TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 4, pp. 1189-1201, September 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

THE DUAL NOTION OF MULTIPLICATION MODULES

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Abstract. Let R be a ring with an identity (not necessary commutative) and let M be a left R-module. In this paper we will introduce the concept of a comultiplication R-module and we will obtain some related results.

1. Introduction

Throughout this paper R will denote a ring with an identity (not necessarily commutative) and all modules are assumed to be left R-modules. Further "$\subset$" will denote the strict inclusion and $\mathbb{Z}$ denote the ring of integers. Let M be a left R-module and let $S := End_R(M)$ be the endomorphism ring of M. Then M has a structure as a right S-module so that M is an $R-S$ bimodule. If $f : M \to M$ and $g : M \to M$, then $fg : M \to M$ defined by $m(fg) = (mf)g$. Also for a submodule N of M,

$$I^N := \{f \in S : Im(f) = Mf \subseteq N\}$$

and

$$I_N := \{f \in S : N \subseteq Ker(f)\}$$

are respectively a left and a right ideal of S. Further a submodule N of M is called ([4]) an open (resp. a closed) submodule of M if $N = N^\circ$, where $N^\circ = \sum_{f\in I^N} Im(f)$ (resp. $N = \bar{N}$, where $\bar{N} = \cap_{f\in I_N} Ker(f)$). A left R-module M is said to be self generated (resp. self cogenerated) if each submodule of M is open (resp. closed).

Let M be an R-module. M is said to be a multiplication (resp. openly multiplication) R-module if for every submodule N of M there exists a two sided ideal I

Received August 3, 2005, accepted March 16, 2006.
Communicated by Shun-Jen Cheng.
2000 *Mathematics Subject Classification*: 13C99.
Key words and phrases: Endomorphism, Multiplication modules, Comultiplication modules, Closedly comultiplication modules.

of R such that $N = IM$ (resp. $N^\circ = IM$). Recently a large body of research has been done about the left mutiplication R-modules haveing right $End_R(M)$-modules structures.

Now let M be an R-module. The purpose of this paper is to introduce the concept of comultiplication (resp. closedly comultiplication) R-modules (the dual notion of multiplication or openly multiplication R-modules). M is said to be a comultiplication (resp. closedly comultiplication) R-module if for every submodule N of M there exists a two sided ideal I of R such that $N = (0 :_M I)$ (resp. $\bar{N} = (0 :_M I)$). It is clear that every comultiplication R-module is closedly comultiplicaion. It is shown that the converse is not true in general. Also we have shown that M is a comultiplication R- module if and only if for each submodule N of M, $N = (0 :_M Ann_R(N))$. Furthermore, we will obtain another characterization for comultiplication R-modules (see 3.10) and it is shown, among the other results, that every submodule of a comultiplication R-module is a comultiplication R-module (see 3.17) and that every cocyclic module over a commutative complete Noetherian ring is a comultiplication module (see. 3.17).

2. Auxilarly Results

In this section we will provide the definitions and results which is necessary in the next section.

Definition 2.1.

(a) M is said to be (see [6]) a multiplication module if for any submodule N of M there exists a two sided ideal I of R such that $IM = N$.

(b) Let N be a non-zero submodule of M. Then N is said to be (see [1]) large or essential (resp. small) if for every non-zero submodule L of M, $N \cap L \neq 0$ (resp. $L + N = M$ implies that $L = M$).

(c) M is said to be (see [1]) couniform if each of its non-zero submodules is small.

(d) A submodule K of M is called fully invariant if $Kf \subseteq K$ for every $f \in End_R(M)$.

(e) Let R be a commutative ring. The non-zero submodule N of M is said to be (see [10]) second submodule of M if for each $a \in R$ the homothety $N \xrightarrow{a} N$ is either surjective or zero. This implies that $Ann_R(M) = P$ is a prime ideal of R.

(f) A non-zero module M over a ring R is said to be (see [2]) prime if the annihilator of M is the same as the annihilator of N for every non-zero submodule N of M.

(g) A non-zero module M over a ring R is said to be (see [2]) coprime if the annihilator of M is the same as the annihilator of Q for every non-zero (left) quotient Q of M.

(h) An R-module M is said to be distributive if the lattice of its submodule is distributive, i.e. $(X+Y)\cap Z = (X\cap Z)+(Y\cap Z)$ for any of its submodules X, Y and Z.

(i) Let R be a commutative ring. An R-module L is said to be cocyclic (see [8] and [9]) if $L \subseteq E(R/P)$ for some maximal ideal P of R.

Remark 2.2. (see [3]). Let R be a commutative Noetherian ring and let E be an injective R-module. Then we have $(0:_E (0:_R I)) = IE$.

Lemma 2.3. *Let R be a commutative ring and M an R-module. Let $S = End_R(M)$ be a domain. Then $Ann_R(M)$ is a prime ideal of R.*

Proof. Let I and J be ideals of the ring R and $IJ \subseteq Ann_R(M)$. Then $IJM = 0$. Now assume that $JM \neq 0$ and $IM \neq 0$. Hence there exist $a \in I$ and $b \in J$ such that $aM \neq 0$ and $bM \neq 0$. Consider the homotheties $M \xrightarrow{f_a} M$ and $M \xrightarrow{g_b} M$ defined respectively by $m \mapsto am$ and $m \mapsto bm$. Then

$$m(f_a g_b) = (mf_a)g_b = (am)g_b = bam = 0.$$

Hence $f_a g_b = 0$. Since S is a domain, $f_a = 0$ or $g_b = 0$. Therefore, $aM = 0$ or $bM = 0$. But this is a contradiction. Hence $IM = 0$ or $JM = 0$ so that $I \subseteq Ann_R(M)$ or $J \subseteq Ann_R(M)$.

3. Main Results

Definition 3.1. An R-module M is said to be a comultiplication module if for any submodule N of M there exists a two sided ideal I of R such that $N = (0:_M I)$.

Example 3.2. Let p be a prime number and consider the $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty)$ (we recall that $\mathbb{Z}$ is the ring of integers). Choose $N = \mathbb{Z}(1/p + \mathbb{Z})$ and Set $I = \mathbb{Z}p^i, i \geq 0$. It is clear that $N = (0:_M I)$. Therefore, $M = \mathbb{Z}(p^\infty)$ as a Z-module is a comultiplication module.

Definition 3.3. An R-module M is said to be a closedly comultiplication module if for any submodule N of M there exists a two sided ideal I of R such that $\bar{N} = (0:_M I)$.

Example 3.4 Let M be a duo R-module (i.e. every submodule of M is fully invariant). Now I_N is a two-sided ideal of S and it is easy to see that

$\bar{N} = (0 :_M I_N)$. Hence M as a right S-module is a closedly comultiplication S-module.

Remark 3.5. It is clear that every comultiplication module is a closedly comultiplication module. But the following example shows that the converse is not true.

Example 3.6. As we will show in example 3.9, $\mathbb{Z}$ as a $\mathbb{Z}$-module is not a comultiplication $\mathbb{Z}$-module. However since every non-zero endomorphism of $\mathbb{Z}$ is a monomorphism, for every $\mathbb{Z}$-submodule N of $\mathbb{Z}$, we have $\bar{N} = 0$ or $\bar{N} = N$. This shows that $\mathbb{Z}$ is a closedly comultiplication $\mathbb{Z}$-module.

Lemma 3.7. *An R module M is a comultiplication module if and only if for each submodule N of M, $N = (0 :_M Ann_R(N))$.*

Proof. The sufficiency is clear. Conversely, suppose that M is a comultiplication module. Then there exists a two sided ideal I of R such that $N = (0 :_M I)$. Then we have $I \subseteq Ann_R(N)$ so that $(0 :_M Ann_R(N)) \subseteq (0 :_M I) = N$. This implies that $N = (0 :_M Ann_R(N))$ as desired.

Example 3.8. Let R be a commutative semi-simple ring and let I be an ideal of R. Then it is clear that R is both injective and Noetherian as R-module. Hence by Remark 2.2, we have $(0 :_R Ann_R(I)) = IR = I$. Thus every semi-simple ring as a module over itself is a comultiplication module by 3.7.

Example 3.9. Let $M = \mathbb{Z}$ (as a $\mathbb{Z}$-module). For a submodule $2\mathbb{Z}$ of $\mathbb{Z}$ we have $(0 : Ann_{\mathbb{Z}}(2\mathbb{Z})) = \mathbb{Z}$. Therefore, $\mathbb{Z}$ is not a comultiplication module.

Theorem 3.10. *Let M be an R-module. Then the following are equivalent.*

(a) *M is a comultiplication module.*

(b) *For every submodule N of M and each two sided ideal C of R with $N \subset (0 :_M C)$, there exists a two sided ideal B of R such that $C \subset B$ and $N = (0 :_M B)$.*

(c) *For every submodule N of M and each two sided ideal C of R with $N \subset (0 :_M C)$, there exists a two sided ideal B of R such that $C \subset B$ and $N \subseteq (0 :_M B)$.*

Proof. $(a) \Rightarrow (b)$. Let N be a submodule of M and let C be a two sided ideal of R such that $N \subset (0 :_M C)$. Since M is a comultiplication module, $N = (0 :_M Ann_R(N))$. We set $B = C + Ann_R(N)$. Since $N = (0 :_M Ann_R(N)) \subset (0 :_M C)$, $Ann_R(N) \not\subset C$. Hence $C \subset B$ and we have

$(0 :_M B) = (0 :_M C + Ann_(RN)) = (0 :_M C) \cap (0 :_M Ann_R(N)) = N.$
The implication $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$. Let N be a submodule of M and let

$$H = \{D : D \ is \ a \ two \ sided \ ideal \ of \ R \ and \ N \subset (0 :_M D)\}$$

Clearly $0 \in H$. Let $\{B_i\}$, $i \in I$, be any non-empty collection of two sided ideals in H. By assumption, $\sum_{i \in I} B_i \in H$. By the Zorn's Lemma, H has a maximal member C so that $N \subseteq (0 :_M C)$. Assume that $N \neq (0 :_M C)$. Then by part (c), there exists a two sided ideal B with $C \subset B$ and $N \subseteq (0 :_M B)$. But this is a contradiction by the choice of C. Thus we have $N = (0 :_M C)$. This shows that M is a comultiplication R-module.

Theorem 3.11. *Let R be a commutative ring and let M be a comultiplication R-module. Then*

(a) *M is a self-cogenerated R-module*

(b) *If N is a submodule of M such that $Ann_R(N)$ is a prime ideal of R, then N is a second submodule of M.*

Proof.

(a) Let N be submodule of a comultiplication R-module M. Then there exists an ideal I of R such that $N = (0 :_M I)$. For each $a \in I$, define the map $f_a : M \to M$ by $m \mapsto am$. Since R is a commutative ring, f_a is an R-endomorphism. It is clear that for each $a \in I$, $N \subseteq Ker(f_a)$ and we have

$$\bar{N} = \cap_{f \in I_N} Ker(f) \subseteq \cap_{a \in I} Ker(f_a) = N.$$

Hence $N = \bar{N}$ as desired.

(b) Set $P = Ann_R(N)$. Since M is a comultiplication R-module, $N = (0 :_M P)$. Let $\phi_a : N \to N$ be the non-zero R-homomorphism defined by $n \mapsto an$. Let $K = Im\phi_a = aN$. It is clear that $0 \neq K \subseteq N$. By Theorem 3.10, there exists a two sided ideal B of R such that $P \subset B$ and $K = (0 :_M B)$. It follows that $Ba \subseteq Ann_R(N)$. Since $Ann_R(N)$ is a prime ideal of R and $P \subset B$, we have $a \in Ann_R(N)$ so that $aN = 0$. This is a contradiction and the proof is completed.

Corollary 3.12. *Let R be a commutative ring and let M be an R-module. Then M is a comultiplication module if and only if it is self cogenerated and closedly comultiplication R-module.*

Proof. This is an immediate consequence of 3.11 (a) and 3.5.

Corollary 3.13. *Let R be a commutative ring and let M be a comultiplication R-module. Futher let N be a submodule of M. Then The following are equivalent.*

(a) *N is second submodule of M.*

(b) *$Ann_R(N)$ is a prime ideal of R.*

Proof. Use 3.11 (b) and 2.1 (e).

Proposition 3.14. *Let M be a comultiplication R-module.*

(a) *Let $\{M_\lambda\}$, $\lambda \in \Lambda$, be a family of submodule of module M with $\cap_{\lambda\in\Lambda} M_\lambda = 0$. Then for every submodule N of M, we have*

$$N = \cap_{\lambda\in\Lambda}(N + M_\lambda).$$

(b) *Let P be a minimal two sided ideal of R such that $(0 :_M P) = 0$. Then M is cyclic.*

Proof.

(a) Let N be a submodule of M. Then

$$\begin{aligned} N &= (0 :_M Ann_R(N)) = (\cap_{\lambda\in\Lambda} M_\lambda :_M Ann_R(N)) \\ &= \cap_{\lambda\in\Lambda}(M_\lambda :_M Ann_R(N)) \supseteq \cap_{\lambda\in\Lambda}(N + M_\lambda) \supseteq N. \end{aligned}$$

It follows that

$$N = \cap_{\lambda\in\Lambda}(N + M_\lambda).$$

(b) Let $0 \neq m \in M$. Since M is a comultiplication R-module, there exists a two sided ideal I of R such that $Rm = (0 :_M I)$ and hence

$$Rm = (0 :_M I) = ((0 :_M P) :_M I) = (0 :_M PI).$$

Now since P is a minimal ideal of R and $0 \subseteq PI \subseteq P$, we have $PI = 0$ or $PI = P$. If $PI = P$, then

$$Rm = (0 :_M PI) = (0 :_M P) = 0.$$

This implies that $m = 0$ which is a contradiction. Hence we have $PI = 0$ so that $Rm = M$ as desired.

Lemma 3.15. *Let M be a faithful comultiplication module over a commutative ring R. Then $W(M) = Z(R)$, where*

$$W(M) = \{a \in R : \textit{ the homothety } M \xrightarrow{a} M \textit{ is not surjective}\}$$

(here $Z(R)$ denotes the set of zero divisors of R).

Proof. Let $a \in W(M)$ and suppose that the homothty $M \xrightarrow{a} M$ defined by $m \mapsto am$ is not surjective. Then Since M is a comultiplication R-module, there exists a two sided ideal I of R such that $aM = (0 :_M I)$. Hence we have $IaM = 0$ so that $Ia \subseteq Ann_R(M) = 0$. Thus $Ia = 0$. It follows that $a \in Z(R)$. Conversely let $a \in Z(R)$. Then there exists $0 \neq b \in R$ such that $ab = 0$. Thus we have $(ab)M = (bR)(aM) = 0$. This implies that $aM \subseteq (0 :_M bR) \neq M$ because M is faithful R-module. Therefore, $aM \neq M$ so that $a \in W(M)$.

Lemma 3.16. *Let R be a ring such that the lattice of two sided ideals of R is distributive and let M be a comultiplication R-module such that for any two sided ideal B and C of R, $(0 :_M B) + (0 :_M C) = (0 :_M B \cap C)$. Then M is a distributive module.*

Proof. Let X, Y, and Z be three submodules of M. Since M is a comultiplication module, there exist two sided ideals B, C and D of R such that $X = (0 :_M B)$, $Y = (0 :_M C)$ and $Z = (0 :_M D)$. Then

$$(X + Y) \cap Z = ((0 :_M B) + (0 :_M C)) \cap (0 :_M D) = (0 :_M B \cap C) \cap (0 :_M D)$$

$$= (0 :_M (B \cap C) + D) = (0 :_M (B + D) \cap (C + D)) = (0 :_M B + D) + (0 :_M C + D)$$

$$= ((0 :_M B) \cap (0 :_M D)) + ((0 :_M C) \cap (0 :_M D)) = (X \cap Z) + (Y \cap Z).$$

Theorem 3.17. *Let M be a comultiplication R-module. Then the following assertions hold.*

(a) *Every submodule of M is fully invariant.*

(b) *If R is a commutative ring, then $End_R(M)$ is a commutative ring.*

(c) *If M is faithful, then M is divisible.*

(d) *Every submodule of M is a comultiplication module.*

(e) *If R is a complete Noetherian local ring, then every cocyclic R-module is a comultiplication R-module.*

Proof.

(a) Let N be a submodule of a comultiplication R-module M. Then there exists a two sided ideal I of R such that $N = (0 :_M I)$. Suppose that $f : M \to M$ be an endomorphism. Since $IN = 0$, $I \subseteq Ann_R(Nf)$ so that

$$(0 :_M Ann_R(Nf)) \subseteq (0 :_M I) = N.$$

This implies that $Nf \subseteq N$.

(b) Let f and g be two endomorphisms of M and let $m \in M$. Then we have $mf \in (Rm)f$ and $mg \in (Rm)g$. But by part (a), $Rm(f) \subseteq Rm$ and $Rm(g) \subseteq Rm$. Thus, $mf, mg \in Rm$. So there exist elements $a, b \in R$ such that $mf = am$ and $mg = bm$. Hence we have

$$m(fg - gf) = mf(g) - mg(f) = am(g) - bm(f)$$
$$= bam - abm = abm - abm = 0.$$

It follows that $End_R(M)$ is a commutative ring.

(c) Let c be a regular element. Then since M is a comultiplication R-module, there exists a two sided ideal I of R such that $cM = (0 :_M I)$. Since M is a faithful R-module, we have $IcM = 0$ so that $Ic = 0$. This implies that $I = 0$ because c is a regular element. Therefore, $cM = M$.

(d) Let M be a comultiplication R- module and let N be a submodule of M. Let K be a submodule of N. Then there exists a two sided ideal I of R such that $K = (0 :_M I)$. But we have $K = (0 :_M I) = (0 :_N I)$. Therefore, N is a comultiplication module.

(e) Let P be the unique maximal ideal of R. Since every cocyclic R-module is a submodule of $E_R(R/P)$, by using part (d), it is enough to prove that $E_R(R/P)$ is a comultiplication R-module. Now by using 3.7, it is enough to prove that for every submodule L of $E_R(R/P)$, $L = (0 :_{E_R(R/P)} Ann_R(L))$. To see this, set $\bar{R} = R/Ann_R(L)$, $\bar{P} = P/Ann_R(L)$, $\bar{E} = E_{\bar{R}}E(\bar{R}/\bar{P})$, and $\bar{H} = (0 :_{E_R(R/P)} Ann_R(L))$. Then $\bar{H}$ has a structure as $\bar{R}$-module and as such is isomorphic to $\bar{E}$. Now, as R and $\bar{R}$ module, $L \subseteq \bar{H}$ and L is a faithful $\bar{R}$-module. Hence by applying $Hom_{\bar{R}}(-, \bar{E})$ to the exact sequence

$$0 \to L \to \bar{H} \to \bar{H}/L \to 0$$

one can see, as in the proof of [6, 2.3], that

$$Hom_{\bar{R}}(\bar{H}/L, \bar{E}) = 0.$$

This implies that $H/L = 0$ as desired.

Proposition 3.18. *Let M be an R-module. Then the following assertions hold.*

(a) *If M is a comultiplication prime R-module, then M is a simple module.*

(b) *If M is a multiplication coprime R-module, then M is a simple module.*

(c) *Let R be a domain and let M be a faithful multiplication and comultiplication R- module. Then M is simple.*

Proof.

(a) Let N be a non-zero submodule of M. Since M is a prime module, we have $Ann_R(N) = Ann_R(M)$. Thus $(0 :_M Ann_R(N)) = (0 :_M Ann_R(M))$. Now by using Lemma 3.7 we have

$$N = (0 :_M Ann_R(N)) = (0 :_M Ann_R(M)) = M.$$

Therefore, M is a simple module.

(b) Let M be a proper submodule of M. Since M is a coprime module, we have $Ann_R(M) = Ann_R(M/N)$. Thus $Ann_R(M)M = Ann_R(M/N)M$. But $Ann_R(M/N)M = N$ by [5]. Hence M is a simple module.

(c) Let N be a submodule of a faithful multiplication and comultiplication R-module M. Then, there exist two sided ideals I and J of R such that $N = (0 :_M J)$ and $N = IM$. It follows that $JN = 0$ so that $JIM = 0$. This implies that $JI \subseteq Ann_R(M) = 0$. So we have $JI = 0$. Since R is a domain, $I = 0$ or $J = 0$. Therefore, $N = M$ or $N = 0$.

Theorem 3.19. *Let M be a closedly comultiplication R-module and $S = End_R(M)$. Then we have the following.*

(a) *If N is a non-zero fully invariant second submodule of M, then I_N is a prime ideal of S.*

(b) *If S is a domain and N is a closed submodule of M, then $I^N = S$ or I^N is a prime ideal of S.*

Proof.

(a) Since $Id_M \in S$ and $Id_M \notin I_N$, $I_N \neq S$. Further since N is a fully invariant submodule of M, I_N is a two sided ideal of S. Now let $fSg \subseteq I_N$, where $f, g \in S$. Then $fg \in I_N$. There exist two sided ideals I and J of R such that $Kerf = (0 :_M I)$ and $Kerg = (0 :_M J)$. Now $fg \in I_N$ implies that $N \subseteq Kerfg$. so that $N(fg) = 0$. Hence $Nf \subseteq Kerg = (0 :_M J)$. It follows that $0 = J(Nf)$ so that

$$JN \subseteq Kerf = (0 :_M I).$$

This implies that $IJN = 0$ so that $IJ \subseteq Ann_R(N)$. So we have $J \subseteq Ann_R(N)$ or $I \subseteq Ann_R(N)$ because $Ann_R(N)$ is a prime ideal of R by 2.1 (e). From this we have

$$N \subseteq (0 :_M J) = Kerg \text{ or } N \subseteq (0 :_M I) = Kerf.$$

Therefore, $f \in I_N$ or $g \in I_N$ as desired.

(b) Let $I^N \neq S$. Then we show that I^N is a prime ideal of S. To see this let $fSg \subseteq I^N$. Since $1 \in S$, $fg \in I^N$. It implies that $(M)fg \subseteq N$. Also there exist two sided ideals I, J, and K of R such that $N = (0 :_M I)$, $Ker(f) = (0 :_M J)$, and $Ker(g) = (0 :_M K)$. Hence we have

$$M(fg) \subseteq N = (0 :_M I).$$

Thus $I(M(fg)) = ((IM)f)g = 0$. This implies that

$$(IM)f \subseteq Ker(g) = (0 :_M K).$$

Hence we have $(KIM)f = 0$ so that

$$(KIM) \subseteq Ker(f) = (0 :_M J).$$

It follows that $JKI \subseteq Ann_R(M)$. Since S is a domain, $Ann_R(M)$ is a prime ideal of R so that $I \subseteq Ann_R(M)$ or $J \subseteq Ann_R(M)$ or $K \subseteq Ann_R(M)$. Hence $N = M$ or $(0 :_M J) = M$ or $(0 :_M K) = M$. So we have $I^N = S$ or $Ker(f) = M$ or $Ker(g) = M$. Since $I^N \neq S$, we have $Ker(f) = M$ or $Ker(g) = M$. If $Ker(f) = M$, then $Mf = 0 \subseteq N$, so $f \in I^N$. If $Ker(g)=M$, then $Mg=0\subseteq N$, so $g\in I^N$. Hence I^N is a prime ideal of S.

Corollary 3.20. *Let M be a closedly comultiplication second R-module. Then $S = End_R(M)$ is a prime ring.*

Proof. It is enough to prove that the zero ideal of S is a prime ideal. But by Theorem 3.19, $I_M = 0$ is a prime ideal of S as desired.

Corollary 3.21. *Let R be a commutative ring and M be a comultiplication R-module. Then the following are equivalent.*

(a) *$S = End_R(M)$ is a domain.*

(b) *$Ann_R(M)$ is a prime ideal of R.*

Proof. Use 2.3, 3.20 and 3.17 (b).

Definition 3.22. Let M be a comultiplication R-module and let I is a two sided ideal of R. Then $(0 :_M I)$ is said to be coidempotent if $(0 :_M I) = (0 :_M I^2)$.

Example 3.23. Let R be a Noetherian ring and I be an ideal of R. Then there exists a positive integer h such that $(0 :_R I^h) = (0 :_R I^{h+i})$ for all $i \geq 0$. Set $I^h = J$. Then we have $(0 :_R J) = (0 :_R J^2)$. Hence R has a coidempotent R-submodule.

Theorem 3.24. *Let M be a comultiplication R-module and let $S = End_R(M)$ be a domain. Then we have the following.*

(a) *Each non-zero endomorphism of M is an epimorphism.*

(b) *M doesn't have any nontrivial open submodule.*

(c) *If R is a commutative ring, then M is a couniform R-module.*

(d) *Each closed maximal submodule of M is coidempotent.*

Proof.

(a) Let $0 \neq f : M \to M$ be an endomorphism of M. Then there exist two sided ideals I and J of R such that $Mf = (0 :_M I)$ and $Ker(f) = (0 :_M J)$. So we have

$$0 = I(0 :_M I) = I(Mf) = (IM)f.$$

This implies that

$$IM \subseteq Ker(f) = (0 :_M J).$$

Therefore, $JIM \subseteq (0 :_M J)J = 0$ so that $JI \subseteq Ann_R(M)$. Since S is a domain by 2.3, we have $J \subseteq Ann_R(M)$ or $I \subseteq Ann_R(M)$. Now by using 3.7, $Ker(f) = (0 :_M J) = M$ or $Mf = (0 :_M I) = M$. Since $f \neq 0$, $Mf = M$.

(b) Suppose that N be a non-zero open submodule of M. Then we have

$$N = N^\circ = \sum_{f \in I^N} Im(f).$$

Since $0 \neq N$, there exists $0 \neq f \in S$ such that $0 \neq Mf \subseteq N$. But by part (a) $Mf = M$. So $N = M$.

(c) Let $N + K = M$, where, N and K are proper submodule of M. But since every comultiplication module over a commutative ring is a self cogenerated by 3.11 (a), there exist $0 \neq f, g \in S$ such that $N \subseteq Ker(f)$ and $K \subseteq Ker(g)$. Now we have $fg \neq 0$ because S is a domain and $f,\ g \neq 0$. Now we have

$$(N + K)(fg) = N(fg) + K(fg) = M(fg).$$

It follows that $K(fg) = M(fg)$ so that

$$M(fg) = K(fg) \subseteq Kg = 0.$$

So we have $fg = 0$. But this is a contradiction. Hence $N = M$ or $K = M$ as desired.

(d) Let N be a closed maximal submodule of M. Then we have,

$$M \neq N = \bar{N} = \cap_{f \in I_N} Ker(f).$$

So there exists $0 \neq f \in S$ such that $N \subseteq Ker(f)$. But $Ker(f) \neq M$ implies that $N = Ker(f)$ because N is a maximal closed submodule of M. On the other hand $Ker f \subseteq Ker f^2 \subseteq M$ yields that $Ker f^2 = M$ or $Ker f^2 = Ker f$. But since S is a domain, $Ker f^2 \neq M$. Thus, $Ker f^2 = Ker f$. Now suppose that I is a two sided ideal of R such that $Ker f = (0 :_M I)$. Then we have $Ker f^2 = (0 :_M I^2)$ because

$$m \in Ker f^2 \Leftrightarrow m(f^2) = 0 \Leftrightarrow mf \in Ker f = (0 :_M I) \Leftrightarrow$$

$$I(mf) = 0 \Leftrightarrow Im \subseteq Ker f$$

$$\Leftrightarrow I^2 m = 0 \Leftrightarrow m \in (0 :_M I^2).$$

Hence $(0 :_M I) = Ker f = Ker f^2 = (0 :_M I^2)$. This implies that N is a coidempotent submodule of M.

Question 3.25. Let R a commutative ring and let M be a cocyclic R-module. Is M a comultiplicatin R-module?

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