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A GENERALIZATION OF NOETHERIAN RINGS

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Abstract. In this paper, we introduce the concept of AFG rings. R is said to be a left AFG ring in case the left annihilator of every nonempty subset of R is a finitely generated left ideal. Some characterizations of AFG rings and applications are obtained.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R $({}_RM)$ denotes a right (left) R-module. For an R-module M, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*. For a subset X of R, the left annihilator of X in R is denoted by $l(X)$. If $X = \{a\}$, we usually abbreviate it to $l(a)$.

We first recall some known notions and facts needed in the sequel.

An R-module M is called *cogenerated by an R-module Q* if M embeds in a direct product of copies of Q. To say that M is *torsionless* is nothing but to say that M is cogenerated by R. Note that M is torsionless if and only if the canonical map $M \to M^{**}$ is a monomorphism. It is easy to check that a cyclic left R-module R/I is torsionless if and only if $I = l(X)$ for some subset $X \subseteq R$.

R is called a *left dual ring* if every left ideal of R is a left annihilator of a nonempty subset of R, equivalently, every cyclic left R-module is torsionless.

An R-module M is said to be a *self-cogenerator* [19] if M cogenerates every factor module of M. For a ring R this means, that ${}_RR$ is a self-cogenerator if and only if R is a left dual ring.

R is called a *left CF ring* if every cyclic left R-module embeds in a free module. Obviously, the left CF rings have a stronger property than the left dual rings since their cyclic left R-modules do not simply embed in a product of copies of ${}_RR$, but in a (finite) direct sum of copies of ${}_RR$.

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R is said to be a *left Noetherian ring* if every left ideal of R is finitely generated. Noetherian rings and their generalizations have been studied extensively by many authors. In this paper, we will introduce a new generalization of Noetherian rings. We will call a ring R to be *left* AFG in case the left annihilator of every nonempty subset of R is finitely generated. In Section 2, we prove that the following are equivalent for a ring R: (1) R is a left AFG ring. (2) Any direct product of copies of R_R is singly projective. (3) Any direct product of singly projective right R-modules is singly projective. (4) Every right R-module has a singly projective preenvelope. (5) Every cyclic right R-module has a projective preenvelope.

Section 3 is devoted to some applications. We get that R is a QF ring if and only if R is a left AFG, left and right dual ring. It is also shown that the following are equivalent for a left AFG ring R: (1) R is a right CF ring. (2) Every right R-module has a monic singly projective preenvelope. (3) Every cyclic right R-module has a monic projective preenvelope. Finally, we prove that the following are equivalent for a left AFG ring R: (1) R is a right PP ring. (2) Every cyclic right R-module has an epic projective preenvelope. (3) Every right R-module has an epic singly projective preenvelope. (4) Every submodule of a singly projective right R-module is singly projective. (5) Every torsionless right R-module is singly projective.

For unexplained concepts and notations, we refer the reader to [1, 7, 15, 19, 20].

2. AFG Rings

Recall that a right R-module M is called *singly projective* [2] in case for every epimorphism $f : N \to M$ and any homomorphism $g : C \to M$ with C a cyclic right R-module, there exists $h : C \to N$ such that $g = fh$.

The following lemma will be used frequently in the sequel.

Lemma 2.1. *The following are equivalent for a right R-module M:*

(1) *M is singly projective.*

(2) *For any cyclic submodule N of M, the inclusion $\iota : N \to M$ factors through a finitely generated free right R-module F, that is, there exist $g : N \to F$ and $h : F \to M$ such that $\iota = hg$.*

(3) *For any cyclic right R-module N and any homomorphism $f : N \to M$, f factors through a finitely generated free right R-module F.*

Proof. It is easy by [2, Proposition 12]. ■

We also recall that a homomorphism $f : M \to P$ is called a *projective preenvelope* of an R-module M [6] if P is projective, and for any homomorphism g from M to any projective R-module P', there exists $h : P \to P'$ such that $g = hf$. Similarly we have the concept of *singly projective preenvelopes*.

Definition 2.2. R is called a *left* AFG *ring* if for any nonempty subset X of R, the left annihilator of X in R is a finitely generated left ideal. Similarly, we have the concept of right AFG rings.

Clearly, any left Noetherian ring is a left AFG ring. But the converse does not hold in general. For example, we can choose R to be a domain (and hence an AFG ring) which is not Noetherian.

Now we characterize left AFG rings as follows.

Theorem 2.3. *The following are equivalent for a ring R:*

(1) *R is a left AFG ring.*

(2) *The dual module $M^* = \mathrm{Hom}_R(M, R)$ of any cyclic right R-module M is finitely generated.*

(3) *Every cyclic torsionless left R-module is finitely presented.*

(4) *Any direct product of copies of R_R is singly projective.*

(5) *Every direct product of singly projective right R-modules is singly projective.*

(6) *Every cyclic right R-module has a projective preenvelope.*

(7) *Every cyclic right R-module has a singly projective preenvelope.*

(8) *Every right R-module has a singly projective preenvelope.*

Proof. $(1) \Leftrightarrow (2)$ Let I be a right ideal of R. Define $\alpha : (R/I)^* \to l(I)$ via

$$f \mapsto f(\overline{1}), \quad f \in (R/I)^*.$$

It is easy to verify that α is well-defined and is an isomorphism. So $(1) \Leftrightarrow (2)$ follows.

$(1) \Leftrightarrow (3)$ follows from the fact that a left ideal I is a left annihilator in R if and only if R/I is a torsionless left R-module.

$(2) \Rightarrow (6)$ Let M be a cyclic right R-module. Since M^* is finitely generated, there exists a generating set $\{f_j \in M^* : 1 \le j \le n\}$. Define $f : M \to R^n$ via

$$x \mapsto (f_1(x), f_2(x), \cdots, f_n(x)), \quad x \in M.$$

We will show that f is a projective preenvelope of M. It is enough to show that for any $m \ge 1$ and any homomorphism $g : M \to R^m$, there is $h : R^n \to R^m$

such that $g = hf$. Let $\pi_i : R^m \to R$ be the ith projection, $1 \leq i \leq m$. Note that $\pi_i g \in M^*$, so there exist $r_{ij} \in R$ $(1 \leq j \leq n)$ such that $\pi_i g = \sum_{j=1}^n r_{ij} f_j$. Define $h_i : R^n \to R$ via

$$(a_1, a_2, \cdots, a_n) \mapsto \sum_{j=1}^{n} r_{ij} a_j, \quad a_j \in R.$$

Then there exists $h : R^n \to R^m$ such that $h_i = \pi_i h$. So $\pi_i h f = h_i f = \pi_i g$ and hence $g = hf$.

(6) $\Rightarrow$ (7) Let M be a cyclic right R-module. Then M has a projective preenvelope $\alpha : M \to P$ by (6). We claim that α is also a singly projective preenvelope. In fact, for any singly projective right R-module N and any homomorphism $f : M \to N$, there exist a finitely generated free right R-module F, $g : M \to F$ and $h : F \to N$ such that $f = hg$ by Lemma 2.1. So there exists $\beta : P \to F$ such that $\beta\alpha = g$. Thus $f = (h\beta)\alpha$, as desired.

(7) $\Rightarrow$ (5) Let $\{M_i\}_{i\in I}$ be a family of singly projective right R-modules and N any cyclic submodule of M_i^I. Write $\iota : N \to M_i^I$ to be the inclusion and $\pi_i : M_i^I \to M_i$ to be the ith projection. Since M_i is singly projective, there exist finitely generated free right R-modules F_i, homomorphisms $g_i : N \to F_i$ and $h_i : F_i \to M_i$ such that $\pi_i \iota = h_i g_i$ by Lemma 2.1. Note that N has a singly projective preenvelope $f : N \to F$ by (7), and so there is $k_i : F \to F_i$ such that $g_i = k_i f$. Thus there exists $g : F \to M_i^I$ such that $\pi_i g = h_i k_i$. Hence $\pi_i \iota = (h_i k_i) f = \pi_i(gf)$, and so $\iota = gf$. Thus M_i^I is singly projective by Lemma 2.1.

(5) $\Rightarrow$ (8) Let N be any right R-module. By [7, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ dependent on Card(N) and Card(R) such that for any homomorphism $g : N \to L$ with L singly projective, there is a pure submodule Q of L such that Card$(Q) \leq \aleph_\alpha$ and $g(N) \subseteq Q$. Thus g has a factorization $N \to Q \to L$ with Q singly projective by [2, Proposition 14]. Now let $(\varphi_i)_{i\in I}$ give all such homomorphisms $\varphi_i : N \to Q_i$ with Card$(Q_i) \leq \aleph_\alpha$ and Q_i singly projective. So any homomorphism $N \to M$ with M singly projective has a factorization $N \to Q_j \to M$ for some $j \in I$. Thus $N \to \Pi_{i\in I} Q_i$ is a singly projective preenvelope since $\Pi_{i\in I} Q_i$ is singly projective by (5).

(8) $\Rightarrow$ (4) follows from [5, Lemma 1].

(4) $\Rightarrow$ (2) Let A be a cyclic right R-module. For every index set I, there is a canonical homomorphism $\alpha : R_R^I \otimes_R A^* \to (A^*)^I$, where α is defined via

$$\alpha((r_j)_{j\in I} \otimes_R \theta) = (\delta_j)_{j\in I}, \delta_j(x) = r_j \theta(x), \quad r_j \in R, \theta \in A^*, x \in A.$$

We will show that α is epic. Indeed, let $(f_j)_{j\in I} \in (A^*)^I$. Then there exists $\beta : A \to R_R^I$ such that $f_j = \pi_j \beta$, where $\pi_j : R_R^I \to R$ is the jth projection. Since R_R^I is singly projective by (4), there exist a finitely generated free right R-module R^n,

$\gamma : A \to R^n$ and $\varphi : R^n \to R^I_R$ such that $\beta = \varphi\gamma$ by Lemma 2.1. Let $p_i : R^n \to R$ be the ith projection and $\lambda_i : R \to R^n$ the ith injection, $i = 1, 2, \cdots, n$. Put $a_i = \varphi\lambda_i(1)$ and $g_i = p_i\gamma$. Then we have

$$f_j(a) = \pi_j\beta(a) = \pi_j\varphi\gamma(a) = \pi_j\varphi\sum_{i=1}^{n}\lambda_i p_i(\gamma(a)) = \pi_j\sum_{i=1}^{n}a_i g_i(a).$$

So $f_j = \pi_j \sum_{i=1}^{n} a_i g_i$, and hence

$$(f_j)_{j\in I} = \alpha(\sum_{i=1}^{n} a_i \otimes g_i).$$

It follows that α is an epimorphism, which means that A^* is a finitely generated left R-module by [18, Lemma 13.1, p. 41]. ■

The next example shows that the definition of AFG rings is not left-right symmetric.

Example 2.4. Let K be a field with a subfield L such that $\dim_L K = \infty$, and there exists a field isomorphism $\varphi : K \to L$ (for instance, $K = \mathbb{Q}(x_1, x_2, x_3, \cdots)$, $L = \mathbb{Q}(x_2, x_3, \cdots)$). Let $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'), \qquad x, y, x', y' \in K.$$

Then it is easy to see that R has exactly three right ideals: 0, R and $(0, K)$. Therefore R is a right Noetherian ring and hence a right AFG ring. On the other hand, let $a = (0, 1) \in R$. Then $l(a)$ is not finitely generated (see [15, Example 4.46 (e)]). Thus R is not a left AFG ring.

The proposition below shows that the concept of AFG rings is left-right symmetric for a left and right pseudo-coherent ring. Recall that R is called a *left pseudo-coherent ring* [3] if the left annihilator of each finite subset of R is a finitely generated left ideal. It is easy to verify that R is left pseudo-coherent if and only if every cyclic submodule of any finitely generated free left R-module is finitely presented.

Proposition 2.5. *The following are equivalent for a left and right pseudo-coherent ring R:*

(1) *R is a left AFG ring.*

(2) *R is a right AFG ring.*

Proof. $(1) \Rightarrow (2)$ Let M be a cyclic torsionless right R-module. Then M^* is finitely generated by Theorem 2.3 since R is a left AFG ring. Thus there exists an exact sequence $F \to M^* \to 0$ with F a finitely generated free left R-module, which induces an exact sequence

$$0 \to M^{**} \to F^*.$$

Thus M embeds in F^* since M is torsionless. Consequently M is finitely presented since R is right pseudo-coherent, and so R is a right AFG ring by Theorem 2.3.

$(2) \Rightarrow (1)$ is similar. ■

3. Applications

In this section, we will give new characterizations of some special rings such as QF rings, CF rings and PP rings using the foregoing results.

Recall that R is called a *left Pseudo-Frobenius ring* [19] if ${}_RR$ is injective and cogenerates every left R-module. Clearly, a left Pseudo-Frobenius ring is left dual.

Proposition 3.1. *The following are equivalent for a ring R:*

(1) *R is a QF ring.*

(2) *R is a left AFG, left and right dual ring.*

(3) *R is a left AFG and left Pseudo-Frobenius ring.*

Proof. $(1) \Rightarrow (2)$ is clear.

$(2) \Rightarrow (1)$ First, R is a left Noetherian ring since R is left AFG and left dual. In addition, R is a left self-injective ring by [9, Lemma 3.1] and [11, Theorem 1]. Thus R is a QF ring.

$(1) \Leftrightarrow (3)$ is easy. ■

The following implications are obvious:

"left Noetherian ring $\Rightarrow$ left AFG rings $\Rightarrow$ left pseudo-coherent rings".

The converses hold if R is a left CF ring as follows.

Proposition 3.2. *The following are equivalent for a left CF ring R:*

(1) *R is a left AFG ring.*

(2) *R is a left pseudo-coherent ring.*

(3) *R is a left Noetherian ring.*

Proof. It is enough to show that (2) $\Rightarrow$ (3). Let I be a left ideal of R. Then there is a monomorphism $f : R/I \to R^n$ for some $n \in \mathbb{N}$ since R is a left CF ring. Put $f(\overline{1}) = (a_1, a_2, \cdots, a_n)$. It is easy to check that $I = l\{a_1, a_2, \cdots, a_n\}$ and so I is finitely generated by (2). Thus R is a left Noetherian ring. ■

In general, a left AFG ring need not be left coherent although it is left pseudo-coherent, where a ring R is called *left coherent* if every finitely generated left ideal is finitely presented. For example, let $x, y_1, y_2, \cdots$ be indeterminates over a field K, $R = K[x^2, x^3, y_i, xy_i]$ and $S = K[x, y_i]$. Then R is a subring of the domain S, hence R is an AFG ring. But R is not a coherent ring by [8, p. 110].

The following result shows that a left AFG ring is left coherent if R is a right FP-injective ring, where R is called a *right FP-injective ring* [17] if $\text{Ext}^1_R(M, R) = 0$ for all finitely presented right R-modules M.

Proposition 3.3. *If R is a left AFG and right FP-injective ring, then R is a left coherent ring.*

Proof. It is clear that $l(a)$ is finitely generated for any $a \in R$. In addition, let I and J be two finitely generated left ideals of R. Then $I = l(X)$ and $J = l(Y)$ for some subsets X and Y of R by [12, Corollary 2.5] since R is a right FP-injective ring. Thus $I \cap J = l(X \cup Y)$ is finitely generated since R is a left AFG ring. So R is a left coherent ring by [4, Theorem 2.2]. ■

Recall that R is called a *left Baer ring* [14] if the left annihilator of each nonempty subset of R is a direct summand of ${}_RR$. It is easy to see that the Baer property is left-right symmetric. Thus any Baer ring is a left and right AFG ring. Note that the ring $\mathbb{Z}_4$ is an AFG ring which is not a Baer ring. However we have the following:

Proposition 3.4. *The following are equivalent for a ring R:*

(1) *R is a Baer and right FP-injective ring.*

(2) *R is a Baer and left FP-injective ring.*

(3) *R is a von Neumann regular and left AFG ring.*

(4) *R is a von Neumann regular and right AFG ring.*

Proof. (1) $\Rightarrow$ (3) and (4) Every finitely generated left ideal I is a left annihilator of a nonempty subset of R since R is right FP-injective, and so I is a direct summand of ${}_RR$ since R is a Baer ring. Thus R is a von Neumann regular ring.

The others are obvious. ■

It is well known that R is a von Neumann regular and left Noetherian ring if and only if R is semisimple Artinian. By Proposition 3.4, we have

Corollary 3.5. *The following are equivalent for a ring R:*

(1) *R is a semisimple Artinian ring.*

(2) *R is a von Neumann regular, left AFG and left dual ring.*

(3) *R is a von Neumann regular, left AFG and right dual ring.*

Lemma 3.6. *The following are equivalent for a ring R:*

(1) *R is a right CF ring.*

(2) *Every injective right R-module is singly projective.*

(3) *The injective envelope of any cyclic right R-module is singly projective.*

Proof. It is straightforward by Lemma 2.1. ∎

Theorem 3.7. *The following are equivalent for a left AFG ring R:*

(1) *R is a right CF ring.*

(2) *R is a right dual ring.*

(3) *Every cyclic right R-module has a monic projective preenvelope.*

(4) *Every cyclic right R-module has a monic singly projective preenvelope.*

(5) *Every right R-module has a monic singly projective preenvelope.*

Proof. $(1) \Rightarrow (5)$ Let M be any right R-module. Then M has a singly projective preenvelope $f : M \to F$ by Theorem 2.3. Note that M embeds in a singly projective right R-module by Lemma 3.6 since M embeds in its injective envelope. So f is a monomorphism.

$(5) \Rightarrow (4)$ is trivial.

$(4) \Rightarrow (3)$ Let M be a cyclic right R-module. Then M has a monic singly projective preenvelope $f : M \to P$ by (4). Thus, by Lemma 2.1, there exist a finitely generated free right R-module F, a monomorphism $g : M \to F$ and a homomorphism $h : F \to P$ such that $f = hg$. It is easy to verify that g is a monic projective preenvelope.

$(3) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (1)$ Let M be a cyclic right R-module. Then there is an exact sequence $0 \to M \to R_R^I$ for some index set I by (2). Note that R_R^I is singly projective by Theorem 2.3. So M embeds in a finitely generated free right R-module, that is, R is a right CF ring. ∎

Remark 3.8. (1) In [3], Björk constructed a two-sided Artinian and one-sided dual ring which is not QF. So a left AFG left CF ring or a left AFG right CF ring need not be QF. However a left AFG two-sided CF ring is QF by

Proposition 3.1. Thus the left AFG left CF rings are different from the left AFG right CF rings.

(2) We note that every cyclic $\mathbb{Z}$-module has a projective preenvelope, but not every cyclic $\mathbb{Z}$-module has a monic projective preenvelope. Indeed, it is easy to see that the cyclic $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ $(n > 1)$ does not have a monic projective preenvelope since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, P) = 0$ for any projective $\mathbb{Z}$-module P.

Recall that R is called a *right PP ring* if every principal right ideal of R is projective. PP rings have been studied in many articles such as [10, 15, 16, 19, 21]. Here we characterize PP rings in terms of (singly) projective preenvelopes.

Theorem 3.9. *The following are equivalent for a left AFG ring R:*

(1) *R is a right PP ring.*

(2) *Any submodule of a singly projective right R-module is singly projective.*

(3) *Every cyclic right R-module has an epic projective preenvelope.*

(4) *Every cyclic right R-module has an epic singly projective preenvelope.*

(5) *Every right R-module has an epic singly projective preenvelope.*

(6) *Every torsionless right R-module is singly projective.*

Proof. $(1) \Rightarrow (2)$ Suppose that N is a submodule of a singly projective right R-module L, and M is a cyclic submodule of N. Let $\lambda : N \to L$ and $\iota : M \to N$ be the inclusions. Since L is singly projective, $\lambda\iota$ factors through a finitely generated free right R-module H by Lemma 2.1. So there exist $g : M \to H$ and $h : H \to L$ such that $\lambda\iota = hg$. It is clear that g is a monomorphism. Without loss of the generality, we may assume that g is an inclusion. Suppose that $\{e_i : 1 \le i \le n\}$ is the basis of H. We will show that M is projective by induction on the number n.

If $n = 1$, then it is clear by (1).

Now suppose that it is true for $n-1$. Let $Q = e_1R + e_2R + \cdots + e_{n-1}R$. For any $x \in M$, there is a unique factorization:

$$x = y + e_n r, \qquad y \in Q, r \in R.$$

Define $\alpha : M \to R$ via

$$\alpha(x) = r, \qquad x \in M.$$

Then α is well-defined, and so we obtain the exact sequence

$$0 \to M \cap Q \to M \to \mathrm{im}(\alpha) \to 0.$$

Note that the sequence is split since $\mathrm{im}(\alpha)$ is projective by (1). So $M \cap Q$ is a direct summand of M, and hence is a cyclic submodule of Q. Thus $M \cap Q$ is projective by the induction hypothesis. It follows that M is projective. Consequently N is singly projective by Lemma 2.1.

(2) $\Rightarrow$ (1) is easy since a cyclic singly projective module is projective.

(4) $\Rightarrow$ (3) Suppose that every cyclic right R-module N has an epic singly projective preenvelope $f : N \to F$. Then f factors through a projective right R-module P, that is, there exist $g : N \to P$ and $h : P \to F$ such that $f = hg$. On the other hand, since P is singly projective, there exists $\alpha : F \to P$ such that $g = \alpha f$. Thus $f = h\alpha f$, and so $h\alpha = 1$ since f is epic. Therefore F is isomorphic to a direct summand of P, and hence is projective. It follows that f is an epic projective preenvelope of N.

(3) $\Rightarrow$ (2) Suppose that N is a submodule of a singly projective right R-module L, and M is a cyclic submodule of N. Let $\lambda : N \to L$ and $\iota : M \to N$ be the inclusions. Then there exist a finitely generated free right R-module H, $g : M \to H$ and $h : H \to L$ such that $\lambda\iota = hg$. By (3), M has an epic projective preenvelope $\beta : M \to Q$. Thus there exists $\gamma : Q \to H$ such that $g = \gamma\beta$, and so $\lambda\iota = h\gamma\beta$. Thus β is a monomorphism and hence an isomorphism. Therefore M is projective, which implies that N is singly projective.

(2) $\Rightarrow$ (5) For any right R-module M, there is a singly projective preenvelope $f : M \to F$ by Theorem 2.3. Note that $\mathrm{im}(f)$ is singly projective by (2), so $M \to \mathrm{im}(f)$ is an epic singly projective preenvelope.

(5) $\Rightarrow$ (6) Let M be a torsionless right R-module. Then there is an exact sequence $0 \to M \to R_R^I$ for some index set I. Note that R_R^I is singly projective by Theorem 2.3. Thus M is singly projective since M has an epic singly projective preenvelope.

(6) $\Rightarrow$ (4) Let M be a cyclic right R-module. Then M has a singly projective preenvelope $\alpha : M \to F$. So there exist a projective right R-module P, $\beta : M \to P$ and $\gamma : P \to M$ such that $\alpha = \gamma\beta$. Note that $\mathrm{im}(\beta)$ is torsionless and hence singly projective by (6), so $M \to \mathrm{im}(\beta)$ is an epic singly projective preenvelope. ∎

We end this paper with the following

Remark 3.10. (1) Recall that R is called a *right* PF *ring* if every principal right ideal of R is flat. Obviously, the concept of PF rings is a generalization of PP rings. The property that R is a PF ring is left-right symmetric (see [13]), but there exists a right PP ring which is not left PP (see [15]). However, if R is a left and right AFG ring, then we claim that R is a right PP ring if and only if R is a left PP ring. In fact, it is enough to note that every principal right or principal left ideal I is finitely presented since R is a left and right AFG ring, and so I is projective if and only if I is flat.

(2) Let $R = \mathbb{Z}_4$. Then R is a commutative QF ring. So projective R-modules coincide with injective R-modules. Thus every cyclic R-module has a projective

preenvelope. But the cyclic R-module $\{0, \overline{2}\}$ does not have an epic projective preenvelope since $\{0, \overline{2}\}$ is not a projective R-module.

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