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FROM RAINBOW TO THE LONELY RUNNER: A SURVEY ON COLORING PARAMETERS OF DISTANCE GRAPHS

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. Motivated by the plane coloring problem, Eggleton, Erdős and Skelton initiated the study of distance graphs. Let D be a set of positive integers. The distance graph generated by D, denoted by $G(\mathbb{Z}, D)$, has all integers $\mathbb{Z}$ as the vertex set, and two vertices x and y are adjacent whenever $|x-y| \in D$. The chromatic number, circular chromatic number and fractional chromatic number of distance graphs have been studied extensively in the past two decades; these coloring parameters are also closely related to some problems studied in number theory and geometry. We survey some research advances and open problems on coloring parameters of distance graphs.

1. Introduction

The long-standing plane coloring problem states: What is the least number of colors needed to paint all the points on the Euclidean plane so that any two points of unit distance apart receive distinct colors? It was more than four decades ago that Moser and Moser [36] proved that four colors is needed, and Hadwiger, Debrunner and Klee [22] showed that seven colors is enough. So far, these bounds remain the best known results for this problem. Figure 1 shows a *rainbow-coloring* (with seven colors) of $\Re^2$.

When restricted to all real numbers $\Re$, the analogy of the plane coloring problem has an immediate answer: One can easily color the real line by two colors such that any two reals with unit absolute difference apart receive different colors. However, when the avoided absolute difference has more than one single value, the problem becomes intricate and has generated various research topics that have been studied extensively in the past two decades.

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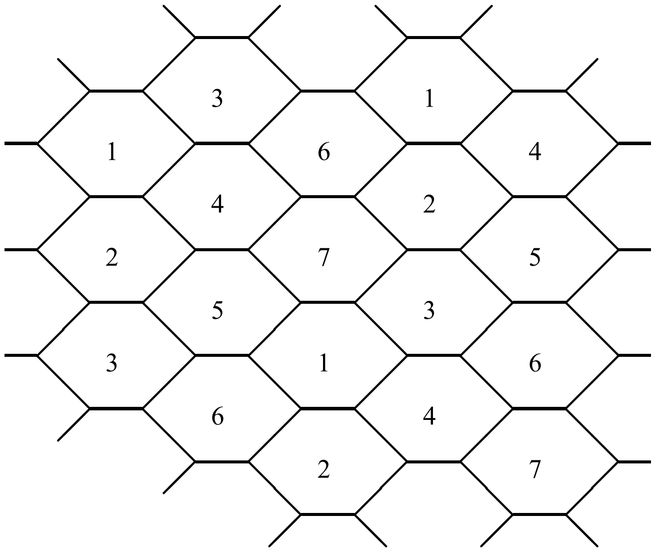

Fig. 1. A 7-coloring of $\Re^2$ such that any two points of unit distance apart receive different colors. Divide $\Re^2$ into cells of hexagons; the distance between any two diagonal points in each cell is slightly smaller than 1. Then, color the points in each cell by the same color as indicated.

In 1985, Eggleton, Erdős and Skilton [17] introduced the notion of coloring the real line. Let D be a subset of $\Re$. Denote $G(\Re, D)$ the graph with the vertex set $\Re$ and edges connecting x and y if $|x-y| \in D$. In [17], the chromatic number of $G(\Re, D)$ for various families of sets D was investigated, such as $D = [1, \delta]$ or $D = (0, \delta)$ for some real $\delta > 1$, with special attention given to finding the minimal subgraph G_0 of $G(\Re, D)$ such that $\chi(G_0) = \chi(G(\Re, D))$.

Also considered in [17] is the case when D is a set of positive integers, in which by isomorphism of components in $G(\Re, D)$, is equivalent to consider the subgraph induced by all the integers $\mathbb{Z}$. Denote such a subgraph by $G(\mathbb{Z}, D)$ and call it *integral distance graph* (or simply, *distance graph*). The set D is called the *distance set*. The chromatic number of distance graphs for different families of distance sets has been studied extensively (cf. [8, 9, 17-20, 25, 26, 32, 40-43, 46, 49]). Among the investigations, for instance, are the subsets of primes. Let $\mathbb{P}$ denote the set of all primes. It was proved in [17] that $\chi(G(\mathbb{Z}, \mathbb{P})) = 4$, and for $D \subseteq \mathbb{P}$ and $|D| = 3$ then $\chi(G(\mathbb{Z}, D)) = 4$ if and only if $D = \{2, 3, 5\}$. For 4-element prime sets, the ones with chromatic number 4 were completely characterized by Voigt and Walther [42]. For prime sets of more than four elements, the problem remains open.

Besides the chromatic number, other coloring parameters of distance graphs such as the fractional chromatic number and the circular chromatic number have also been studied in the past decade (cf. [7, 8, 29, 30, 33-35, 45]). These coloring parameters provide more information on the structure of distance graphs and are useful to determine the chromatic number of distance graphs. Moreover, these coloring parameters of distance graphs are found closely related to some problems studied in number theory and geometry.

The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is the minimum ratio m/n ($m, n \in \mathbb{Z}^+$) of an (m/n)-coloring, where an (m/n)-coloring is a function on $V(G)$ to n-element subsets of $[m] = \{1, 2, \cdots, m\}$ such that if $uv \in E(G)$ then $f(u) \cap f(v) = \emptyset$. There are several equivalent definitions of the fractional chromatic number; we refer the readers to the book by Scheinerman and Ullman [39].

Let r, x, y be reals with $0 \leqslant x, y < r$. The *circular difference modular* r for x and y, denoted as $|x-y|_r$, is defined by

$$|x-y|_r = \min \{|x-y|,\ r-|x-y|\}.$$

Let k, d be positive integers with $k \geqslant 2d$. A (k, d)-*coloring* of a graph G is a mapping, $c : V(G) \to \{0, 1, \cdots, k-1\}$, such that $|c(x)-c(y)|_k \geqslant d$ for any $xy \in E(G)$. The *circular chromatic number* $\chi_c(G)$ is the minimum ratio (k/d) among all (k, d)-colorings of G. For research advances on circular coloring, we refer the reader to the comprehensive survey articles by Zhu [47, 48].

Denote the independence number and clique size of G by $\alpha(G)$ and $\omega(G)$, respectively. The following are known (cf. [47]):

$$\max\left\{\omega(G), \tfrac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G), \quad \lceil \chi_c(G) \rceil = \chi(G). \tag{1.1}$$

The fractional chromatic number of distance graphs is closely related to the problem called "density of integral sequences with missing differences" studied in number theory. For a set D of positive integers, a sequence S of non-negative integers is called a D-*sequence* if $|x-y| \notin D$ for any $x, y \in S$. Let $S(n)$ denote as $|S \cap \{0, 1, 2, \cdots, n-1\}|$. The *density* of S, $\delta(S)$, is defined by

$$\delta(S) := \lim_{n\to\infty} \frac{S(n)}{n}.$$

The parameter of interest is the density of D, $\mu(D)$, defined by

$$\mu(D) := \sup\ \{\ \delta(S) :\ S \text{ is a } D\text{-sequence}\}.$$

The problem of determining or estimating $\mu(D)$ was initially posed by Motzkin in an unpublished problem collection (cf. [6]). In 1975, Cantor and Gordon [6] proved the existence of $\mu(D)$ for any D. The exact values of $\mu(D)$ for several families of sets D have been later on studied by Haralambis [23]. The parameter $\mu(D)$ is also closely related to the channel assignment problem (or T-coloring).

Griggs and Liu [21] indicated that $\mu(D)$ is equivalent to the asymptotic ratio of T-colorings for cliques and gave a different proof for the existence of $\mu(D)$, using directed graphs. This implies that the earlier work by Rabinowitz and Proulx [37] on the asymptotic ratio of T-coloring has equivalently determined the exact values of $\mu(D)$ for several families of sets D, and the authors also posted a conjecture on the value of $\mu(D)$ for $D = \{a, b, a+b\}$. In Section 3, we shall discuss with more details on this conjecture and its confirmation.

To simplify the notations, throughout the article we denote $\chi(X, D)$, $\chi_f(X, D)$, $\chi_c(X, D)$, $\alpha(X, D)$ and $\omega(X, D)$ as the corresponding parameters for the distance graph $G(X, D)$.

For any D, the density of D and the fractional chromatic number of the corresponding distance graph $G(\mathbb{Z}, D)$ are indeed identical, as proved by Chang, Liu and Zhu [8]:

Theorem 1. *For any finite set D of positive integers, $\chi_f(\mathbb{Z}, D) = 1/\mu(D)$.*

The circular chromatic number of distance graphs is closely related to the parameter involved in the Wills conjecture [44]. For a real number x, let $||x||$ denote the distance from x to the nearest integer, i.e.,

$$||x|| = \min\{x - \lfloor x \rfloor,\ \lceil x \rceil - x\}.$$

For a set $D \subseteq \Re$ and $t \in \Re$, let $||tD|| = \inf\{||tx|| : x \in D\}$, and define

$$\kappa(D) = \sup\ \{\ ||tD||\ : t \in\ \Re\ \}.$$

The parameter $\kappa(D)$ is studied in the Diophantine approximations by Wills [44], Y.G. Chen [10-12], the View Obstruction Problems by Cusick [13-15] and Cusick and Pomerance [16], and problems concerning flows and colorings of graphs by Bienia et al. [4]. A long-standing open question concerning $\kappa(D)$ is the conjecture of Wills [44]:

Conjecture 1. For any finite set D of positive integers, $\kappa(D) \geqslant 1/(|D|+1)$.

Conjecture 1 has been given a poetic name *lonely runner conjecture* by Goddyn [4]: Suppose m runners run laps on a circular track of unit length. Each runner maintains a constant speed, and the speeds of all the runners are distinct. A runner is called lonely if the distance on the circular track between him or her and every other runner is at least $1/m$. Equivalently, the conjecture asserts that for each runner, there is a time when he or she is lonely. Conjecture 1 was confirmed for $|D| = 3$ by Betke and Wills [3] and Cusick [13-15]; for $|D| = 4$ by Cusick and Pomerance [16] and later by Bienia et al. [4] with a simpler proof, and also by Y. G. Chen [10] who considered a more general format of this conjecture; for $|D| = 5$ by Bohman, Holzman and Kleitman [5] and a simpler proof by Renault [38].

It is known [47] that for any set D, $\chi_c(\mathbb{Z}, D) \leqslant 1/\kappa(D)$. Combining this with Theorem 1 and (1.1), we have

$$\omega(\mathbb{Z}, D) \leqslant 1/\mu(D) = \chi_f(\mathbb{Z}, D) \leqslant \chi_c(\mathbb{Z}, D) \leqslant 1/\kappa(D). \tag{1.2}$$

Inequalities in (1.2) layout intriguing connections among coloring parameters of distance graphs and the two number theory problems. These inequalities are also useful in the study of coloring parameters for distance graphs. For instance, if $\mu(D) = \kappa(D)$ holds for some D then the last two equalities in (1.2) hold, $\chi_f(\mathbb{Z}, D) = \chi_c(\mathbb{Z}, D) = 1/\mu(D)$. If one can determine the fractional chromatic number – or equivalently $\mu(D)$ - and $\kappa(D)$, or one can get bounds for these parameter, sometimes these bounds are sharp enough to determine the chromatic number of the corresponding distance graphs. For example, let $m > k$ be positive integers. Denote $D_{m,k} = \{1, 2, \cdots, m\} - \{k\}$. The chromatic number of $G(\mathbb{Z}, D_{m,k})$ was first studied by Eggleton et al. [17], and later on by Kemnitz and Kolberg [25], and Liu [32]. Partial results for special values of k and m were obtained. Using fractional chromatic number as a lower bound, Chang et al. [8] solved this problem completely. This family of distance graphs will be further discussed in Section 4.

Moreover, inequalities in (1.2) generate interesting research problems. A natural question is when those inequalities are sharp (or strict)? For instance, as $\mu(D) \geqslant \kappa(D)$, the question whether the equality holds in general was first raised by Cantor and Gordon [6] (although the parameter $\kappa(D)$ was not introduced explicitly there), and then discussed by Haralambis [23]. An infinite family of sets D for which $\mu(D) > \kappa(D)$ was given in [23]. In later sections of this survey, we will further discuss results that lead to sharpness or strictness of the inequalities in (1.2).

The article is organized as: In Section 2, we introduce some commonly used tools in the study of coloring parameters of distance graphs; and in other sections, restricting to special families of distance sets, we survey research advances and open problems on the parameters involved in (1.2), with a main focus on the coloring parameters of distance graphs. Many of the families of sets D that have been studied are inspired by (1.2) or are generalized from the ones initially investigated by Eggleton et al. [17].

Notice that if $D = \{d_1, d_2, \cdots d_n\}$ and $\gcd(d_1, d_2, \cdots, d_n) = d$, then the graph $G(\mathbb{Z}, D)$ consists of d components, each isomorphic to $G(\mathbb{Z}, \{d_1/d, d_2/d, \cdots, d_n/d\})$. Hence, throughout the article we assume, unless indicated, that $\gcd(D) = 1$.

2. Distance Graphs v.s. Circulant Graphs

A *graph homomorphism* from a graph G to a graph H is an edge preserving function from $V(G)$ to $V(H)$. If such a function exists, we say G is homomorphic to H and denote this by $G \to H$. The chromatic number of G is the minimum n such that $G \to K_n$; the clique size of G is the maximum w such that $K_w \to G$.

The fractional chromatic number of a graph G is indeed the minimum ratio n/m such that $G \to K(n,m)$, where $K(n,m)$ is the *Kneser graph* with the vertex set $\binom{[n]}{m}$ (m-element subsets of $[n]$) and two vertices A and B are adjacent if $A \cap B = \emptyset$. From this point of view, $K(n,m)$ plays essentially the same role as the one that the clique K_n does in vertex-coloring.

For positive integers p, q with $p \geqslant 2q$, the *circular clique* $K_{p/q}$ has as the vertex set $Z_p = \{0, 1, \cdots, p-1\}$ where two vertices u and v are adjacent if $|u-v|_p \geqslant q$. The circular chromatic number of G is the minimum ratio p/q such that $G \to K_{p/q}$. From this point of view, $K_{p/q}$ plays essentially the same role as the one that K_n does in vertex-coloring.

By composition of functions, if $G \to H$ then $g(G) \leqslant g(H)$, where the parameter g can be χ, χ_f, χ_c, $\omega(G)$ etc.

For a positive integer n and a subset D of $[n]$, the n-vertex *circulant graph* generated by D, denoted by $G(Z_n, D)$, has as the vertex set $Z_n = \{0, 1, 2 \cdots, n-1\}$ and vertices a and b are adjacent if $|a-b|_n \in D$. The circular clique $K_{p/q}$ is a circulant graph, $K_{p/q} = G(Z_p, [q-1])$.

For any D, $G(\mathbb{Z}, D) \to G(Z_n, D)$ holds for all $n \geqslant 2 \max D$. Hence, a coloring parameter of the circulant graph $G(Z_n, D)$ provides an upper bound for $G(\mathbb{Z}, D)$. It is known that for a vertex-transitive graph G, $\chi_f(G) = |V(G)|/\alpha(G)$. As circulant graphs are vertex-transitive, we have the upper bound $\chi_f(\mathbb{Z}, D) \leqslant n/\alpha(Z_n, D)$ for any $n \geqslant 2 \max D$. For a lower bound, one may consider the subgraph of $G(\mathbb{Z}, D)$ induced by the vertex set $\{0, 1, 2 \cdots, n-1\}$ (note, this subgraph may not be isomorphic to $G(Z_n, D)$). The coloring parameters of this subgraph provides a lower bound for $G(\mathbb{Z}, D)$. See Example 1 below for a straightforward application of this method. A more complicated example will be introduced in Section 5.

A discrete version of $\kappa(D)$ for finite sets D is given as follows. For a set of D, let $\lambda, p \leqslant 2 \max D$ be positive integers, $\gcd(\lambda, p) = 1$. Denote

$$(\lambda D)_p = \min\{|\lambda d|_p : d \in D\}.$$

Then we have

Proposition 2. *For a finite set D of positive integers,*

$$\kappa(D) = \max\ \{(\lambda D)_p/p :\ 1 \leqslant p,\ \lambda \leqslant 2 \max D, \gcd(\lambda, p) = 1\}.$$

A direct proof of $\mu(D) \geqslant \kappa(D)$ can be derived from a result of Cantor and Gordon [Theorem 1, [6]]. Suppose there exist positive integers λ, p with $\gcd(\lambda, p) = 1$. Assume $(\lambda D)_p = q$. Let S be the set of integers defined by

$$S = \{n \in Z : (\lambda n \text{ mod } p) \in \{0, 1, 2, \cdots, q-1\}\}.$$

Then S is a D-sequence, implying $\mu(D) \geqslant q/p$. By Proposition 2, $\mu(D) \geqslant \kappa(D)$.

For an example, consider $D = \{a, b\}$ with $\gcd(a, b) = 1$. If both a and b are odd, then $G(\mathbb{Z}, D)$ is a bipartite graph and $(D)_2 = 1$, so $\mu(D) = \kappa(D) = 1/2$. If a and b are of different parity we have the following:

Example 1. Let $D = \{a, b\}$ where a and b are relative primes of different parity. Then $\mu(D) = \kappa(D) = (a + b - 1)/2(a + b)$.

Proof. As $\gcd(a, b) = 1$, the following is an odd cycle of length $a + b$ in $G(\mathbb{Z}, D)$:

$$[0, a, 2a, 3a, \cdots, (a + b - 1)a] \text{ (mod } a + b).$$

It is known that $\chi_f(C_{2k+1}) = (2k + 1)/k$ (cf. [47]), so

$$1/\kappa(D) \geqslant 1/\mu(D) = \chi_f(\mathbb{Z}, D) \geqslant 2(a + b)/(a + b - 1).$$

As $\gcd(b + a, b - a) = \gcd(a, b) = 1$, there exist x and y such that $x(b - a) = y(b + a) + 1$. Also, because $x(b + a) \equiv 0$ (mod $b + a$), we get $2xb \equiv 1$ (mod $b + a$). This implies that $xb \equiv (b + a + 1)/2$ (mod $a + b$). Similarly, we get $2xa \equiv -1$ (mod $b + a$), and so $xa \equiv (b + a - 1)/2$ (mod $a + b$). Hence, $(xD)_{a+b} = (a + b - 1)/2$, implying $\kappa(D) \geqslant (a + b - 1)/2(a + b)$. ■

The result of the above example was shown in [6], while the proof of the first part - using fractional chromatic number - was from [8].

The relation between $\chi_c(\mathbb{Z}, D)$ and $\kappa(D)$ can be established by the following method which also provides an upper bound for $\chi_c(Z_n, D)$. Suppose $\kappa(D) = q/p$. Then there exist $p, \lambda \leqslant 2 \max D$, $\gcd(\lambda, p) = 1$, such that $(\lambda D)_p = q$. The permutation π on Z_p defined by $\pi(x) = \lambda x$ is indeed a homomorphism from $G(Z_p, D)$ to the circular clique $K_{p/q}$. Therefore,

$$\chi_c(\mathbb{Z}, D) \leqslant \chi_c(Z_p, D) \leqslant p/q = 1/\kappa(D).$$

The permutation π is called the *multiplier method* (cf. Zhu [47]) which has been used in several articles on coloring parameters for distance graphs [8, 7, 46, 49] or coloring parameters for circulant graphs [28].

The multiplier method provides an upper bound for the circular chromatic number of distance graph and often the bound is sharp enough to determine the chromatic number. On the other hand, the multiplier method does not always provide the exact value for $\chi_c(\mathbb{Z}, D)$, a particular case is when $\chi_c(\mathbb{Z}, D) < \kappa(D)$. In the next few sections, we will encounter several examples like this.

3. Distance Graphs with Large Clique Size

If $|D| = 1$ or if $D = \{a, b\}$ where a and b are both odd, then $G(\mathbb{Z}, D)$ is a bipartite graph and so $\mu(D) = \kappa(D) = 1/2$. For the rest of 2-element sets D, the solution is included in Example 1.

For $|D| = 3$, the chromatic number of $G(\mathbb{Z}, D)$ was studied by Eggleton et al. [17], J.-J. Chen el al. [9], and Voigt [40], and was completely settled by Zhu [49] (after confirming a conjecture posted in [9, 40]).

Theorem 3. ([49]) *Let $D = \{a, b, c\}$, $a < b < c$ and $\gcd(a, b, c) = 1$. Then*

$$\chi(\mathbb{Z}, D) = \begin{cases} 2, & \text{if } a, b, c \text{ are all odd;} \\ 4, & \text{if either } c = a + b \text{ and } b - a \not\equiv 0 \pmod 3, \\ & \quad \text{or } D = \{1, 2, 3k\}; \\ 3, & \text{otherwise.} \end{cases}$$

Comparing to the complete solution of the chromatic number, the values of $\chi_f(\mathbb{Z}, D)$ and $\chi_c(\mathbb{Z}, D)$ for 3-element sets D are known only for some special families.

The sets $D = \{a, b, a+b\}$ were considered by Rabinowitz and Proulx [37] and by Haralambis [23]. If none of a, b or $a + b$ is a multiple of 3, then it is easy to see that $\kappa(D) = \mu(D) = 1/3$ (since all multiples of 3 form a D-sequence and $(D)_3 = 1$). If $a = 1$, i.e. $D = \{1, b, b+1\}$, then the value of $\mu(D)$ was determined in [23]. A complete solution to this problem was settled by Liu and Zhu [34].

Theorem 4. ([34]) *If $D = \{a, b, c\}$, where $c = a + b$, $\gcd(a, b) = 1$, and exactly one of a, b, c is a multiple of 3, then*

$$\mu(D) = \min\left\{ \frac{\lfloor (a+c)/2 \rfloor}{a+c}, \frac{\lfloor (b+c)/2 \rfloor}{b+c} \right\}.$$

Using circulant graphs, one inequality ($\geqslant$) of Theorem 4 was proved by Rabinowitz and Proulx [37]. The argument is by showing that for $n = a + c$ or $b + c$, then $\alpha(Z_n, D) \geqslant \lfloor n/2 \rfloor$, implying $\chi_f(\mathbb{Z}, D) \leqslant \chi_f(Z_n, D) \leqslant n/\lfloor n/2 \rfloor$. In the same article the authors conjectured that the other direction of the inequality ($\leqslant$) also holds. This conjecture was confirmed in [34] by using the following result of Haralambis as a major tool.

Lemma 5. ([23]) *Let D be a set of positive integers, t a real number in the interval $(0, 1]$. If for every D-sequence S there exists some $n \geqslant 0$ such that $S(n)/(n + 1) \leqslant t$, then $\mu(D) \leqslant t$.*

In viewing of (1.2), it is natural to consider distance graphs with large clique size. By definition of distance graphs, the largest value of $\omega(\mathbb{Z}, D)$ is $|D|+1$, which happens to be when $D = \{1, 2, \cdots, d\}$. For this case, it can be easily seen that all the equalities in (1.2) hold with $\omega(\mathbb{Z}, D) = d + 1 = 1/\kappa(D)$.

Hence, the next case to consider are the sets D with $\omega(\mathbb{Z}, D) = |D|$, which are called *almost difference closed sets* [34]. Notice that the set D in Theorem 4 is almost difference closed. A complete characterization of almost difference closed sets was given by Kemnitz and Marangio [26].

Theorem 6. ([26]) *Suppose D is a finite set of positive integers,* $\gcd(D) = 1$. *Then D is almost difference closed if and only if D is one of the following:*

(*Type A.1*) $D = \{a, 2a, 3a, \cdots, (m-1)a, b\}$.

(*Type A.2*) $D = \{a, b, a+b\}$ *for some* $b \neq 2a$.

(*Type A.3*) $D = \{x, y, y-x, y+x\}$ *for some* $y > x$.

For almost difference closed sets, the chromatic numbers were completely settled [9, 25, 26, 40], and most of the values of $\mu(D)$ and $\kappa(D)$ were determined [11, 34], except only for one subcase of A.3 in which $\kappa(D)$ were determined while bounds for $\mu(D)$ were given [34]. The following is the result for A.1 sets D.

Theorem 7. ([34]) *Suppose* $D = \{a, 2a, \cdots, (m-1)a, b\}$, *where* $\gcd(a, b) = 1$. *Then*

$$\mu(D) = \kappa(D) = \begin{cases} \dfrac{1}{m}, & \text{if } a \geqslant 2 \text{ or } b \text{ is not a multiple of } m; \\ \dfrac{k}{km+1}, & \text{if } a = 1 \text{ and } b = km \text{ for some } k. \end{cases}$$

For A.2 sets in Theorem 6, the values of $\kappa(D)$ were determined by Y. G. Chen [11]. By Theorem 4, we have:

Theorem 8. ([11, 34]) *Suppose* $D = \{a, b, c\}$, $0 < a < b$, $c = a + b$, *and* $\gcd(a, b) = 1$. *Then*

$$\mu(D) = \kappa(D) = \min\left\{\frac{\lfloor (a+c)/2 \rfloor}{a+c}, \frac{\lfloor (b+c)/2 \rfloor}{b+c}\right\}.$$

Notice that the chromatic number for the sets in Theorem 8 were determined by J.-J. Chen et al. [9] and by Voigt [40]. Partial results on the circular chromatic number and the fractional chromatic number were obtained by Zhu [49].

Type A.3 sets turned out to be more complicated.

Theorem 9. ([34]) *If* $D = \{x, y, y-x, x+y\}$, $y > x$, *and x, y are of different parity, then* $\mu(D) = 1/4$.

Theorem 10. ([34]) *Suppose* $D = \{x, y, y-x, y+x\}$*, where* $y > x$*,* $x = 2k+1$*,* $y = 2m+1$ *and* $\gcd(x, y) = 1$*. Then* $\mu(D) \geqslant \frac{(k+1)m}{4(k+1)m+1}$.

The equality in Theorem 10 holds for $x = 1$ and is conjectured to hold for the general case [34]. The following is an upper bound for this family of sets D.

Theorem 11. ([34]). *Suppose* $D = \{x, y, y - x, y + x\}$*, where* $y > x$*,* $x = 2k + 1$*,* $y = 2m + 1$ *and* $\gcd(x, y) = 1$*. Let* $\delta = \frac{1}{k^2+2km+3k+m+1}$*. Then* $\mu(D) \leqslant \frac{1}{4+\delta}$.

By Theorems 7 and 8, $\mu(D) = \kappa(D)$ for Types A.1 and A.2. For Type A.3, however, the equality does not always hold (see Theorem 13 below). An easier case for Type A.3, $D = \{x, y, y - x, x + y\}$, is when x and y are of distinct parity and none of x, y is a multiple of 4, then we have $||(1/4)D|| = 1/4$, by Theorem 9, $\mu(D) = \kappa(D) = 1/4$.

It is known and not hard to show (cf. [23]) that $\kappa(D)$ is a fraction whose denominator always divides the sum of some pair of elements in D. Indeed, suppose $\kappa(D) = ||tD|| = p/q$, then there exist $a, b \in D$ such that $at = k_1 + p/q$ and $bt = k_2 - p/q$, for some integers k_1 and k_2 (for otherwise, one may increase or decrease t by a small amount so that $||tD||$ increases). This implies that $t = (k_1+k_2)/(a+b)$, and hence $q|(a + b)$. By this fact and some calculation, we obtain the values of $\kappa(D)$ for all Type A.3 sets.

Theorem 12. ([34]). *Suppose* $D = \{x, y, y - x, y + x\}$*, where* $\gcd(x, y) = 1$*. Then*

$$\kappa(D) = \begin{cases} \phi_4(2y + x), & \textit{if } x \equiv 0 \pmod 4 \textit{ and } y \equiv 3 \pmod 4\textit{, or} \\ & \quad x \equiv 1 \pmod 4 \textit{ and } y \equiv 0 \pmod 4\textit{, or} \\ & \quad x \equiv 3 \pmod 4 \textit{ and } y \equiv 1, 3 \pmod 4\textit{;} \\ \phi_4(2x + y), & \textit{if } x \equiv 0 \pmod 4 \textit{ and } y \equiv 1 \pmod 4\textit{, or} \\ & \quad x \equiv 1 \pmod 4\textit{, } y \equiv 3 \pmod 4\textit{, and } y < 3x\textit{;} \\ \phi_4(2y - x), & \textit{if } x \equiv 3 \pmod 4 \textit{ and } y \equiv 0 \pmod 4\textit{, or} \\ & \quad x \equiv y \equiv 1 \pmod 4\textit{, or} \\ & \quad x \equiv 1 \pmod 4\textit{, } y \equiv 3 \pmod 4\textit{, and } y \geqslant 3x\textit{,} \end{cases}$$

where $\phi_4(n)$ *denotes* $\lfloor \frac{n}{4} \rfloor / n$.

By Theorems 9, 10 and Corollary 12, we conclude that

Theorem 13. ([34]). *Let* $D = \{x, y, y - x, y + x\}$*,* $\gcd(D) = 1$*. Then* $\mu(D) = \kappa(D)$ *if and only if one of the following holds:*

- $x = 1$ *and* y *is odd,*
- x *and* y *are of different parity and none of them is a multiple of* 4.

Note that by Theorems 9, 11, and (1.2), we can determine the chromatic number for all Type A.3 sets, which was obtained by Kemnitz and Kolberg [25] and Kemnitz and Marangioand [26] by different approaches.

Corollary 14. *Suppose* $D = \{x, y, y - x, x + y\}$*, where* $\gcd(x, y) = 1$*. If* x *and* y *are of distinct parity then* $\chi_f(\mathbb{Z}, D) = \chi_c(\mathbb{Z}, D) = \chi(\mathbb{Z}, D) = 4$*. If* x *and* y *are both odd then* $\chi(\mathbb{Z}, D) = 5$*.*

Barajas and Serra [1] proved that: For any sets D with $|D| = 4$, $\chi(\mathbb{Z}, D) = 5$ if and only if either D belongs to Type A.1 or it is the set with $\chi(\mathbb{Z}, D) = 5$ mentioned in Corollary 14. This partially confirmed the following conjecture of Zhu [49], for 4-element sets D.

Conjecture 2. ([49]). If $\omega(\mathbb{Z}, D) < |D|$ then $\chi(\mathbb{Z}, D) \leqslant |D|$.

As mentioned in the previous paragraph, Conjecture 2 has been confirmed for sets D with $|D| \leqslant 4$. (For $|D| = 3$, see Theorem 3.)

By Theorems 7, 8 and 13, all the values of $\chi_c(\mathbb{Z}, D)$ for almost difference closed sets are determined, except the following case:

Problem 1. What are the values of $\chi_c(\mathbb{Z}, D)$ where $D = \{x, y, y - x, y + x\}$, $x < y$, x and y are both odd and $x \neq 1$, or one of x and y is a multiple of 4 and the other is odd?

Another open problem is about 3-element sets D. So far, all the known results for $|D| = 3$ have $\mu(D) = \kappa(D)$. Whether this is true for all 3-element sets D remains an open problem.

Problem 2. Does there exist a 3-element set D with $\mu(D) > \kappa(D)$?

Problem 3. Does there exist a 3-element set D with $\chi_c(\mathbb{Z}, D) < 1/\kappa(D)$?

4. The Punched Sets $D = [m] - \{k, 2k, \cdots, sk\}$

By (1.2), $\chi_f(\mathbb{Z}, D) = \chi_c(\mathbb{Z}, D)$ holds if $\mu(D) = \kappa(D)$. On the other hand, there also exist sets D with $\chi_f(\mathbb{Z}, D) < \chi_c(\mathbb{Z}, D)$. In this section, we introduce a family of such sets. Indeed, we will show families of sets D such that all the inequalities of (1.2) are strict.

Recall, for integers $k < m$, $D_{m,k} = [m] - \{k\}$. The chromatic number of $G(\mathbb{Z}, D_{m,k})$ was first studied by Eggleton, Erdős and Skilton [17] and later on by several authors. In [17], Eggleton et al. obtained the solution for $k = 1$, and partial solutions for $k = 2$: $\chi(\mathbb{Z}, D_{m,1}) = \lfloor \frac{m+3}{2} \rfloor$ for any $m \geqslant 2$; $\chi(\mathbb{Z}, D_{m,2}) = \lfloor \frac{m+4}{2} \rfloor$ when $m \not\equiv 3 \pmod 4$, and $\lfloor \frac{m+3}{2} \rfloor \leqslant \chi(\mathbb{Z}, D_{m,2}) \leqslant \lfloor \frac{m+5}{2} \rfloor$ for any $m \geqslant 4$ with $m \equiv 3 \pmod 4$. For $3 \leqslant k < m$, the same authors provided the following bounds:

$$\max\{k, \lfloor \tfrac{1}{2}(\tfrac{m}{k-1} + 1) \rfloor t\} \leqslant \chi(\mathbb{Z}, D_{m,k}) \leqslant \min\{m, \lfloor \tfrac{1}{2}(\tfrac{m}{k} + 3) \rfloor k\},$$

where $t = 2$ if $k = 3$ and $t = k - 2$ if $k \geqslant 4$. The result for the case $k = 1$ was also proven by Kemnitz and Kolberg in [25], by a different approach. The lower bound of $\chi(\mathbb{Z}, D_{m,k})$ in the above was improved to $\lceil \frac{m+k+1}{2} \rceil$ by Liu [32], who also showed that the new bound is sharp for all pairs of integers (m, k) where k is odd. Furthermore, complete solutions for $k = 2$ and 4, and partial solutions for other even integers k are given in [32].

By using fractional chromatic number, Chang et al. [8] completely solved this problem.

Theorem 15. [8] *If $2k > m$, then*

$$\omega(\mathbb{Z}, D_{m,k}) = \chi_f(\mathbb{Z}, D_{m,k}) = \chi_c(\mathbb{Z}, D_{m,k}) = \chi(\mathbb{Z}, D_{m,k}) = k.$$

If $2k \leqslant m$, then $\chi_f(\mathbb{Z}, D_{m,k}) = (m + k + 1)/2$. Suppose $2k \leqslant m$. Write $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. Then

$$\chi(\mathbb{Z}, D_{m,k}) = \begin{cases} \frac{m+k+1}{2}, & \text{if } r > s; \\ \lceil \frac{m+k+2}{2} \rceil, & \text{otherwise.} \end{cases}$$

The circular chromatic number for $G(\mathbb{Z}, D_{m,k})$ for $2k \leqslant m$ was studied in [8] and later on completely determined by Chang, Huang and Zhu [7].

Theorem 16. [7] *Suppose $2k \leqslant m$. Write $m + k + 1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. Then*

$$\chi_c(\mathbb{Z}, D_{m,k}) = \begin{cases} \dfrac{m+k+2}{2}, & \text{if } r \leqslant s \text{ and } \gcd(m+k+1, k) \neq 1; \\ \dfrac{m+k+1}{2}, & \text{otherwise.} \end{cases}$$

The sets $D_{m,k}$ can be extended to a more general setting. For positive integers m, k and s with $m > ks$, let $D_{m,k,s}$ be the set $[m] - \{k, 2k, \cdots, sk\}$. The set $D = [m] - \{k\}$ is a special case when $s = 1$, $D_{m,k,1}$.

For the general case $s \geqslant 2$, the fractional chromatic number and the chromatic number were given by Liu and Zhu [33]. The circular chromatic number turned out to be more difficult, since for some sets D in this family we have $\chi_c(\mathbb{Z}, D) < 1/\kappa(D)$ (see Theorems 17 and 18). That is, the multiplier method for circulant graphs does not provide sharp bounds for $\chi_c(\mathbb{Z}, D)$. Partial results on $\chi_c(\mathbb{Z}, D)$ were given by Liu and Zhu [33] and by Huang and Chang [24]. Complete solutions were shown by Zhu [50].

Theorem 17. [50] *If $m < (s+1)k$, then $\mu(D_{m,k,s}) = \kappa(D_{m,k,s}) = 1/k$. If $m \geqslant (s+1)k$, then $\chi_f(\mathbb{Z}, D_{m,k,s}) = \frac{m+sk+1}{s+1}$ and*

$$\chi_c(\mathbb{Z}, D_{m,k,s}) = \begin{cases} \dfrac{m+sk+1}{s+1}, & \text{if } d = 1 \text{ or } d(s+1)|(m+sk+1); \\ \dfrac{m+sk+2}{s+1}, & \text{otherwise,} \end{cases}$$

where $d = \gcd(m+1, k)$.

As all the fractional chromatic numbers and circular chromatic numbers for $G(\mathbb{Z}, D_{m,k,s})$ are settled, the next question is the values of $\kappa(D)$? We answer this question in the following:

Theorem 18. *Suppose $D = [m] - \{k, 2k, \cdots, sk\}$ for some $m \geqslant (s+1)k$. Let t be the smallest positive integer such that* $\gcd(m+t, k) = 1$. *Then*

$$\kappa(D) = \begin{cases} 1/(2k), & \text{if } m = (s+2)k - 1 \text{ and } s \text{ is even;} \\ (s+1)/(m+sk+t), & \text{otherwise.} \end{cases}$$

Proof. For the first case, $m = (s+2)k - 1$ and s is even, one can see that $(D)_{2k} = 1$ as $\gcd(m, 2k) = 1$, so $\kappa(D) \geqslant 1/(2k)$. Also, by Theorem 17, $\mu(D) = (s+1)/(m+sk+1) = 1/(2k)$, so the result follows.

For the remaining case, we first claim that $\kappa(D) \geqslant (s+1)/(m+sk+t)$. Since $\gcd(m+sk+t, k) = \gcd(m+t, k) = 1$, there exists some $\lambda \leqslant 2m$ such that $\lambda k \equiv \pm 1 \pmod{m+sk+t}$. This implies that for $i = \pm 1, \pm 2, \cdots, \pm s$, $\lambda ik \equiv \pm i \pmod{m+sk+t}$. Hence, $(\lambda D)_{m+sk+t} = s+1$. By Proposition 2, $\kappa(D) \geqslant (s+1)/(m+sk+t)$.

Next we verify that $\kappa(D) \leqslant (s+1)/(m+sk+t)$. Let $\kappa(D) = q/p$. By Theorem 17,

$$(s+1)/(m+sk+t) \leqslant q/p \leqslant (s+1)/(m+sk+1).$$

Hence the result follows for $t = 1$. Assume $t \geqslant 2$. So, $k \geqslant 2$. Because $m \geqslant (s+1)k$, by direct calculation, the following holds for all $0 \leqslant i < j \leqslant s$:

$$(i+1)/(m+ik+1) \leqslant (j+1)/(m+jk+k). \tag{4.1}$$

By Proposition 2, there exist some $\lambda \leqslant 2m$ and $d \in D$ such that $\gcd(\lambda, p) = 1$ and $|\lambda d|_p = q$. Suppose $p \geqslant m+1$. Let $p = m + xk + r$ for some $x \geqslant 0$ and $1 \leqslant r \leqslant k$. By (4.1), it suffices to show that $q \leqslant x+1$. As $\gcd(\lambda, p) = 1$ and $\lambda \leqslant 2m$, there exists $\lambda' \leqslant m$ such that $\lambda'\lambda \equiv \pm 1$ (mod p). If $\lambda' \in D$, then $q = 1 \leqslant x+1$.

Assume $\lambda' \notin D$. Then $\lambda' = yk$ for some $y \leqslant s$. As $\lambda'\lambda \equiv \pm 1$ (mod p), it must be $\gcd(p, k) = 1$. This implies that $p - ck \in D$ for $c \geqslant x+1$. Note, if $|\lambda ck|_p = i$ for some $c \leqslant s$ then $|\lambda(p-ck)|_p = -i$ (mod p). Therefore, at most s numbers (and x numbers, respectively) from the set $\{\pm 1, \pm 2, \cdots, \pm s\}$ will be avoided by $|\lambda(ck)|_p$ for $1 \leqslant c \leqslant s$ (and by $|\lambda(p-ck)|_p$ for $1 \leqslant c \leqslant x$, respectively). So $q \leqslant x+1$.

It remains to consider the case $p \leqslant m$. Then $p = ak$ for some $2 \leqslant a \leqslant s$ and $m < (s{+}a)k$ (for otherwise, $q = 0$). Since $\gcd(k, ak) = k > 1$ and $\gcd(\lambda, ak) = 1$, we have $\{\lambda i : i = 1, 2, \cdots, ak-1\} = \{1, 2, \cdots, ak-1\}$ (mod ak) and $\lambda bk \not\equiv \pm 1$ (mod ak) for $b = 1, 2, \cdots, a-1$. So, $q = 1$.

Therefore, $q/p = 1/(ak) \leqslant (s+1)/(m+sk+1)$, equivalently, $m+sk+1 \leqslant ak(s{+}1)$. If $m{+}sk{+}1 < ak(s{+}1)$, then by the choice of t, $m{+}sk{+}t < ak(s{+}1)$, inducing $1/(ak) < (s+1)/(m+sk+t)$. So the result follows.

Assume $m{+}sk{+}1 = ak(s{+}1)$. Since $m < (s{+}a)k$, so $a \leqslant 2$, implying $a = 2$ and so $m = (s+2)k - 1$. Then s must be even (for if s is odd, then $(s+1)k \equiv 0$ (mod $2k$), contradicting $q = 1$). This is the first case considered at the beginning of the proof. ■

We conclude that $\kappa(D_{m,k,s}) > 1/|D_{m,k,s}|$ (stronger than the lonely runner conjecture which has $|D|+1$ in the denominator), except when $D = \{1, 3, 4\}$ in which $\kappa(D) = 2/7$.

The complete solutions of $\mu(D)$, $\kappa(D)$ and $\chi_c(\mathbb{Z}, D)$ for this family of sets D also provide examples for which the inequalities in (1.2) are strict. For instance, let $D = [26] - \{6\}$. By Theorems 17 and 18, $\chi_f(\mathbb{Z}, D) = 33/2$, $\chi_c(\mathbb{Z}, D) = 34/2$, and $1/\kappa(D) = 35/2$.

5. Union of Two Intervals

Another extension of the family $D = [m] - \{k\}$ is $D = [m] \setminus [a, b]$ for some $1 \leqslant a \leqslant b \leqslant m$. Denote this family of distance sets by $D_{m,[a,b]} = [a{-}1] \cup [b{+}1, m]$. The special case $D = [m] - \{a\}$ is when $a = b$. Another special case is when $a = 1$, which is the punched set discussed in the previous section, i.e., $D_{m,[1,b]} = D_{m,1,b}$.

For the general case $2 \leqslant a < b$, the coloring parameters for $G(\mathbb{Z}, D_{m,[a,b]})$ have been studied by several authors. Both the fractional chromatic number and the chromatic number for $G(\mathbb{Z}, D_{m,[a,b]})$ were studied by Wu and Lin [45], Lam, Lin and Song [30] and Lam and Lin [29]. An easy case is when $m \leqslant 2a-1$, in which

$\mu(D) = \kappa(D) = 1/a$. The case for $b < 2a$ was considered in [30, 45], in which the authors completely determined the fractional chromatic number; partial results and bounds on the circular chromatic number were also presented and the bounds are sharp enough to determine the chromatic number.

Theorem 19. ([45]). *Let $G = G(\mathbb{Z}, D_{m,[a,b]})$ for some $a+1 \leqslant b \leqslant 2a-1$. Then*

$$\chi_f(G) = \begin{cases} \chi_c(G) = (m+1)/2, & \text{if } 2a \leqslant m < 2b; \\ (m+a+1)/2, & \text{if } m \geqslant 2b. \end{cases}$$

Theorem 20. ([30, 45]). *Let $G = G(\mathbb{Z}, D_{m,[a,b]})$ for some $a+1 \leqslant b \leqslant 2a-1$. Suppose $m \geqslant 2b$. Let $m+a+1 = 2^r m'$ and $a = 2^s a'$, where r, s are non-negative integers and m', a' are odd integers. Then*

(a) *If $1 \leqslant r \leqslant s$, then $\frac{m+a+1}{2} + \frac{1}{3} \leqslant \chi_c(G) \leqslant \frac{m+a+2}{2}$.*
(b) *If $r = 0$ and $\gcd(m+1, a) \neq 1$, then $\chi_c(G) = \frac{m+a+2}{2}$.*
(c) *Otherwise, $\chi_c(G) = \frac{m+a+1}{2}$.*

The values of $\kappa(D_{m,[a,b]})$ for the above family are determined as follows.

Theorem 21. *Let $D = D_{m,[a,b]}$ for some $a+1 \leqslant b \leqslant 2a-1$. Then*

$$\kappa(D) = \begin{cases} 2/(m+1), & \text{if } 2a \leqslant m < 2b; \\ 2/(m+a+t), & \text{if } m \geqslant 2b, \end{cases}$$

where t is the minimum positive integer such that there exists some y, $a \leqslant y \leqslant \min\{a+t-1, b\}$, with $\gcd(m+a+t, y) = 1$.

Proof. We first consider the case $2a \leqslant m < 2b$. Assume m is odd. Let $m = 2m'+1$. Then $a \leqslant m' < b$, implying $a < m'+1 \leqslant b$. As $m < 2b$, we have $(D)_{m'+1} = 1$. By Proposition 2, Theorem 19 and (1.2), $\kappa(D) = 1/(m'+1) = 2/(m+1)$.

Assume m is even. Let $m = 2m'$. Then $a \leqslant m' < b$ and $a < m'+1 \leqslant b$. That is, $m', m'+1 \notin D$. Because $\gcd(m', 2m'+1) = 1$, there exists some integer x such that $\gcd(x, 2m'+1) = 1$ and $xm' \equiv 1 \pmod{2m'+1}$. This implies $x(m'+1) \equiv -1 \pmod{2m'+1}$. Hence, $(xD)_{2m'+1} \geqslant 2$, so $\kappa(D) \geqslant 2/(2m'+1) = 2/(m+1)$. By Theorem 19 and (1.2), the proof for the case $2a \leqslant m < 2b$ is complete.

We now consider the remaining case, $m \geqslant 2b$. Let $\kappa(D) = q/p$. Because $m \geqslant 2b$ and $b < 2a$, for every $x \in [a, b]$ it holds that $2x \in D$. Hence, $p \geqslant m+1$ and $q \leqslant 2$. Let t be the minimum positive integer such that there exists some y, $a \leqslant y \leqslant \min\{a+t-1, b\}$, such that $\gcd(m+a+t, y) = 1$. (Note, $t \leqslant a-1$

as we can have $y = a$.) Then there exists some $\lambda \leqslant m$ such that $\lambda y \equiv 1 \pmod{m+a+t}$. Since $y \leqslant \min\{a+t-1, b\}$, we have $(\lambda D)_{m+a+t} \geqslant 2$. By Proposition 2, $q/p \geqslant 2/(m+a+t)$. Next we verify that $\kappa(D) \leqslant 2/(m+a+t)$. By Theorem 19 and (1.2),

$$\frac{2}{m+a+t} \leqslant \frac{q}{p} \leqslant \frac{2}{m+a+1}.$$

Hence the result follows if $t = 1$. Assume $t \geqslant 2$. It can be seen that $p \geqslant m+a+1$. For if $p \leqslant m+a$ then $q \leqslant 1$, which is impossible as $p \geqslant m+1$ and $1/(m+1) \leqslant 2/(m+a+t)$. Assume $p = m+a+x$ for some $1 \leqslant x < t$. Then by the choice of t, we have $q \leqslant 1$, a contradiction. This completes the proof. ∎

The case for $b \geqslant 2a$ turned out to be much more complicated. Lam and Lin [29] obtained partial results on both the fractional chromatic number and the circular chromatic number. Complete solutions for the fractional chromatic number were recently obtained by Liu and Zhu [35].

Theorem 22. ([35]). *For integers $1 < a \leqslant b < m$. Let $G = G(\mathbb{Z}, D_{m,[a,b]})$, $\Delta = m+1-b$, $s = \lfloor b/a \rfloor$, and $q = \lfloor (m+1)/\Delta \rfloor$.*

- *If $\Delta \geqslant 2a$, then $\chi_f(G) = (m+sa+1)/(s+1)$.*
- *If $\Delta \leqslant a$, then $\chi_f(G) = \max\{a, (m+1)/(s+1)\}$.*
- *If $a < \Delta < 2a$, then*

$$\chi_f(G) = \begin{cases} \dfrac{m+sa+1}{s+1}, & \text{if } 2qa-1 \leqslant m < a+q\Delta-1 \text{ or} \\ & \quad m \geqslant (2q+1)a-1; \\ \dfrac{m+1}{q}, & \text{if } m < \min\{q\Delta+a-1, 2qa-1\}; \\ \dfrac{(2q-1)(m+1)+a}{2q^2}, & \text{if } q\Delta+a-1 \leqslant m < (2q+1)a-1. \end{cases}$$

The cases for $\Delta \geqslant 2a$ and $\Delta \leqslant a$ in Theorem 22 were first proved in [29]. For the remaining case, $a < \Delta < 2a$, the upper bounds were established by defining a weight function to the independent sets in $G(\mathbb{Z}, D)$, while the lower bounds were by the *periodic D-sequences*. To provide more insight into these formulas, in the following we describe these sequences.

Let $S = (s_1, s_2, \cdots, s_n, \cdots)$ be a D-sequence where $s_i < s_{i+1}$. Equivalently, S is an independent set in $G(\mathbb{Z}, D)$. Let $\delta_i = s_{i+1} - s_i$. The sequence $\Omega(S) = (\delta_1, \delta_2, \cdots)$ is called the *gap sequence* of S. Observe the following:

- *A sequence* $(\delta_1, \delta_2, \cdots)$ *is a* $D_{m,[a,b]}$*-gap sequence if and only if*

  (1) $\delta_i \geqslant a$ *for each* i*; and*

  (2) *for any* $j \leq j'$*, either* $\sum_{i=j}^{j'} \delta_i \leqslant b$ *or* $\sum_{i=j}^{j'} \delta_i \geqslant m+1$.

By definition,

$$\mu(D) = \max \lim_{n\to\infty} \frac{|S \cap [0, n-1]|}{n},$$

where the maximum is taken over all D-sequences S. Hence

$$\chi_f(G) = \frac{1}{\mu(D)} = \min \lim_{n\to\infty} \frac{n}{|S \cap [0, n-1]|} = \min \lim_{k\to\infty} \sum_{i=1}^{k} \frac{\delta_i}{k}.$$

Again, the minimum is taken over all D-sequences S with gap sequence $(\delta_1, \delta_2, \cdots)$.

The D-set S is *periodic* if $\Omega(S)$ is periodic. That is, if there exists some k such that $\delta_i = \delta_{i+k}$ for every $i \geqslant 1$. We denote a periodic Ω sequence by $\Omega = \langle \delta_1, \delta_2, \cdots, \delta_k \rangle$.

To simplify the notations, we denote consecutive repeated terms in Ω using $\otimes$. For instance, $\langle 3 \otimes 2, 5\rangle = \langle 2, 2, 2, 5\rangle$. The following are the corresponding gap sequences for the sets in Theorem 22.

- For $\Delta \geqslant 2a$, the gap sequence is:
$$\Omega = \langle s \otimes a, m+1\rangle.$$
- For $\Delta \leqslant a$:
$$\Omega = \langle a \rangle \text{ or } \langle (m+1)/(s+1)\rangle.$$
- For $a < \Delta < 2a$, $2qa - 1 \leqslant m < a + q\Delta - 1$ or $m \geqslant (2q+1)a - 1$:
$$\Omega = \langle s \otimes a, m+1\rangle.$$
- For $a < \Delta < 2a, m < \min\{q\Delta + a - 1, 2qa - 1\}$:
$$\Omega = \langle (q-1) \otimes \Delta, m+1-((q-1)\Delta)\rangle.$$
- For $a < \Delta < 2a, q\Delta + a - 1 \leqslant m < (2q+1)a - 1$:
$$\Omega = \langle Y'_q, Y_{q-1}, Y'_{q-1}, Y_{q-2}, Y'_{q-2}, \cdots, Y_1, Y'_1, a\rangle.$$

where

$$\begin{aligned}
Y_i &= i \otimes \Delta, a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta), \\
Y'_i &= (i-1) \otimes \Delta, \Delta + a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta), \\
Y'_q &= (q-1) \otimes \Delta, m - (q-1)\Delta.
\end{aligned}$$

It is not hard to check that each sequence is a periodic D-sequence. The ratio $\frac{\sum_{i=1}^{|\Omega|}\delta_i}{|\Omega|}$ of each Ω gave a lower bound for $\chi_f(\mathbb{Z}, D)$. These bounds were proved to be the exact values [35].

So far, the values of $\chi_c(\mathbb{Z}, D)$ of this family of sets D are known only for some special cases (see [29, 30, 45]).

6. A Concluding Remark

Although the lonely runner conjecture is widely open for eight or more runners [1], an analogy for the parameter $\mu(D)$, $\mu(D) \geqslant 1/(|D|+1)$, is quite straightforward. As it is known [9] that $\chi(\mathbb{Z}, D) \leqslant |D|+1$ (by first-fit coloring the vertices of positive integers sequentially), so $1/\mu(D) = \chi_f(\mathbb{Z}, D) \leqslant \chi(\mathbb{Z}, D) \leqslant |D|+1$. So far, the bound $\chi_f(\mathbb{Z}, D) \leqslant |D|+1$ is known to be sharp only for some sets D with $\omega(\mathbb{Z}, D) \geqslant |D|$ (see Section 3). Are there any other sets attaining this bound remains an open problem. In particular, the following was posted in [34]:

Conjecture 3. Let D be a finite set of positive integers with $\omega(\mathbb{Z}, D) \leqslant |D|-1$. Then $\chi_f(\mathbb{Z}, D) \leqslant |D|$ (or equivalently, $\mu(D) \geqslant 1/|D|$).

Notice that Conjecture 3 is weaker than Conjecture 2. As 2 has been confirmed for $|D| \leqslant 4$, so does Conjecture 3.

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