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ON THE STRONGLY $p-$SUMMING SUBLINEAR OPERATORS

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Abstract. Let $\mathcal{SB}(X, Y)$ be the set of the bounded sublinear operators from a Banach space X into a complete Banach lattice Y. In the present paper, we introduce to this category the concept of strongly $p-$summing sublinear operators. We give an analogue to Pietsch's domination theorem and study some comparisons between linear and sublinear operators.

0. INTRODUCTION

Pietsch has shown in [Pie 67, p. 338] that the identity from l_1 into l_2 is $2-$absolutely summing but the adjoint operator is not $2-$absolutely summing. For this, the concept of strongly $p-$summing linear operators ($1 \leq p < \infty$) was introduced by J. S. Cohen [6] as a characterization of the conjugates of absolutely p^*-summing linear operators. An operator u between two Banach spaces X, Y is strongly $p-$summing for ($1 < p < \infty$) if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$, we have

$$\sum_{1\leq i\leq n} |\langle u(x_i), y_i^*\rangle| \leq C(\sum_{1\leq i\leq n} \|(x_i)\|^p)^{\frac{1}{p}} \sup_{y\in B_Y} (\sum_{1\leq i\leq n} |\langle y_i^*, y\rangle|^{p^*})^{\frac{1}{p^*}}. \tag{0.1}$$

The smallest constant C which is noted by $d_p(u)$, such that the inequality (0.1) holds, is called the strongly $p-$summing norm on the space $\mathcal{D}_p(X, Y)$ of all strongly $p-$summing linear operators from X into Y which is a Banach space. We have $\mathcal{D}_1(X, Y) = \mathcal{B}(X, Y)$, the vector space of all bounded linear operators from X into Y.

In this paper, we generalize this notion for the sublinear maps, and give an analogue to Pietsch's domination theorem for this category of operators which is

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one of the main result of this work. Cohen deduced the domination theorem simply from the adjoint operator which is p^*-summing. That is not the case for sublinear operators because we do not know the adjoint of a sublinear operator. We show it directly by using Ky Fan's lemma. We end this work by studying some relations between linear and sublinear operators concerning this notion.

In the first section, we give some basic definitions and terminology concerning Banach lattices. We also give some standard notations. In the second section, we announce some definitions and properties concerning sublinear operators. We introduce the definition of positive p-summing operators investigated by O. Blasco [3, 4, 5]. We generalize in the third section, the class of strongly $p-$summing operators introduced by Cohen in [6] to the sublinear operators. This category verifies a domination theorem similar to the linear case, which is the principal result. We use Ky Fan's lemma to show this property. In the linear case, it is obviously obtained because the adjoint operator is p^*-summing and consequently verifies the Pietsch's domination theorem.

We end in section four, by studying some relation between the strongly $p-$ summing sublinear operators T and the linear operators $u \in \nabla T$, where $\nabla T = \{u \in \mathcal{L}(X,Y) : u \leq T\}$ ($\mathcal{L}(X,Y)$ is the space of all linear operators from X into Y). We show that if T is strongly $p-$summing then u is positive strongly $p-$summing and consequently u^* is positive p^*-summing. For the converse, we add one condition concerning T.

1. Basic Definitions and Terminology

In this section we introduce some terminology concerning the Banach lattices. For more details, the interested reader can consult the references [8, 9].

We recall the abstract definition of Banach lattice. Let X be a Banach space. If X is a vector lattice and $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ ($|x| = \sup\{x, -x\}$) we say that X is a Banach lattice. If the lattice is complete, we say that X is a complete Banach lattice. Note that this implies obviously that for any $x \in X$ the elements x and $|x|$ have the same norm. We denote by $X_+ = \{x \in X : x \geq 0\}$. An element x of X is positive if $x \in X_+$.

The dual X^* of a Banach lattice X is a complete Banach lattice endowed with the natural order

$$x_1^* \leq x_2^* \Longleftrightarrow \langle x_1^*, x\rangle \leq \langle x_2^*, x\rangle, \quad \forall x \in X_+ \tag{1.1}$$

where $\langle ., .\rangle$ denotes the bracket of duality.

By a sublattice of a Banach lattice X we mean a linear subspace E of X so that $\sup\{x, y\}$ belongs to E whenever $x, y \in E$. The canonical embedding $i : X \longrightarrow X^{**}$ such that $\langle i(x), x^*\rangle = \langle x^*, x\rangle$ of X into its second dual X^{**} is an order isometry from X onto a sublattice of X^{**}, see [8, Proposition 1.a.2]. If we consider X as a sublattice of X^{**} we have for $x_1, x_2 \in X$

$$x_1 \leq x_2 \Longleftrightarrow \langle x_1, x^* \rangle \leq \langle x_2, x^* \rangle, \quad \forall x^* \in X_+^*. \tag{1.2}$$

The space $C(K)$ is a Banach lattice. The L_p $(1 \leq p \leq \infty)$ are complete Banach lattices.

Any reflexive Banach lattice is a complete Banach lattice.

Now let us give some standard notations. Let X be a Banach space and $1 \leq p \leq \infty$. We denote by $l_p(X)$ (resp. $l_p^n(X)$) the space of all sequences (x_i) in X with the norm

$$\|(x_i)\|_{l_p(X)} = (\sum_1^\infty \|x_i\|^p)^{\frac{1}{p}}$$
$$(\text{resp.} \left\|(x_i)_{1\leq i\leq n}\right\|_{l_p^n(X)} = (\sum_1^n \|x_i\|^p)^{\frac{1}{p}})$$

and by $l_p^\omega(X)$ (resp. $l_p^{n\ \omega}(X)$) the space of all sequences (x_i) in X with the norm

$$\|(x_n)\|_{l_p^\omega(X)} = \sup_{\|\xi\|_{X^*}=1} (\sum_1^\infty |\langle x_i, \xi \rangle|^p)^{\frac{1}{p}}$$
$$(\text{resp.} \|(x_n)\|_{l_p^{n\ \omega}(X)} = \sup_{\|\xi\|_{X^*}=1} (\sum_1^n |\langle x_i, \xi \rangle|^p)^{\frac{1}{p}}).$$

We know (see [7, p. 33]) that $l_p(X) = l_p^\omega(X)$ for some $1 \leq p < \infty$ if and only if $dim(X)$ is finite. If $p = \infty$, we have $l_\infty(X) = l_\infty^\omega(X)$. We have also if $1 < p \leq \infty$, $l_p^\omega(X) \equiv B(l_{p^*}, X)$ isometrically and $l_1^\omega(X) \equiv B(c_O, X)$ isometrically (where p^* is the conjugate of p *i.e.* $\frac{1}{p} + \frac{1}{p^*} = 1$). In other words, let $v : l_{p^*} \longrightarrow X$ be a linear operator such that $v(e_i) = x_i$ (namely $v = \sum\limits_1^\infty e_j \otimes x_j$, e_j denotes the unit vector basis of l_p) then

$$\|v\| = \|(x_n)\|_{l_p^\omega(X)}. \tag{1.3}$$

2. Sublinear Operators

For the convenience of the reader, we recall in this section some elementary definitions and fundamental properties relative to sublinear operators. For more details see [1, 2, 10].

Definition 2.1. A mapping T from a Banach space X into a Banach lattice Y is said to be sublinear if for all x, y in X and λ in $\mathbb{R}_+$, we have

(i) $T(\lambda x) = \lambda T(x)$ (i.e. positively homogeneous),

(ii) $T(x + y) \leq T(x) + T(y)$ (i.e. subadditive).

Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator.

Let us denote by

$$\mathcal{SL}(X, Y) = \{ \text{ sublinear mappings } T : X \longrightarrow Y\}$$

and we equip it with the natural order induced by Y

$$T_1 \leq T_2 \Longleftrightarrow T_1(x) \leq T_2(x), \quad \forall x \in X \tag{2.1}$$

and

$$\nabla T = \{u \in L(X, Y) : u \leq T(i.e. \forall x \in X,\, u(x) \leq T(x))\}.$$

The set ∇T is not empty by Proposition 2.3 below. As a consequence

$$u \leq T \Longleftrightarrow -T(-x) \leq u(x) \leq T(x), \quad \forall x \in X \tag{2.2}$$

and

$$\lambda T(x) \leq T(\lambda x), \quad \forall x \in \mathbb{R}. \tag{2.3}$$

Also we say that a sublinear operator T:
is symmetrical if for all x in X, $T(x) = T(-x)$,
is positive if for all x in X, $T(x) \geq 0$,
is increasing if for all x, y in X, $T(x) \leq T(y)$ when $x \leq y$.

The symmetry implies the positivity, the converse is false. Also, there is no relation between positivity and increasing like the linear case (a linear operator $u \in \mathcal{L}(X, Y)$ is positive if $u(x) \geq 0$ for $x \geq 0$).

Example.

(1) If $u : X \rightarrow Y$ is a linear operator from a Banach space X into a Banach lattice Y, then $T(x) = |u(x)|$ is a symmetrical sublinear operator.

(2) Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the torus equipped with the invariant measure $d\theta$ and X be the Hilbert space $L_2(\mathbb{T}, d\theta)$. For all r such that $0 < r \leq \pi$ and for all $f \in X$, we define a function $2\pi-$periodic $S_r\,(f) \geq 0$ by

$$S_r\,(f)\,(x) = \frac{1}{2\pi r}\int_{x-r}^{x+r} |f(y)|^2\, dy, \quad \text{for every } x \in \mathbb{R}.$$

Put $T_r\,(f) = \sqrt{S_r\,(f)}$. The operator T_r is sublinear and the operator T defined by $T\,(f) = \sup\{T_r\,(f) : 0 < r < \pi\}$, is a positive sublinear operator from $L_2(\mathbb{T}, d\theta)$ into $L_1(\mathbb{T}, d\theta)$. $T\,(f)$ is the square root of the maximal function Mf^2 (the Hardy-Littlewood maximal operator).

(3) Let X be a Banach space, Y be a Banach lattice. Consider T in $\mathcal{SL}(X,Y)$. If we put $\varphi(x) = \sup\{T(x), T(-x)\}$ then φ is a symmetrical sublinear operator.

Let T be a sublinear operator from a Banach space X into a Banach lattice Y. Then we have, T is continuous if and only if there is $C > 0$ such that for all $x \in X$, $\|T(x)\| \leq C\|x\|$.

In this case we also say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\|_{B_X} = 1\}$$

where B_X denotes the closed unit ball of X. We will denote by $\mathcal{SB}(X,Y)$ the set of all bounded sublinear operators from X into Y.

We will need the following remark.

Remark 2.2. Let X be an arbitrary Banach space. Let Y, Z be Banach lattices.

(i) Consider T in $\mathcal{SL}(X,Y)$ and u in $\mathcal{L}(Y,Z)$. Assume that u is positive (i.e., $u(x) \geq 0$ for every $x \in X_+$). Then, $u \circ T \in \mathcal{SL}(X,Z)$.

(ii) Consider u in $\mathcal{L}(X,Y)$ and T in $\mathcal{SL}(Y,Z)$. Then, $T \circ u \in \mathcal{SL}(X,Z)$.

(iii) Consider S in $\mathcal{SL}(X,Y)$ and T in $\mathcal{SL}(Y,Z)$. Assume that S is increasing. Then, $S \circ T \in \mathcal{SL}(X,Z)$.

The following proposition, will be used implicitly in the sequel. For the proof see [1, Proposition 2.3].

Proposition 2.3. *Let X be a Banach space and let Y be a complete Banach lattice. Let $T \in \mathcal{SL}(X,Y)$. Then, for all x in X there is $u_x \in \nabla T$ such that, $T(x) = u_x(x)$, ($i.e.$ the supremum is attained, $T(x) = \sup\{u(x) : u \in \nabla T\}$).*

As an immediate consequence of Proposition 2.3, we have for all $x \in X$

$$\|T(x)\| \leq \sup_{u \in \nabla T} \|u(x)\| \leq \|T(x)\| + \|T(-x)\| \tag{2.4}$$

and consequently

$$\|T\| \leq \sup_{u \in \nabla T} \|u\| \leq 2\|T\|. \tag{2.5}$$

Hence, the operator T is bounded if and only if for all $u \in \nabla T$, u is bounded.

We continue by giving the notion of positive $p-$summing operators as defined by Blasco in [3] generalized to sublinear operators (we can consult also [7, p.

343]). For the definition of $p-$summing sublinear operators and related properties, the reader can see [1].

Let X, Y be Banach lattices. Let $T : X \longrightarrow Y$ be a sublinear operator. We will say that T is "*positive* $p-$*summing*" ($0 \leq p \leq \infty$) (we write $T \in \Pi_p^+ (X, Y)$), if there exists a positive constant C such that for all $n \in \mathbb{N}$ and all $\{x_1, ..., x_n\} \subset X_+$, we have

$$\|(T(x_i))\|_{l_p^n(Y)} \leq C \|(x_i)\|_{l_p^{n\,\omega}(X)}. \tag{2.6}$$

We put

$$\pi_p^+(T) = \inf \{C \text{ verifying the above inequality}\}.$$

By using (1.3), the above definition can be reformulated as follow: the operator T is $p-$summing and $\pi_p(T) \leq C$, if and only if, for every n in $\mathbb{N}$ and every linear operator $v : l_{p^*}^n \longrightarrow X$ such that $v(e_i) \in X_+$, we have

$$\left(\sum_1^n \|Tv(e_i)\|^p\right)^{\frac{1}{p}} \leq C \|v\|. \tag{2.7}$$

Theorem 2.4. *A sublinear operator between a Banach space X and a Banach lattice Y is positive $p-$ summing ($1 \leq p < \infty$) if and only if there exists a positive constant $C > 0$ and a Borel probability μ on $(B_{X^*}^+, \sigma(X^*, X))$ ($B_{X^*}^+ = B_{X^*} \cap X_+^*$) such that*

$$\|T(x)\| \leq C(\int_{B_{X^*}^+} |\langle |x|, x^*\rangle|^p \, d\mu(x^*))^{\frac{1}{p}} \tag{2.8}$$

for every $x \in X$. Moreover, in this case

$$\pi_p^+(T) = \inf \{C > 0 : \textit{ for all } C \textit{ verifying the inequality } (2.8)\}.$$

Proof. It is similar to the linear case (see [3] and [12, p. 244]). ■

Consequently, positive p_1-summing implies positive p_2-summing for $p_1 \leq p_2$. If T is positive $p-$summing then u is positive $p-$summing for all $u \in \nabla T$ and by (2.4), we have $\pi_p(u) \leq 2\pi_p(T)$. We do not know if the converse is true.

3. Strongly p-Summing Sublinear Operators

We introduce the following generalization of the class of strongly $p-$summing linear operators defined in 1973 by Cohen [6]. We give the domination theorem like the linear case but the proof is totally different because in the linear case, Cohen obtained it from the adjoint operator which is p^*-summing. In the sublinear case it is not possible. We use Ky Fan's lemma for showing it.

Definition 3.1. Let X be a Banach space and Y be a Banach lattice. A sublinear operator $T : X \longrightarrow Y$ is strongly $p-$summing ($1 < p < \infty$), if there is a positive constant C such that for any $n \in \mathbb{N}, x_1, ..., x_n \in X$ and $y_1^*, y_2^*, ..., y_n^* \in Y^*$, we have

$$\|(\langle T(x_i), y_i^* \rangle)\|_{l_1^n} \leq C \|(x_i)\|_{l_p^n(X)} \|(y_i^*)\|_{l_{p^*}^n \omega}. \tag{3.1}$$

We denote by $\mathcal{D}_p(X, Y)$ the class of all strongly $p-$summing sublinear operators from X into Y and $d_p(T)$ the smallest constant C such that the inequality (3.1) holds. For the definition of strongly positive $p-$summing, we replace Y^* by Y_+^* and $d_p(T)$ by $d_p^+(T)$.

Let $T \in \mathcal{SB}(X, Y)$ and $v : l_p^n \longrightarrow Y^*$ be a bounded linear operator. The sublinear operator T is strongly $p-$summing if and only if

$$\sum_{i=1}^{n} |\langle T(x_i), v(e_i) \rangle| \leq C(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \|v\|. \tag{3.2}$$

For $p = 1$, we have $\mathcal{D}_1(X, Y) = \mathcal{SB}(X, Y)$.

Now, we give an example of a strongly $p-$summing sublinear operator. Let $1 < p < \infty$ and $n, N \in \mathbb{N}$. Let u be a linear operator from l_2^n into l_p^N. Then the sublinear operator $T(x) = |u(x)|$ is strongly $2-$summing. Indeed, let m be an integer. Consider $(x_j)_{1 \leq j \leq m} \subset l_2^n$ and $\left(y_j^*\right)_{1 \leq j \leq m} \subset l_{p^*}^N$ ($\frac{1}{p} + \frac{1}{p^*} = 1$). Using (1.1) and (1.2), we have by taking $x_j = \sum_{i=1}^{n} a_{ij} e_i$

$$\begin{aligned} x_j &= \sum_{j=1}^{m} \left| \left\langle T(x_j), y_j^* \right\rangle \right| \\ &\leq \sum_{i,j=1}^{m,n} |a_{ij}| \left\langle T(e_i), \left| y_j^* \right| \right\rangle. \end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned} &\sum_{j=1}^{m} \left| \left\langle T(x_j), y_j^* \right\rangle \right| \\ &\leq (\sum_{i,j=1}^{m,n} |a_{ij}|^2)^{\frac{1}{2}} (\sum_{i,j=1}^{m,n} \left\langle T(e_i), \left| y_j^* \right| \right\rangle^2)^{\frac{1}{2}} \\ &\leq (\sum_{j=1}^{m} \|x_j\|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} \|u(e_i)\|^2 \sum_{j=1}^{m} (\left\langle \frac{T(e_i)}{\|u(e_i)\|}, \left| y_j^* \right| \right\rangle)^2)^{\frac{1}{2}} (\|u(e_i)\| = \|T(e_i)\|) \\ &\leq (\sum_{j=1}^{m} \|x_j\|^2)^{\frac{1}{2}} \pi_2(u) \sup_{\|y^{**}\|_{l_p^N} = 1} (\sum_{j=1}^{m} \left| \left\langle y^{**}, \left| y_j^* \right| \right\rangle \right|^2)^{\frac{1}{2}} \\ &\leq (\sum_{j=1}^{m} \|x_j\|^2)^{\frac{1}{2}} \pi_2(u) C\left(l_p^N\right) \sup_{\|y^{**}\|_{l_p^N} = 1} (\sum_{j=1}^{m} \left| \left\langle y^{**}, y_j^* \right\rangle \right|^2)^{\frac{1}{2}}. \end{aligned}$$

This concludes that $d_2(T) \leq \pi_2(u)C\left(l_p^N\right)$, because we have u of finite rank and consequently $2-$summing ($C\left(l_p^N\right)$ is a constant depending only on l_p^N). We can add that, for every q^*-summing ($1 < q < \infty$) linear operator u from l_q into l_p^N then the sublinear operator T is strongly $q-$summing.

Proposition 3.2. *Let X be Banach space and Y, Z be two Banach lattices. Let $T \in \mathcal{SB}(X,Y)$, R be a positive operator in $\mathcal{B}(Y,Z)$ and $S \in \mathcal{B}(E,X)$.*

(a) *If T is strongly $p-$summing sublinear operator, then $R \circ T$ is strongly $p-$summing sublinear operator and $d_p(R \circ T) \leq \|R\| \, d_p(T)$.*

(b) *If T is strongly $p-$summing sublinear operator, then $T \circ S$ is strongly $p-$summing sublinear operator and $d_p(T \circ S) \leq \|S\| \, d_p(T)$.*

Proof. (a). Let $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $z_1^*, ..., z_n^* \in Z^*$. It suffices by (3.2) to show that

$$\sum_{i=1}^{n} |\langle R \circ T\left(x_i\right), z_i^*\rangle| \leq C(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \|v\|$$

where $v : Z \longrightarrow l_{p^*}^n$ such that $v(z) = \sum\limits_{i=1}^{n} z_i^*(z)e_i$. We have

$$\begin{aligned} \sum_{i=1}^{n} |\langle R \circ T\left(x_i\right), z_i^*\rangle| &= \sum_{i=1}^{n} |\langle T\left(x_i\right), R^*(z_i^*)\rangle| \\ &\leq (\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \|w\| \end{aligned}$$

where

$$\begin{aligned} w(y) &= \sum_{i=1}^{n} \langle R^*(z_i^*), y\rangle \, e_i \\ &= \sum_{i=1}^{n} \langle z_i^*, R(y)\rangle \, e_i \\ &= \|R(y)\| \sum_{i=1}^{n} \left\langle z_i^*, \tfrac{R(y)}{\|R(y)\|} \right\rangle e_i. \end{aligned}$$

This implies

$$\begin{aligned} \|w\| &\leq \|R\| \sup_{z \in B_Z} \left\|\left(z_i^*(z)\right)_{1 \leq i \leq n}\right\|_{l_{p^*}^n} \\ &\leq \|R\| \, \|v\| \, . \end{aligned}$$

(b). Let $n \in \mathbb{N}$, $e_1, ..., e_n \in E$ and $y_1^*, ..., y_n^* \in Y^*$. We have

$$\begin{aligned}
&\sum_{i=1}^{n} \left|\langle T\circ S\left(e_i\right), y_i^*\rangle\right| \\
&= \sum_{i=1}^{n} \left|\langle T\left(S\left(e_i\right)\right), y_i^*\rangle\right| \\
&\leq d_p(T)(\sum_{i=1}^{n} \left\|(S\left(e_i\right))\right\|_X^p)^{\frac{1}{p}} \left\|v\right\| (v(y) = \sum_{i=1}^{n} y_i^*(y)e_i) \\
&\leq d_p(T) \left\|S\right\| (\sum_{i=1}^{n} \left\|e_i\right\|_E^p)^{\frac{1}{p}} \left\|v\right\|.
\end{aligned}$$

This implies that $d_p(T\circ S) \leq \|S\| \, d_p(T)$ and this ends the proof. ■

We now present the domination theorem concerning this class of sublinear operators. Before this, we first announce the Ky Fan's lemma. For the proof the reader can consult [7, p. 190].

Lemma 3.3. *Let E be a Hausdorff topological vector space and let $\mathcal{C}$ be a compact convex subset of E. Let M be a set of functions on $\mathcal{C}$ with values in $(-\infty, \infty]$ having the following properties:*

(a) *Each $f \in M$ is convex and lower semicontinuous.*

(b) *If $g \in conv(M)$, there is an $f \in M$ with $g(x) \leq f(x)$, $\forall x \in \mathcal{C}$.*

(c) *There is an $r \in \mathbb{R}$ such that each $f \in M$ has a value $\leq r$.*

Then there is an $x_0 \in \mathcal{C}$ such that $f(x_0) \leq r$ for all $f \in M$.

Theorem 3.4. *Let X be a Banach space and Y be a Banach lattice. An operator $T \in \mathcal{SB}(X, Y)$ is strongly $p-$ summing ($1 < p < \infty$) if and only if there exists a positive constant $C > 0$ and Radon probability measure μ on $B_{Y^{**}}$ such that for all $x \in X$, we have*

$$|\langle T(x), y^*\rangle| \leq C\|x\| \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}))^{\frac{1}{p^*}}. \tag{3.3}$$

Moreover, in this case

$$d_p(T) = \inf\{C > 0 : \text{ for all } C \text{ verifying the inequality } (3.3)\}.$$

Proof. First we prove the converse. Let $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$. We have by (3.3)

$$|\langle T(x_i), y_i^*\rangle| \leq C\|x_i\| \left(\int_{B_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**})\right)^{\frac{1}{p^*}}$$

for all $1 \leq i \leq n$. By using Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle T(x_i), y_i^*\rangle| \\
&\leq C \sum_{i=1}^{n} (\|x_i\| (\int_{B_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**}))^{\frac{1}{p^*}}) \\
&\leq C(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} (\int_{B_{Y^{**}}} |y_i^*(y^{**})|^{p^*} d\mu(y^{**})))^{\frac{1}{p^*}} \\
&\leq C(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \sup_{y\in B_Y} \left\| (y_i^*(y))_{1\leq i\leq n} \right\|_{l_{p^*}^n}.
\end{aligned}$$

This implies that $T \in \mathcal{D}_p(X,Y)$ and $d_p(T) \leq C$.

To prove the first implication, let $K = B_{Y^{**}}$. Consider the set $\mathcal{C}$ of probability measures on $C(K)^*$. It is a convex compact of $C(K)^*$ endowed with its weak* topology $\sigma(C(K)^*, C(K))$. Let M be the set of all functions on $\mathcal{C}$ with values in $\mathbb{R}$ of the form

$$\begin{aligned}
&f_{((x_i),(y_i^*))}(\mu) \\
&= \sum_{i=1}^{k} (|\langle T(x_i), y_i^*\rangle| - \tfrac{C}{p} \|x_i\|^p - \tfrac{C}{p^*} \int_K |\langle y_i^*, y^{**}\rangle|^{p^*} d\mu(y^{**}))
\end{aligned} \tag{3.4}$$

where $(x_i)_{1\leq i\leq n} \subset X$, and $(y_i^*)_{1\leq i\leq n} \subset Y^*$.

Clearly these functions are convex and continuous. We now apply the Ky Fan's lemma with $E = C(K)^*$. Let f, g be in M and $\alpha \in [0,1]$ such that

$$\begin{aligned}
&f_{((x_i'),(y_i'^*))}(\mu) \\
&= \sum_{i=1}^{k} (|\langle T(x_i'), y_i^*\rangle| - \tfrac{C}{p} \|x_i'\|^p - \tfrac{C}{p^*} \int_K |\langle y_i'^*, y^{**}\rangle|^{p^*} d\mu(y^{**}))
\end{aligned}$$

and

$$\begin{aligned}
&g_{((x_i''^j),(y_i''^*))}(\mu) \\
&= \sum_{i=1}^{l} (|\langle T(x_i''), y_i^*\rangle| - \tfrac{C}{p} \|x_i''\|^p - \tfrac{C}{p^*} \int_K |\langle y_i''^*, y^{**}\rangle|^{p^*} d\mu(y^{**})).
\end{aligned}$$

It follows that (because T is positively homogeneous)

$$\begin{aligned}
&\alpha f \\
&= \sum_{i=1}^{k} \alpha(|\langle T(x_i'), y_i^*\rangle| - \tfrac{C}{p} \|x_i'\|^p - \tfrac{C}{p^*} \int_K |\langle y_i'^*, y^{**}\rangle|^{p^*} d\mu(y^{**})) \\
&= \sum_{i=1}^{k} (\left| \left\langle T(\alpha^{\frac{1}{p}} x_i'), \alpha^{\frac{1}{p^*}} y_i^* \right\rangle \right| \\
&- \tfrac{C}{p} \left\| \alpha^{\frac{1}{p}} (x_i') \right\|^p - \tfrac{C}{p^*} \int_K \left| \left\langle \alpha^{\frac{1}{p^*}} y_i'^*, y^{**} \right\rangle \right|^{p^*} d\mu(y^{**}))
\end{aligned}$$

and

$$
\begin{aligned}
&(1-\alpha)\, g \\
&= \sum_{i=1}^{l} (1-\alpha) \left(\left| \left\langle T\left(x_i''\right), y_i^* \right\rangle \right| - \tfrac{C}{p} \left\| x_i'' \right\|^p \right. \\
&- \tfrac{C}{p^*} \int_K \left| \left\langle (1-\alpha)^{\frac{1}{p^*}} y_i''^*, y^{**} \right\rangle \right|^{p^*} d\mu\left(y^{**}\right)) \\
&= \sum_{i=1}^{l} (\left| \left\langle T\left((1-\alpha)^{\frac{1}{p}} x_i'' \right), (1-\alpha)^{\frac{1}{p^*}} y_i^* \right\rangle \right| \\
&= \tfrac{C}{p} \left\| (1-\alpha)^{\frac{1}{p}} x_i''^j \right\|^p - \tfrac{C}{p^*} \int_K \left| \left\langle (1-\alpha)^{\frac{1}{p^*}} y_i''^*, y^{**} \right\rangle \right|^{p^*} d\mu\left(y^{**}\right)).
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
&\alpha f + (1-\alpha)\, g \\
&= \sum_{i=1}^{n} (\left| \left\langle T\left(x_i\right), y_i^* \right\rangle \right| - \tfrac{C}{p} \left\| x_i \right\|^p - \tfrac{C}{p^*} \int_K \left| \left\langle y_i^*, y^{**} \right\rangle \right|^{p^*} d\mu\left(y^{**}\right))
\end{aligned}
$$

with $n = k + l$,

$$
x_i = \begin{cases} \alpha^{\frac{1}{p}} x_i' & \text{if} \quad 1 \leq i \leq k, \\ (1-\alpha)^{\frac{1}{p}} x_i'' & \text{if} \quad k+1 \leq i \leq n, \end{cases}
$$

and

$$
y_i^* = \begin{cases} \alpha^{\frac{1}{p^*}} y_i'^* & \text{if} \quad 1 \leq i \leq k, \\ (1-\alpha)^{\frac{1}{p^*}} y_i''^* & \text{if} \quad k+1 \leq i \leq n. \end{cases}
$$

This shows that M is convex and consequently the condition (b) of Lemma 3.3 is satisfied.

For the condition (c), let $y_0 \in B_Y$ such that

$$
\sup_{\|y\|=1} \left(\sum_{i=1}^{n} \left| \left\langle y_i^*, y \right\rangle \right|^{p^*} \right)^{\frac{1}{p^*}} = \left(\sum_{i=1}^{n} \left| \left\langle y_i^*, y_0 \right\rangle \right|^{p^*} \right)^{\frac{1}{p^*}}
$$

and f of the form (3.4)

$$
\begin{aligned}
&f\left(\delta_{y_0}\right) \\
&\quad = \sum_{i=1}^{n} (\left| \left\langle T\left(x_i\right), y_i^* \right\rangle \right| - \tfrac{C}{p} \left\| x_i \right\|^p - \tfrac{C}{p^*} \int_K \left| \left\langle y_i^*, y^{**} \right\rangle \right|^{p^*} d\delta_{y_0}\left(y^{**}\right)) \\
&\quad = \sum_{i=1}^{n} (\left| \left\langle T\left(x_i\right), y_i^* \right\rangle \right| - \frac{C}{p} (\left\| x_i \right\|^p + \frac{C}{p^*} \left| \left\langle y_i^*, y_0 \right\rangle \right|^{p^*})) \\
&\quad = \sum_{i=1}^{n} \left| \left\langle T\left(x_i\right), y_i^* \right\rangle \right| - C (\frac{\sum_{i=1}^{n} \left\| x_i \right\|^p}{p} + \frac{\sum_{i=1}^{n} \left| \left\langle y_i^*, y_0 \right\rangle \right|^{p^*}}{p^*}).
\end{aligned}
$$

Using the elementary identity

$$\forall\alpha,\beta\in\mathbb{R}_+^* \quad \alpha\beta=\inf_{\epsilon>0}\left\{\frac{1}{p}\left(\frac{\alpha}{\epsilon}\right)^p+\frac{1}{p^*}\left(\epsilon\beta\right)^{p^*}\right\} \tag{3.5}$$

we find by taking $\alpha=(\sum_{i=1}^{n}\|x_i\|^p)^{\frac{1}{p}}, \beta=(\sum_{i=1}^{n}|\langle y_i^*,y_0\rangle|^{p^*})^{\frac{1}{p^*}}$ and $\epsilon=1$

$$\begin{aligned}&\sum_{i=1}^{n}(|\langle T\left(x_i\right),y_i^*\rangle|-\tfrac{C}{p}(\|x_i\|^p+\tfrac{C}{p^*}\,|\langle y_i^*,y_0\rangle|^{p^*}))\\ \leq\ &\sum_{i=1}^{n}(|\langle T\left(x_i\right),y_i^*\rangle|-C(\sum_{i=1}^{n}\|x_i\|^p)^{\frac{1}{p}}(\sum_{i=1}^{n}|\langle y_i^*,y_0\rangle|^{p^*})^{\frac{1}{p^*}}.\end{aligned}$$

The last quantity is less or equal to zero (by hypothesis because T is strongly $p-$summing) and hence the condition (c) is verified by taking $r=0$. By Ky Fan's lemma, there is $\mu\in\mathcal{C}$ such that $f(\mu)\leq 0$ for all $f\in M$. If we take $x\in X$ and $y^*\in Y^*$ we have

$$f(\mu)=f_{(x,y^*)}(\mu)=|\langle T\left(x\right),y^*\rangle|-\tfrac{C}{p}\,\|x\|^p-\tfrac{C}{p^*}\int_K|\langle y^*,y^{**}\rangle|^{p^*}\,d\mu\leq 0.$$

Hence

$$|\langle T\left(x\right),y^*\rangle|\leq C(\frac{1}{p}\,\|x\|^p+\frac{1}{p^*}\int_K|\langle y^*,y^{**}\rangle|^{p^*}\,d\mu).$$

Replacing x by $\dfrac{1}{\epsilon}x, y^*$ by ϵy^* and taking the infimum over all $\epsilon>0$ (see (3.5)), we find

$$\begin{aligned}|\langle T\left(x\right),y^*\rangle|\ &\leq\ C(\tfrac{1}{p}\,\|x\|^p+\tfrac{1}{p^*}\textstyle\int_K|\langle y^*,y^{**}\rangle|^{p^*}\,d\mu)\\ &\leq\ C(\tfrac{1}{p}\left\|\frac{x}{\epsilon}\right\|^p+\tfrac{1}{p^*}\textstyle\int_K|\langle\epsilon y^*,y^{**}\rangle|^{p^*}\,d\mu)\\ &\leq\ C(\tfrac{1}{p}\left(\frac{\|x\|}{\epsilon}\right)^p+\tfrac{1}{p^*}\left(\epsilon(\textstyle\int_K|\langle y^*,y^{**}\rangle|^{p^*})^{\frac{1}{p^*}}\right)^{p^*}\\ &\leq\ C\,\|x\|\textstyle\int_K|\langle y^*,y^{**}\rangle|^{p^*})^{\frac{1}{p^*}}.\end{aligned}$$

This implies

$$|\langle T\left(x\right),y^*\rangle|\leq C\,\|x\|\,\|y^*\|_{L_{p^*(B_{Y^{**}},\mu)}}$$

which concludes the proof. ■

Corollary 3.5. *Consider* $1<p_1,p_2<\infty$ *such that* $p_1\leq p_2$. *If* $T\in\mathcal{D}_{p_2}(X,Y)$ *then* $T\in\mathcal{D}_{p_1}(X,Y)$ *and*

$$d_{p_1}(T)\leq d_{p_2}(T).$$

4. Relation Between Linear and Sublinear Operators

In this section we study the relation between T and ∇T concerning the notion of strongly $p-$summing. We use the result of section 3.

Proposition 4.1. *Let X be a Banach lattice and Y be a complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Suppose that T is strongly $p-$summing ($1 < p < \infty$). Then for all $u \in \nabla T$, u is strongly positive $p-$summing and hence u^* is positive p^*-summing.*

Proof. We have by (1.2) for all x in X and y^* in Y_+^*

$$\langle u(x), y^* \rangle \leq \langle T(x), y^* \rangle$$

and consequently

$$-\langle u(x), y^* \rangle \leq \langle T(-x), y^* \rangle .$$

This implies that

$$\begin{aligned} |\langle u(x), y^* \rangle| &\leq \sup \{ \langle T(x), y^* \rangle , \langle T(-x), y^* \rangle \} \\ &\leq \sup \{ |\langle T(x), y^* \rangle| , |\langle T(-x), y^* \rangle| \} \\ &\leq |\langle T(x), y^* \rangle| + |\langle T(-x), y^* \rangle| \end{aligned}$$

and by (3.3)

$$|\langle u\,(x)\,, y^* \rangle| \leq 2 d_p(T) \, \|x\| \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}))^{\frac{1}{p^*}}. \tag{4.1}$$

Hence, we obtain

$$\|u^*(y^*)\| \leq 2 d_p(T) \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}))^{\frac{1}{p^*}}.$$

Thus the operator u^* is positive p^*-summing and $\pi_+\,(u^*) \leq 2 d_p(T)$. ■

Remark 4.2. If y^* is negative we have also (4.1). We do not know if u is strongly $p-$summing.

We now study the converse of the precedent proposition.

Theorem 4.3. *Let X be Banach space and Y be a complete Banach lattice. Let $T : X \to Y$ be a sublinear operator. Suppose that there is a constant $C > 0$, a set I, an ultrafilter $\mathcal{U}$ on I and $\{u_i\}_{i \in I} \subset \nabla T$ such that for all x in X,*

$$|\langle u_i\,(x)\,, y^* \rangle| \underset{\mathcal{U}}{\longrightarrow} |\langle T\,(x)\,, y^* \rangle|$$

and $d_p(u_i) \leq C$ *uniformly. Then,*

$$T \in \mathcal{D}_p(X,Y) \text{and } d_p(T) \leq C.$$

Proof. Since u_i is strongly $p-$summing, by Theorem 3.4 then there is a Radon probability measure μ_i on $B_{Y^{**}}$ such that for all $x \in X$, we have

$$|\langle u_i(x), y^*\rangle| \leq d_p(u_i) \|x\| \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu_i(y^{**}))^{\frac{1}{p^*}}.$$

As we have for all x in X and $y^* \in Y^*$,

$$|\langle u_i(x), y^*\rangle| \underset{\mathcal{U}}{\longrightarrow} |\langle T(x), y^*\rangle|$$

thus we obtain for all x in X and $y^* \in Y^*$,

$$|\langle T(x), y^*\rangle| \leq \lim_{\mathcal{U}} d_p(u_i) \|x\| \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu_i(y^{**}))^{\frac{1}{p^*}}.$$

The unit ball $B_{Y^{**}}$ is weak* compact, hence μ_i converge weak* to a probability μ on $B_{Y^{**}}$ and consequently

$$|\langle T(x), y^*\rangle| \leq C \|x\| \int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}))^{\frac{1}{p^*}}$$

for all x in X and $y^* \in Y^*$.

This implies that $d_p(T) \leq C$. ■

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