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THE HAMILTON-WATERLOO PROBLEM FOR TWO EVEN CYCLES FACTORS

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. This paper investigates the problem of factoring $K_{2n} - I_n$ into 2-factors of two kinds or three kinds: (1) C_t-factors and C_{2t}-factors, (2) C_4-factors and C_{2t}-factors, (3) C_4-factors, C_8-factors and C_{16}-factors.

1. Introduction

The well-known Oberwolfach problem, formulated by Ringel in 1967, asks for a 2-factorization of the complete graph K_{2n+1} into 2-factors each of which isomorphic to a given 2-factor D. If D consists of cycles of lengths $m_1, m_2, \ldots, m_t$ with $m_1 + m_2 + \ldots + m_t = 2n + 1$, then the Oberwolfach problem is denoted by OP$(2n + 1; m_1, m_2, \ldots, m_t)$. It is known that the cases OP$(9; 4, 5)$ and OP$(11; 3, 3, 5)$ do not exist. Therefore, it has been conjectured that a solution to OP$(2n + 1; m_1, m_2, \ldots, m_t)$ exists except the above two counterexamples. So far, the conjecture has been verified for the case when $m_1 = m_2 = \ldots = m_t$, i.e., all components of the isomorphic 2-factor D are cycles of the same odd length $[3, 4]$. However, the Oberwolfach problem remains unsolved in general. It is common to extend the Oberwolfach problem to the case by considering 2-factorization of the complete graph K_{2n} with a 1-factor I_n removed, denoted by $K_{2n} - I_n$. It has also been verified that a solution to OP$(2n; m_1, m_2, \ldots, m_t)$ exists when $m_1 = m_2 = \ldots = m_t$ except that OP$(6; 3, 3)$ and OP$(12; 3, 3, 3, 3)$ have no solution $[2, 3, 6]$.

The Hamilton-Waterloo problem (HWP) is a generalization of the Oberwolfach problem which asks for a 2-factorization of K_{2n+1} in which r of the 2-factors are isomorphic to a given 2-factor D_1 and the remaining s of the 2-factors are

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isomorphic to another given 2-factor D_2, where $r+s=n$. Again, one may ask the similar problem by considering 2-factorizations of $K_{2n}-I_n$.

For convenience, we introduce the following notations. Let I_n be a 1-factor of K_{2n}. Suppose H is a subgraph of G. Let mH be the edge-disjoint union of m copies of H. An H-factor of a graph G is a spanning subgraph of G in which each component is isomorphic to H. An $\{H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t}\}$-factorization of a graph G is a factorization which consists precisely of m_i H_i-factors. If there is such a factorization of G, then we say that $(G; H_1^{m_1}, H_2^{m_2}, \ldots, H_t^{m_t})$ exists. Let HWP$(v; m, n)$ be the set of pairs (r,s) such that $(G; C_m^r, C_n^s)$ exists for $G=K_v$ if v is odd and $G=K_v-I_{\frac{v}{2}}$ if v is even, where C_t denotes a cycle of length t. The cases $(m,n)=(3,v)$, $(m,n)=(3,4)$ and $(m,n)\in\{(4,6),(4,8),(4,16),(8,16),(3,5),(3,15),(5,15)\}$ are considered in $[7,5,1]$, respectively. In this paper, we completely determine the sets HWP$(2n; t, 2t)$ for even integers $t\geq 4$ and HWP$(2n; 4, 2t)$ for $t\geq 3$. Moreover, we also show that $(K_{2n}-I_n; C_4^r, C_8^s, C_{16}^t)$ exists for all pairs (r,s,t) with $r+s+t=n-1$.

2. Basic Constructions

In this section we present the idea and some basic constructions in order to prove our main results. Since the following lemmas are easy to see, we omit the proofs.

Lemma 1. *Suppose G_1 and G_2 are two vertex-disjoint graphs. If $(G_1; C_m^r, C_n^s)$ and $(G_2; C_m^r, C_n^s)$ both exist, then $(G_1\cup G_2; C_m^r, C_n^s)$ exists.*

Lemma 2. *Suppose G_1 and G_2 are two edge-disjoint graphs with the same vertex set. If $(G_1; C_m^{r_1}, C_n^{s_1})$ and $(G_2; C_m^{r_2}, C_n^{s_2})$ both exist, then $(G_1\cup G_2; C_m^{r_1+r_2}, C_n^{s_1+s_2})$ exists.*

The trivial necessary condition for K_{2n} having a C_{2t}-factor is $2n=2tk$ or $n=tk$. On the other hand, if $n=tk$, then $K_{2n}-I_n=k(K_{2t}-I_t)\bigcup(2k-2)(kK_{t,t})=(kK_{2t}-I_n)\bigcup(2k-2)(kK_{t,t})$, we shall prove that $(K_{2t}-I_t; C_m^{r_1}, C_{2t}^{s_1})$ exists for $r_1+s_1=t-1$ and $m=4$ or t, and $(K_{t,t}; C_m^{\frac{t}{2}})$ exists for $m=t$ or $2t$. Then, by Lemma 1, $(k(K_{2t}-I_t); C_m^{r_1}, C_{2t}^{s_1})$ and $(kK_{t,t}; C_m^{\frac{t}{2}})$ both exist. Let (r,s) be a pair of nonnegative integers with $r+s=n-1=tk-1$. It is not difficult to see that there are pairs of nonnegative integers (r_1,s_1) with $r_1+s_1=t-1$ and (r_2,s_2) with $r_2+s_2=2k-2$ such that $(r,s)=(r_1,s_1)+\frac{t}{2}(r_2,s_2)$. Now, factor $k(K_{2t}-I_t)=kK_{2t}-I_n$ into r_1 C_t-factors and s_1 C_{2t}-factors and factor r_2 $kK_{t,t}$'s into $\frac{t}{2}r_2$ C_t-factors and the remaining s_2 $kK_{t,t}$'s into $\frac{t}{2}s_2$ C_{2t}-factors. By Lemma 2, $(K_{2n}-I_n; C_t^r, C_{2t}^s)$ exists, i.e., $(r,s)\in$ HWP$(2n;t,2t)\subseteq\{(r,s): r+s=n-1\}$.

Therefore, we have $\text{HWP}(2n; t, 2t) = \{(r, s) : r+s = n-1\}$ for even integers t. By the same argument, we also obtain that $\text{HWP}(2n; 4, 2t) = \{(r, s) : r + s = n - 1\}$ for integers $t \geq 3$.

The following lemma is a well-known result.

Lemma 3. *The complete graph K_{2t} is 1-factorable and it can be decomposed into $t - 1$ Hamilton cycles and one 1-factor. Moreover, by ordering the vertices of K_{2t}, one of the $t - 1$ Hamilton cycles is of the form $v_0v_1 \cdots v_{2t-1}v_0$ and the 1-factor is $\{v_0v_t, v_iv_{2t-i} : 1 \leq i \leq t-1\}$.*

Before proving the next lemma, we define a new graph. Suppose G is a graph. The *duplicate graph* of G, denoted by DG, obtained from G by replacing each vertex v_i of G by two new vertices x_i and y_i and replacing each edge v_iv_j of G by four edges x_ix_j, x_iy_j, y_ix_j and y_iy_j, i.e., each edge v_iv_j of G corresponds to a $K_{2,2} = C_4$ in DG. In what follows, let the vertex sets and edge sets of duplicate graphs are the same as above if no confusion occurs.

Lemma 4. *Let $C = v_0v_1v_2 \cdots v_{t-1}v_0$ be an even cycle. Then $(DC; C_m^2)$ exists for $m = 4$, t and $2t$. Moreover, let $I_t = \{x_iy_i : 0 \leq i \leq t-1\}$ and $G = DC \bigcup I_t$. Then there is a 1-factor M of G such that $(G-M; C_m^1, C_{2t}^1)$ exists for $m=4$ and t.*

Proof. Since t is even, $F_1 = \{v_iv_{i+1} : i \text{ is odd}\}$ and $F_2 = \{v_iv_{i+1} : i \text{ is even}\}$ are two 1-factors of C. Hence, DF_1 and DF_2 are two C_4-factors of DC. Next, by directed construction, $\{x_0x_1x_2 \cdots x_{t-1}x_0, y_0y_1y_2 \cdots y_{t-1}y_0\}$ and $\{x_0y_1x_2y_3 \cdots x_{t-2}y_{t-1}x_0, y_0x_1y_2x_3 \cdots y_{t-2}x_{t-1}y_0\}$ are two C_t-factors of DC and $\{x_0x_1 \cdots x_{t-1}y_0y_1 \cdots y_{t-1}x_0\}$ and $\{x_0y_1x_2y_3 \cdots x_{t-2}y_{t-1}y_0x_1y_2x_3 \cdots y_{t-2}x_{t-1} x_0\}$ are two C_{2t}-factors of DC. Let

$$Q_1 = \{x_0x_1 \cdots x_{\frac{t}{2}-1}y_{\frac{t}{2}-1}y_{\frac{t}{2}-2} \cdots y_1y_0x_0, x_{\frac{t}{2}}x_{\frac{t}{2}+1} \cdots x_{t-1}y_{t-1}y_{t-2} \cdots y_{\frac{t}{2}}x_{\frac{t}{2}}\},$$

$$Q_2 = \{x_0y_1x_2y_3 \cdots x_{t-2}y_{t-1}y_0x_1y_2x_3 \cdots y_{t-2}x_{t-1}x_0\} \text{ and}$$

$$M_1 = G - (Q_1 \bigcup Q_2).$$

It is routine to verify that Q_1 is a C_t-factor, Q_2 is a C_{2t}-factor and M_1 is a 1-factor of G, respectively.

Let $F_3 = \{x_ix_{i+1}y_{i+1}y_ix_i : i \text{ is even}\}$, Q_2 be the same as above and $M_2 = G - (F_3 \bigcup Q_2)$. It is easy to see that F_3 is a C_4-factor, Q_2 is a C_{2t}-factor and M_2 is a 1-factor of G, respectively. ■

Lemma 5. *Suppose G is a graph consisting of an even cycle $C = v_0v_1v_2 \cdots v_{t-1}v_0$ and a 1-factor $M_1 = \{v_0v_{\frac{t}{2}}, v_iv_{t-i} : 1 \leq i \leq \frac{t}{2} - 1\}$. Let $H = DG \bigcup I_t$, where $I_t = \{x_iy_i : 0 \leq i \leq t-1\}$. Then there is a 1-factor M of H such that $(H - M; C_t^r, C_{2t}^s)$ exists for all pairs (r, s) with $r + s = 3$.*

Proof. Let $F_1 = \{x_0x_1x_{t-1}x_{t-2}x_2 \cdots x_{\frac{t}{2}}x_0, y_0y_1y_{t-1}y_{t-2}y_2 \cdots y_{\frac{t}{2}}y_0\}$,

$F_2 = \{x_0y_{t-1}x_1y_2x_{t-2} \cdots y_{\frac{t}{2}}x_0, y_0x_{t-1}y_1x_2y_{t-2} \cdots x_{\frac{t}{2}}y_0\}$,

$Q_1 = \{x_0x_1x_{t-1}x_{t-2}x_2 \cdots x_{\frac{t}{2}}y_0y_1y_{t-1}y_{t-2}y_2 \cdots y_{\frac{t}{2}}x_0\}$ and

$Q_2 = \{x_0y_{t-1}x_1y_2x_{t-2} \cdots y_{\frac{t}{2}}y_0x_{t-1}y_1x_2y_{t-2} \cdots x_{\frac{t}{2}}x_0\}$.

Then F_1 and F_2 are two C_t-factors of H, Q_1 and Q_2 are two C_{2t}-factors of H and $F_1 \bigcup F_2 = Q_1 \bigcup Q_2$. Let $R_1 = DG - (F_1 \bigcup F_2)$. Then $R_1 = \{x_0y_1y_2x_3x_4 \cdots x_{t-1}x_0, y_0x_1x_2y_3y_4 \cdots y_{t-1}y_0\}$ which is a C_t-factor of H if $t \equiv 0$ (mod 4) and $R_1 = \{x_0y_1y_2x_3x_4 \cdots y_{t-1}y_0x_1x_2y_3y_4 \cdots x_{t-1}x_0\}$ which is a C_{2t}-factor of H if $t \equiv 2$ (mod 4).

For $t \equiv 0$ (mod 4), let $R_2 = (R_1 - \{x_0y_1, y_0x_1\}) \bigcup \{x_0y_0, x_1y_1\}$ which is a C_{2t}-factor of H and $M_2 = (I_t - \{x_0y_0, x_1y_1\}) \bigcup \{x_0y_1, y_0x_1\}$ which is a 1-factor of H. Then $\{F_1, F_2, R_1, I_t\}$, $\{F_1, F_2, R_2, M_2\}$, $\{R_1, Q_1, Q_2, I_t\}$ and $\{R_2, Q_1, Q_2, M_2\}$ are the desired four factorizations.

For $t \equiv 2$ (mod 4), let $R_3 = (R_1 - \{x_0x_{t-1}, y_0y_{t-1}, x_{\frac{t}{2}-1}y_{\frac{t}{2}}, x_{\frac{t}{2}}y_{\frac{t}{2}-1}\}) \bigcup \{x_iy_i : i = 0, \frac{t}{2}-1, \frac{t}{2}, t-1\}$ which is a C_t-factor of H and $M_3 = (I_t - \{x_iy_i : i = 0, \frac{t}{2}-1, \frac{t}{2}, t-1\}) \bigcup \{x_0x_{t-1}, y_0y_{t-1}, x_{\frac{t}{2}-1}y_{\frac{t}{2}}, x_{\frac{t}{2}}y_{\frac{t}{2}-1}\}$ which is a 1-factor of H. Then $\{F_1, F_2, R_3, M_3\}$, $\{F_1, F_2, R_1, I_t\}$, $\{R_3, Q_1, Q_2, M_3\}$ and $\{R_1, Q_1, Q_2, I_t\}$ are the desired four factorizations. ■

Lemma 6. *Suppose $t \geq 4$ is an even integer. Then HWP$(2t; t, 2t) = \{(r, s) : r + s = t - 1\}$.*

Proof. By definition, HWP$(2t; t, 2t) \subseteq \{(r, s) : r + s = t - 1\}$. Conversely, let (r, s) be a pair with $r + s = t - 1$. By Lemma 3, we have $(0, t-1) \in$ HWP$(2t; t, 2t)$. Now, suppose $r > 0$. It is easy to see that $K_{2t} - I_t = DK_t$. Since t is even, by Lemma 3, K_t can be decomposed into $\frac{t}{2} - 1$ Hamilton cycles, denoted by HC for short, and one 1-factor F. Hence, $DK_t = (\frac{t}{2} - 2)DHC \bigcup DHC^* \bigcup DF^*$ with the particular DHC^* and DF^* stated in Lemma 3. Let $G = DHC^* \bigcup DF^* \bigcup I_t$. If r is even, by Lemma 5, then $(G - M_1; C_t^2, C_{2t}^1)$ exists, where M_1 is some 1-factor of G which is also a 1-factor of K_{2t}. By Lemmas 4 and 2, $((\frac{r}{2} - 1)DHC; C_t^{r-2})$ and $((\frac{t}{2} - 1 - \frac{r}{2})DHC; C_{2t}^{t-2-r})$ both exist. Hence, $((\frac{t}{2} - 2)DHC; C_t^{r-2}, C_{2t}^{t-2-r})$ exists and then $(K_{2t} - M_1; C_t^r, C_{2t}^{t-1-r})$ exists. Thus, $(r, t-1-r) \in$ HWP$(2t; t, 2t)$. If r is odd, by Lemma 5, then $(G - M_2; C_t^1, C_{2t}^2)$ exists, where M_2 is some 1-factor both of G and K_{2t}. By Lemmas 4 and 2, $(\frac{r-1}{2}DHC; C_t^{r-1})$ and $((\frac{t}{2} - 2 - \frac{r-1}{2})DHC; C_{2t}^{t-3-r})$ both exist. Hence, $((\frac{t}{2} - 2)DHC; C_t^{r-1}, C_{2t}^{t-3-r})$ exists and then $(K_{2t} - M_2; C_t^r, C_{2t}^{t-1-r})$ exists. Thus, $(r, t-1-r) \in \text{HWP}(2t; t, 2t)$. Therefore, $\text{HWP}(2t; t, 2t) = \{(r, s) : r + s = t - 1\}$. ■

The following result can be found in [8].

Lemma 7. ([8]). *There is a 2-factorization of $K_{n,n}$ in which each 2-factor is the vertex disjoint union of m cycles of lengths $t_1, t_2, \ldots, t_m$ if and only if n is even, $t_i \geq 4$ is even for $1 \leq i \leq m$ and $t_1 + t_2 + \ldots + t_m = 2n$, except there is no C_6-factorization of $K_{6,6}$. In particular, $(K_{t,t}; C_m^{\frac{t}{2}})$ exists for even integers t and $m = 4, t$ or $2t$, except $m = t = 6$.*

Since $(K_{6,6}; C_6^3)$ does not exist, we can not obtain $\text{HWP}(2n; 6, 12)$ directly by applying Lemma 7. However, by a minor modification, we also can completely determine the set $\text{HWP}(2n; 6, 12)$.

Lemma 8. *Suppose $n \equiv 0$ (mod 12). Then HWP$(2n; 6, 12) = \{(r, s) : r + s = n - 1\}$.*

Proof. By Lemma 6, $(K_{12} - I_6; C_6^r, C_{12}^s)$ exists for all pairs (r, s) with $r + s = 5$. By Lemma 7, $(K_{12,12}; C_m^6)$ exists for $m = 6$ or 12. Let (a, b) be a pair with $a + b = 11$. Then $(a, b) = (a_1, b_1) + 6(a_2, b_2)$, where $a_1 + b_1 = 5$ and $a_2 + b_2 = 1$. Since $K_{24} - I_{12} = 2(K_{12} - I_6) \bigcup K_{12,12}$, $(K_{12} - I_6; C_6^{a_1}, C_{12}^{b_1})$ and $(K_{12,12}; C_6^{6a_2}, C_{12}^{6b_2})$ both exist, by Lemmas 1 and 2, $(K_{24} - I_{12}; C_6^a, C_{12}^b)$ exists. Now, if (r, s) is a pair with $r + s = n - 1 = 12k - 1$, then $(r, s) = (r_1, s_1) + 6(r_2, s_2)$, where $r_1 + s_1 = 11$ and $r_2 + s_2 = 2k - 2$. Since $K_{2n} - I_n = K_{24k} - I_{12k} = k(K_{24} - I_{12}) \bigcup (2k-2)(kK_{12,12})$, $(K_{24} - I_{12}; C_6^{r_1}, C_{12}^{s_1})$ exists and, $(r_2(kK_{12,12}); C_6^{6r_2})$ and $(s_2(kK_{12,12}); C_{12}^{6s_2})$ both exist by Lemmas 1 and 2, we have $(K_{2n} - I_n; C_6^r, C_{12}^s)$ exists. Therefore, $(r, s) \in \text{HWP}(2n; 6, 12)$ and then $\text{HWP}(2n; 6, 12) = \{(r, s) : r + s = n - 1\}$. ■

For the case that $n \equiv 6$ (mod 12), we need the following. Let $K_{u(g)}$ be the complete u-partite graph with g vertices in each partite set.

Lemma 9. ([4])/ *The graph $K_{u(g)}$ is C_3-factorable if and only if $(u-1)g$ is even and $ug \equiv 0$ (mod 3).*

Lemma 10. *Let $n \equiv 6$ (mod 12). Then HWP$(2n; 6, 12) = \{(r, s) : r + s = n - 1\}$.*

Proof. Let $n = 6k$, where k is odd. Then $K_{2n} - I_n = k(K_{12} - I_6) \bigcup K_{k(12)}$. By Lemma 9, $K_{k(12)}$ is $K_{4,4,4}$-factorable, i.e., $K_{k(12)} = \dfrac{3(k-1)}{2}(kK_{4,4,4})$, where

$kK_{4,4,4}$ is a $K_{4,4,4}$-factor. It is not difficult to see that $K_{4,4,4} = DK_{2,2,2}$ which can be decomposed into two DC_6. Since (DC_6, C_m^2) exists for $m = 6$ or 12, by Lemmas 4 and 2, $(K_{4,4,4}; C_m^4)$ exists. By Lemma 1, $(kK_{4,4,4}; C_m^4)$ exists. If (r, s) is a pair with $r + s = n - 1 = 6k - 1$, then $(r, s) = (r_1, s_1) + 4(r_2, s_2)$, where $r_1 + s_1 = 5$ and $r_2 + s_2 = \dfrac{3(k-1)}{2}$. Since $(K_{12} - I_6; C_6^{r_1}, C_{12}^{s_1})$ exists by Lemma 6 and, $(r_2(kK_{4,4,4}); C_6^{4r_2})$ and $(s_2(kK_{4,4,4}); C_{12}^{4s_2})$ both exist by Lemma 2, we have $(K_{2n} - I_n; C_6^r, C_{12}^s)$ exists. Therefore, $(r, s) \in$ HWP$(2n; 6, 12)$ and then HWP$(2n; 6, 12) = \{(r, s) : r + s = n - 1\}$. ■

Combining Lemmas 8 and 10, we have

Corollary 11. *Suppose* $n \equiv 0$ *(mod 6). Then HWP*$(2n; 6, 12) = \{(r, s) : r + s = n - 1\}$.

3. Main Results

Now, we are ready to prove our main results.

Theorem 12. *Suppose* $t \geq 4$ *is even and* $n \equiv 0$*(mod t). Then HWP*$(2n; t, 2t) = \{(r, s) : r + s = n - 1\}$.

Proof. By Corollary 11, the assertion holds for $t = 6$. Now, suppose $t \neq 6$. Since $n \equiv 0$ (mod t), we have $n = tk$ and $K_{2n} = kK_{2t} \bigcup (2k - 2)(kK_{t,t})$. Let (r, s) be a pair of nonnegative integers with $r + s = n - 1 = tk - 1$. Then $(r, s) = (r_1, s_1) + \frac{t}{2}(r_2, s_2)$ for some pairs (r_1, s_1) with $r_1 + s_1 = t - 1$ and (r_2, s_2) with $r_2 + s_2 = 2k - 2$. By Lemma 6, $(K_{2t} - I_t; C_t^{r_1}, C_{2t}^{s_1})$ exists. Hence, by Lemma 1, $(k(K_{2t} - I_t); C_t^{r_1}, C_{2t}^{s_1}) = (kK_{2t} - I_n; C_t^{r_1}, C_{2t}^{s_1})$ exists. By Lemma 7, $(K_{t,t}; C_m^{\frac{t}{2}})$ exists for $m = t$ or $2t$. Hence, by Lemma 1, $(kK_{t,t}; C_m^{\frac{t}{2}})$ exists. By Lemma 2, $(r_2(kK_{t,t}); C_t^{\frac{t}{2}r_2})$ and $(s_2(kK_{t,t}); C_{2t}^{\frac{t}{2}s_2})$ both exist and then $(K_{2n} - I_n; C_t^r, C_{2t}^s)$ exists, i.e., $(r, s) \in$ HWP$(2n; t, 2t)$. Therefore, HWP$(2n; t, 2t) = \{(r, s) : r + s = n - 1\}$. ■

In what follows, we study HWP$(2n; 4, 2t)$ for $t \geq 3$. The necessary condition for the existence of $(K_{2n} - I_n; C_4^r, C_{2t}^s)$ with $r + s = n - 1$ is that $2n$ is divisible by 4 and $2t$. Hence, we may assume that $n = tk$ is even. We also need the following result.

Lemma 13. ([6]). *A* C_k*-factorization of* $K_{2n} - I_n$ *exists if and only if* k *divides* $2n$ *except that* $K_6 - I_3$ *and* $K_{12} - I_6$ *do not admit a* C_3*-factorization.*

Theorem 14. *For an integer* $t \geq 3$, *HWP*$(2n; 4, 2t) = \{(r, s) : r+s = n-1\}$.

Proof. The assertion holds for $t = 3$ which is proved in [1]. Suppose $t \geq 4$ is even. Let (r, s) be a pair with $r + s = n - 1$. By Lemma 3, $(0, n - 1) \in \text{HWP}(2n; 4, 2t)$. Let $r > 0$. It is easy to see that $K_{2n} - I_n = DK_n$. Since n is even, by Lemma 13, $K_n - I_{\frac{n}{2}}$ is C_t-factorable. Let $K_n - I_{\frac{n}{2}} = \bigcup_{i=1}^{\frac{n}{2}-1} F_i$, where each F_i is a C_t-factor. It is clear that $DI_{\frac{n}{2}}$ corresponds to a C_4-factor in K_{2n}. Let C be a t-cycle of F_i. Since t is even, by Lemma 4, (DC, C_4^2) and (DC, C_{2t}^2) both exist. Hence, (DF_i, C_4^2) and (DF_i, C_{2t}^2) exist. If r is odd, then $(D(\bigcup_{i=1}^{\frac{r-1}{2}} F_i \bigcup I_{\frac{n}{2}}); C_4^r)$ and $(D(\bigcup_{i=\frac{r+1}{2}}^{\frac{n}{2}-1} F_i); C_{2t}^s)$ both exist. By Lemma 2, $(K_{2n} - I_n; C_4^r, C_{2t}^s)$ exists. Hence, $(r, s) \in \text{HWP}(2n; 4, 2t)$. If r is even, by Lemma 4, $((DF_1 \bigcup I_n) - M_1; C_4^1, C_{2t}^1)$ exists for some 1-factor M_1 of $DF_1 \bigcup I_n$ which is also a 1-factor of K_{2n}. Hence, $(((DF_1 \bigcup I_n) - M_1) \bigcup DI_{\frac{n}{2}}; C_4^2, C_{2t}^1)$ exists. Since $(D(\bigcup_{i=2}^{\frac{r}{2}-1} F_i); C_4^{r-2})$ and $(D(\bigcup_{i=\frac{r}{2}+1}^{\frac{n}{2}-1} F_i); C_{2t}^s)$ both exist, by Lemma 2, $(K_{2n} - M_1; C_4^r, C_{2t}^s)$ exists. Hence, $(r, s) \in \text{HWP}(2n; 4, 2t)$. Therefore, the assertion holds for t is even.

Now, suppose $t \geq 5$ is odd. Since $n = kt$ is even, n is divisible by $2t$. Again, by Lemma 13, $K_n - I_{\frac{n}{2}}$ is C_{2t}-factorable. By Lemma 4, Lemma 2 and a similar argument as above, $(K_{2n} - I_n; C_4^r, C_{2t}^s)$ exists. Therefore, the assertion holds for $t \geq 5$ being odd and then we complete the proof. ■

4. Concluding Remark

So far, we study the Hamilton-Waterloo problem for (1) C_t-factors and C_{2t}-factors if t is even and (2) C_4-factors and C_{2t}-factors if $t \geq 3$. However, by using the similar argument in Lemma 6, we are able to deal with the Hamilton-Waterloo problem for cycle size 4, 6 and 8. Here is the result.

Theorem 15. *Suppose* $n \equiv 0$ *(mod 8). Then* $(K_{2n} - I_n; C_4^r, C_8^s, C_{16}^t)$ *exists for all pairs* (r, s, t) *with* $r + s + t = n - 1$.

Proof. Suppose $n = 8k$. Then $K_{2n} - I_n = (kK_{16} - I_n) \bigcup (2k - 2)(kK_{8,8})$. By a similar argument as in Lemma 6, $(K_{16} - I_8; C_4^{r_1}, C_8^{s_1}, C_{16}^{t_1})$ exists for all pairs (r_1, s_1, t_1) with $r_1 + s_1 + t_1 = 7$. Hence, by Lemma 2, $(kK_{16} - I_n; C_4^{r_1}, C_8^{s_1}, C_{16}^{t_1})$ exists for $r_1 + s_1 + t_1 = 7$. By Lemma 7, $(K_{8,8}; C_m^4)$ exists for $m = 4, 8$ or 16. By

Lemma 1, $(kK_{8,8}; C_m^4)$ exists. If (r, s, t) is a pair with $r + s + t = n - 1 = 4 \cdot (2k - 2) + 7$, it is not difficult to see that $(r, s, t) = (r_1, s_1, t_1) + 4(r_2, s_2, t_2)$, where $r_1 + s_1 + t_1 = 7$ and $r_2 + s_2 + t_2 = 2k - 2$. Now, factor $kK_{16} - I_n$ into r_1 C_4-factors, s_1 C_8-factors and t_1 C_{16}-factors. By Lemma 2, we can factor $r_2(kK_{8,8})$ into $4r_2$ C_4-factors, $s_2(kK_{8,8})$ into $4s_2$ C_8-factors and $t_2(kK_{8,8})$ into $4t_2$ C_{16}-factors. Hence, by Lemma 2, $(K_{2n} - I_n; C_4^r, C_8^s, C_{16}^t)$ exists for $r + s + t = n - 1$. ■

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