, y) p_i) + \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i], \cdots,\\
&\quad f_k(x, y) + s(x|B_k) - y^T w_k - \sum_{i=1}^{k} \lambda_i [y^T (\nabla_y f_i(x, y)\\
&-w_i + \nabla_{yy} f_i(x, y) p_i) + \frac{1}{2} p_i^T \nabla_{yy} f_i(x, y) p_i]\Big)
\end{aligned} \tag{25}$$

subject to $$-\sum_{i=1}^{k} \lambda_i [\nabla_y f_i(x, y) - w_i + \nabla_{yy} f_i(x, y) p_i] \in C_2^*,$$

$$x \in C_1, \quad w_i \in D_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,$$

$$(\mathbf{WD}) \quad \text{Maximize}$$

$$
\begin{aligned}
& D(u,v,\lambda,z,r) \\
& = \Big(f_1(u,v) - s(v|D_1) + u^T z_1 - \sum_{i=1}^{k} \lambda_i [u^T (\nabla_x f_i(u,v) \\
& + z_i + \nabla_{xx} f_i(u,v) r_i) + \frac{1}{2} r_i^T \nabla_{xx} f_i(u,v) r_i], \cdots, \\
& \quad f_k(u,v) - s(v|D_k) + u^T z_k - \sum_{i=1}^{k} \lambda_i [u^T (\nabla_x f_i(u,v) \\
& + z_i + \nabla_{xx} f_i(u,v) r_i) + \frac{1}{2} r_i^T \nabla_{xx} f_i(u,v) r_i] \Big) \\
& \text{subject to} \sum_{i=1}^{k} \lambda_i [\nabla_x f_i(u,v) + z_i + \nabla_{xx} f_i(u,v) r_i] \in C_1^*, \\
& \qquad v \in C_2, \quad z_i \in B_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,
\end{aligned}
\tag{26}
$$

where

(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,

(4) K is a closed convex cone in $\mathbb{R}^k$ with $intK \neq \emptyset$ and $\mathbb{R}^k_+ \subset K$,

(5) $r_i, z_i (i = 1, \cdots, k)$ are vectors in $\mathbb{R}^n$, $p_i, w_i (i = 1, \cdots, k)$ are vectors in $\mathbb{R}^m$,

(6) $e = (1, \cdots, 1)^T$ is a vector in $\mathbb{R}^k$,

(7) B_i and $D_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Now we establish the symmetric duality theorems of $(\mathbf{WP})$ and $(\mathbf{WD})$.

Theorem 4.1. (Weak Duality). *Let (x,y,λ,w,p) and (u,v,λ,z,r) be feasible solutions of $(\mathbf{WP})$ and $(\mathbf{WD})$, respectively. Assume that,*

(27) $\sum_{i=1}^{k} \lambda_i [f_i(\cdot, y) + (\cdot)^T z_i]$ *is second order invex in the first variable for fixed y with respect to* η_1,

(28) $-\sum_{i=1}^{k}\lambda_i[f_i(x,\cdot)-(\cdot)^T w_i]$ *is second order invex in the second variable for fixed x with respect to*η_2,

(29) $$\eta_1(x,u)+u\in C_1,$$

(30) $$\eta_2(v,y)+y\in C_2. Then$$

$$D(u,v,\lambda,z,r)-P(x,y,\lambda,w,p)\notin intK.$$

Proof. By assumptions (27) (28) (29) and (30) and applying constraints (25) and (26), we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i[f_i(u,v)-v^T w_i+u^T z_i\\
&-\sum_{i=1}^{k}\lambda_i\{u^T(\nabla_x f_i(u,v)+z_i+\nabla_{xx}f_i(u,v)r_i)+\frac{1}{2}r_i^T\nabla_{xx}f_i(u,v)r_i\}]\\
&-\sum_{i=1}^{k}\lambda_i[f_i(x,y)+x^T z_i-y^T w_i\\
&-\sum_{i=1}^{k}\lambda_i\{y^T(\nabla_y f_i(x,y)-w_i+\nabla_{yy}f_i(x,y)p_i)+\frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i\}]\\
\leqq\ & 0.
\end{aligned}$$

Using $x^T z_i \leqq s(x|B_i)$ and $v^T w_i \leqq s(v|D_i)$ for $i=1,\cdots,k$, we get

$$\begin{aligned}
&\sum_{i=1}^{k}\lambda_i[f_i(u,v)-s(v|D_i)+u^T z_i\\
&-\sum_{i=1}^{k}\lambda_i\{u^T(\nabla_x f_i(u,v)+z_i+\nabla_{xx}f_i(u,v)r_i)+\frac{1}{2}r_i^T\nabla_{xx}f_i(u,v)r_i\}]\\
&-\sum_{i=1}^{k}\lambda_i[f_i(x,y)+s(x|B_i)-y^T w_i\\
&-\sum_{i=1}^{k}\lambda_i\{y^T(\nabla_y f_i(x,y)-w_i+\nabla_{yy}f_i(x,y)p_i)+\frac{1}{2}p_i^T\nabla_{yy}f_i(x,y)p_i\}]\\
&\leqq 0.
\end{aligned} \tag{31}$$

But suppose that

$$D(u,v,\lambda,z,r)-P(x,y,\lambda,w,p)\in intK.$$

Since $\lambda \in intK^*$, it becomes $\lambda^T[D(u,v,\lambda,z,r) - P(x,y,\lambda,w,p)] > 0$, which is a contradiction to the inequality (31). Hence the result holds. ■

Theorem 4.2. (Strong Duality). *Let $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ be a K-weakly efficient solution of* $(\mathbf{WP})$. *Fix $\lambda = \overline{\lambda}$ in* $(\mathbf{WD})$. *Assume that*

(i) *$\nabla_{yy} f_i$ is positive definite for $i = 1, \cdots, k$ and $\sum_{i=1}^{k} \overline{\lambda}_i \overline{p}_i^T [\nabla_y f_i - \overline{w}_i] \geqq 0$;or*

$\nabla_{yy} f_i$ is negative definite for $i = 1, \cdots, k$ and $\sum_{i=1}^{k} \overline{\lambda}_i \overline{p}_i^T [\nabla_y f_i - \overline{w}_i] \leqq 0$,

(ii) *the set $\{\nabla_y f_i - \overline{w}_i,\ i = 1, \cdots, k\}$ is linearly independent, where $f_i = f_i(\overline{x},\overline{y})$ for $i = 1, \cdots, k$.*

Then there exists $\overline{z}_i \in B_i (i = 1, \cdots, k)$ such that $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r} = 0)$ is a feasible solution of $(\mathbf{WD})$ *and objective values of* $(\mathbf{WP})$ *and* $(\mathbf{WD})$ *are equal. Furthermore, under the assumptions of Theorem 4.1, $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r} = 0)$ is a K-weakly efficient solution of* $(\mathbf{WD})$.

Proof. Since $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ is a K-weakly efficient solution of $(\mathbf{WP})$, by Lemma 3.1, there exist $\alpha \in K^*, \beta \in C_2, \delta \in C_1^*$ and $\rho \in K$ such that

$$\begin{aligned}(32)\qquad &\sum_{i=1}^{k} \alpha_i (\nabla_x f_i + z_i) + (\beta - (\alpha^T e)\overline{y})^T \sum_{i=1}^{k} \overline{\lambda}_i \nabla_{yx} f_i \\ &+ \sum_{i=1}^{k} (\beta - (\alpha^T e)\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p}_i)^T \overline{\lambda}_i \nabla_x (\nabla_{yy} f_i \overline{p}_i) - \delta = 0,\end{aligned}$$

$$\begin{aligned}(33)\qquad &\sum_{i=1}^{k} (\alpha_i - (\alpha^T e)\overline{\lambda}_i)(\nabla_y f_i - \overline{w}_i) \\ &+ \sum_{i=1}^{k} (\beta - (\alpha^T e)\overline{p}_i - (\alpha^T e)\overline{y})^T \overline{\lambda}_i \nabla_{yy} f_i \\ &+ \sum_{i=1}^{k} (\beta - (\alpha^T e)\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p}_i)^T \overline{\lambda}_i \nabla_y (\nabla_{yy} f_i \overline{p}_i) = 0,\end{aligned}$$

$$(34)\qquad (\beta - (\alpha^T e)\overline{y})^T (\nabla_y f_i - \overline{w}_i + \nabla_{yy} f_i \overline{p}_i) - \frac{1}{2}(\alpha^T e)\overline{p}_i^T \nabla_{yy} f_i \overline{p}_i - \rho_i = 0, \quad i = 1, \cdots, k,$$

$$(35)\qquad \alpha_i \overline{y} + (\beta - (\alpha^T e)\overline{y})\overline{\lambda}_i \in N_{D_i}(\overline{w}_i), \quad i = 1, \cdots, k,$$

$$(\beta-(\alpha^Te)\overline{y}-(\alpha^Te)\overline{p}_i)^T\overline{\lambda}_i\nabla_{yy}f_i=0,\quad i=1,\cdots,k, \tag{36}$$

$$\beta^T\sum_{i=1}^{k}\overline{\lambda}_i(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)=0, \tag{37}$$

$$\delta^T\overline{x}=0, \tag{38}$$

$$\rho^T\overline{\lambda}=0, \tag{39}$$

$$z_i\in B_i,\quad z_i^T\overline{x}=s(\overline{x}|B_i),\quad i=1,\cdots,k, \tag{40}$$

$$(\alpha,\beta,\delta,\rho)\neq 0. \tag{41}$$

As $\overline{\lambda}>0$, it follows from (39), that $\rho=0$. Hence from (34), we obtain

$$(\beta-(\alpha^Te)\overline{y})^T(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)-\frac{1}{2}(\alpha^Te)\overline{p}_i^T\nabla_{yy}f_i\overline{p}_i=0,\quad i=1,\cdots,k. \tag{42}$$

As $\nabla_{yy}f_i$ is positive or negative definite for $i=1,\cdots,k$, it follows from (36),

$$\beta=(\alpha^Te)(\overline{y}+\overline{p}_i),\quad i=1,\cdots,k. \tag{43}$$

If $\alpha_i=0$ for $i=1,\cdots,k$, then $\delta=0$ and $\beta=0$ from (32) and (43), respectively. This contradicts (41). So, $\alpha_i>0$ for $i=1,\cdots,k$. Using (43), (42) implies

$$(\alpha^Te)\overline{p}_i^T(\nabla_y f_i-\overline{w}_i+\nabla_{yy}f_i\overline{p}_i)-\frac{1}{2}(\alpha^Te)\overline{p}_i^T\nabla_{yy}f_i\overline{p}_i=0,\quad i=1,\cdots,k.$$

Since $\alpha^Te>0$, the above equality becomes

$$\overline{p}_i^T(\nabla_y f_i-\overline{w}_i+\frac{1}{2}\nabla_{yy}f_i\overline{p}_i)=0,\quad i=1,\cdots,k.$$

Using $\overline{\lambda}>0$, it follows that

$$\sum_{i=1}^{k}\overline{\lambda}_i\overline{p}_i^T(\nabla_y f_i-\overline{w}_i)+\sum_{i=1}^{k}\frac{1}{2}\overline{\lambda}_i(\overline{p}_i^T\nabla_{yy}f_i\overline{p}_i)=0. \tag{44}$$

We now prove that $\overline{p}_i=0$ for $i=1,\cdots,k$. Otherwise, the assumption (i) implies that

$$\sum_{i=1}^{k}\overline{\lambda}_i\overline{p}_i^T(\nabla_y f_i-\overline{w}_i)+\sum_{i=1}^{k}\frac{1}{2}\overline{\lambda}_i(\overline{p}_i^T\nabla_{yy}f_i\overline{p}_i)\neq 0,$$

which contradicts (44). Hence $\overline{p}_i = 0$ for $i = 1, \cdots, k$. Thus (43) implies

$$\beta = (\alpha^T e)\overline{y}. \tag{45}$$

Consequently, $\overline{y} \in C_2$. From (33), we obtain

$$\sum_{i=1}^{k} (\alpha_i - (\alpha^T e)\overline{\lambda}_i)(\nabla_y f_i - \overline{w}_i) = 0.$$

By the assumption (ii), we get

$$\alpha_i = (\alpha^T e)\overline{\lambda}_i, \quad i = 1, \cdots, k. \tag{46}$$

From (32),

$$\sum_{i=1}^{k} \alpha_i(\nabla_x f_i + z_i) = \delta \in C_1^*. \tag{47}$$

Using (46) and $\alpha^T e > 0$, it follows that

$$\sum_{i=1}^{k} \overline{\lambda}_i(\nabla_x f_i + z_i) \in C_1^*.$$

Taking $\overline{z}_i := z_i \in B_i$ for $i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for $(\mathbf{WD})$.

Moreover, from (35), we get $\overline{y} \in N_{D_i}(\overline{w}_i)$ for $i = 1, \cdots, k$, which implies $\overline{y}^T \overline{w}_i = s(\overline{y}|D_i)$ for $i = 1, \cdots, k$. Multiplying (47) by $\overline{x}$ and using (38), we get

$$\overline{x}^T \sum_{i=1}^{k} \overline{\lambda}_i(\nabla_x f_i + \overline{z}_i) = 0.$$

And from (37) and (45), we obtain

$$\overline{y}^T \sum_{i=1}^{k} \overline{\lambda}_i(\nabla_y f_i - \overline{w}_i + \nabla_{yy} f_i \overline{p}_i) = 0.$$

So, using (40)

$$\begin{aligned} & f_i + s(\overline{x}|B_i) - \overline{y}^T \overline{w}_i - \sum_{i=1}^{k} \overline{\lambda}_i[\overline{y}^T(\nabla_y f_i - \overline{w}_i + \nabla_{yy} f_i \overline{p}_i) + \frac{1}{2}\overline{p}_i^T \nabla_{yy} f_i \overline{p}_i] \\ & = f_i - s(\overline{y}|D_i) + \overline{x}^T \overline{z}_i - \sum_{i=1}^{k} \overline{\lambda}_i[\overline{x}^T(\nabla_x f_i + \overline{z}_i + \nabla_{xx} f_i \overline{r}_i) + \frac{1}{2}\overline{r}_i^T \nabla_{xx} f_i \overline{r}_i], \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \cdots, k. \end{aligned}$$

Thus objective values of $(\mathbf{WP})$ and $(\mathbf{WD})$ are equal.

We will now show that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0)$ is a K-weakly efficient solution of $(\mathbf{WD})$, otherwise, there exists a feasible solution $(u, v, \bar{\lambda}, z, r = 0)$ of $(\mathbf{WD})$ such that

$$D(u, v, \bar{\lambda}, z, r = 0) - D(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0) \in intK.$$

Since objective values of $(\mathbf{WP})$ and $(\mathbf{WD})$ are equal, it follows that

$$D(u, v, \bar{\lambda}, z, r = 0) - P(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0) \in intK,$$

which contradicts weak duality. Hence the results hold. ■

Remark 4.1. ([19]). If we replace (i) and (ii) of Theorem 4.2 by

(i) the matrix $\nabla_{yy}(\bar{\lambda}^T f)$ is non-singular,

(ii) the vectors $\nabla_y f_1 - \bar{w}_1, \cdots, \nabla_y f_k - \bar{w}_k$ are linearly independent,

(iii) the vector $\bar{p}_i^T \nabla_y(\nabla_{yy}(\bar{\lambda}^T f)\bar{p}_i) = 0$ implies that $\bar{p}_i = 0 (i = 1, 2, \cdots, k)$, and then the same results also hold.

Theorem 4.3. (Converse Duality). *Let* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}, \bar{r})$ *be a* K*-weakly efficient solution of* $(\mathbf{WD})$*. Fix* $\lambda = \bar{\lambda}$ *in* $(\mathbf{WP})$*. Assume that*

(i) $\nabla_{xx} f_i$ *is positive definite for* $i = 1, \cdots, k$ *and* $\sum_{i=1}^{k} \bar{\lambda}_i \bar{r}_i^T [\nabla_x f_i + \bar{z}_i] \geqq 0$; *or*

$$\nabla_{xx} f_i \text{ is negative definite for } i = 1, \cdots, k \text{ and } \sum_{i=1}^{k} \bar{\lambda}_i \bar{r}_i^T [\nabla_x f_i + \bar{z}_i] \leqq 0,$$

(ii) *the set* $\{\nabla_x f_i + \bar{z}_i, \ \ i = 1, \cdots, k\}$ *is linearly independent, where* $f_i = f_i(\bar{u}, \bar{v})$ *for* $i = 1, \cdots, k$.

Then there exists $\bar{w}_i \in D_i (i = 1, \cdots, k)$ *such that* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ *is a feasible solution of* $(\mathbf{WP})$ *and objective values of* $(\mathbf{WP})$ *and* $(\mathbf{WD})$ *are equal. Furthermore, under the assumptions of Theorem 4.1,* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ *is a* K*-weakly efficient solution of* $(\mathbf{WP})$.

Proof. It follows on the lines of Theorem 4.2. ■

Remark 4.2. ([19]). If we replace (i) and (ii) of Theorem 4.3 by

(i) the matrix $\nabla_{xx}(\bar{\lambda}^T f)$ is non-singular,

(ii) the vectors $\nabla_x f_1 + \bar{z}_1, \cdots, \nabla_x f_k + \bar{z}_k$ are linearly independent,

(iii) the vector $\bar{r}_i^T \nabla_x(\nabla_{xx}(\bar{\lambda}^T f)\bar{r}_i) = 0$ implies that $\bar{r} = 0$, and then the same results also hold.

We now prove the following self duality theorem for the primal $(\mathbf{WP})$ and the dual $(\mathbf{WD})$ on the lines of Mond and Weir [14]. We describe $(\mathbf{WP})$ and $(\mathbf{WD})$ as dual programs, if the conclusions of Theorem 4.2 hold.

Theorem 4.4. (Self Duality). *Assume that $m = n$, $C_1 = C_2$ and $B = D$. If f is skew-symmetric, then the program $(\mathbf{WP})$ is self dual. Furthermore, if $(\mathbf{WP})$ and $(\mathbf{WD})$ are dual programs with K-weakly efficient solutions as $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r})$, respecively, then $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ and $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{w}, \bar{r})$ are K-weakly efficient solutions of $(\mathbf{WP})$ and $(\mathbf{WD})$, respectively. Also common objective value of the objective functions is 0.*

Proof. Rewriting the dual as in [14], we have

$(\mathbf{WD}')$ Minimize

$$\Big(f_1(v,u) + s(v|D_1) - u^T z_1 - \sum_{i=1}^k \lambda_i [u^T(\nabla_x f_i(v,u) - z_i + \nabla_{xx} f_i(v,u) r_i) + \frac{1}{2} r_i^T \nabla_{xx} f_i(v,u) r_i], \cdots, \quad f_k(v,u) + s(v|D_k) - u^T z_k - \sum_{i=1}^k \lambda_i [u^T(\nabla_x f_i(v,u) - z_i + \nabla_{xx} f_i(v,u) r_i) + \frac{1}{2} r_i^T \nabla_{xx} f_i(v,u) r_i]\Big)$$

$$\text{subject to} \quad -\sum_{i=1}^k \lambda_i [\nabla_x f_i(v,u) - z_i + \nabla_{xx} f_i(v,u) r_i] \in C_1^*,$$

$$v \in C_2, \quad z_i \in B_i, \quad \lambda \in intK^*, \quad \lambda^T e = 1,$$

which is just $(\mathbf{WP})$. Thus, if $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r})$ is K-weakly efficient solution of $(\mathbf{WD})$, then $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ is K-weakly efficient solution of $(\mathbf{WP})$, and hence by symmetric duality, also $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{w}, \bar{r})$ is K-weakly efficient solution of $(\mathbf{WD})$. Therefore,

$$\begin{aligned}
&P(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0) \\
&= (f_1(\bar{x}, \bar{y}) + s(\bar{x}|B_1) - \bar{y}^T \bar{w}_1, \cdots, f_k(\bar{x}, \bar{y}) + s(\bar{x}|B_k) - \bar{y}^T \bar{w}_k) \\
&= (f_1(\bar{y}, \bar{x}) + s(\bar{y}|D_1) - \bar{x}^T \bar{z}_1, \cdots, f_k(\bar{y}, \bar{x}) + s(\bar{y}|D_k) - \bar{x}^T \bar{z}_k) \\
&= (-f_1(\bar{x}, \bar{y}) - s(\bar{x}|B_1) + \bar{y}^T \bar{w}_1, \cdots, -f_k(\bar{x}, \bar{y}) - s(\bar{x}|B_k) + \bar{y}^T \bar{w}_k) \\
&= D(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{r} = 0).
\end{aligned}$$

This implies

$$P(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) = 0 = D(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r}). \qquad \blacksquare$$

5. Special Cases

We give some special cases of our symmetric duality.

(1) If $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then $(\mathbf{MP})$ and $(\mathbf{MD})$ become the pair of Mond-Weir symmetric dual programs considered in X.M. Yang et al.[20] for the same B and D.

(2) If $B_i = \{0\}$ and $D_i = \{0\}$, $i = 1, \cdots, k$, then $(\mathbf{MP})$ and $(\mathbf{MD})$ reduced to the second order symmetric dual programs in S.K. Mishra and K.K. Lai [12].

(3) If $p = r = 0$, then we get the first order symmetric dual programs which studied by M.H. Kim and D.S. Kim [7].

(4) If $B_i = \{0\}$, $D_i = \{0\}$ and $p_i = r_i = 0$, $i = 1, \cdots, k$, then $(\mathbf{MP})$ and $(\mathbf{MD})$ become the pair of symmetric dual programs considered in Seema Khurana [6].

(5) If $B_i = \{0\}$, $D_i = \{0\}$ and $p_i = r_i = 0$, $i = 1, \cdots, k$, then $(\mathbf{WP})$ and $(\mathbf{WD})$ reduced to the first order multiobjective symmetric dual programs in D.S. Kim et al.[9].

(6) If $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then $(\mathbf{MP})$, $(\mathbf{MD})$, $(\mathbf{WP})$ and $(\mathbf{WD})$ become the pair of Mond-Weir and Wolfe type symmetric dual programs considered in D.S. Kim et al.[8] for the same B, D, p and r.

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