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WEAK CONVERGENCE THEOREM FOR NEW NONEXPANSIVE MAPPINGS IN BANACH SPACES AND ITS APPLICATIONS

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Abstract. A new nonexpansive mapping in a Banach space which is called generalized nonexpansive was introduced by the authors [4]. In this paper, we prove a weak convergence theorem for finding a fixed point of a generalized nonexpansive mapping in a Banach space. Moreover, using this result, we consider a proximal-type algorithm and the feasibility problem.

1. Introduction

Let C be a closed convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. We denoted by $F(T)$ the set of fixed points of T. In 1953, Mann [8] introduced an iteration method for finding a fixed point of a mapping T in a Banach space as follows: $x_0 \in C$ and

(1.1) $$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n, \quad n = 0, 1, 2, \dots,$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [11] discussed this iteration sequence in a uniformly convex Banach space with a Fréchet differentiable norm and obtained that the sequence $\{x_n\}$ converges weakly to a fixed point of T under $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Motivated by Kohsaka and Takahashi [7], Matsushita and Takahashi [9] also studied an iteration sequence for relatively nonexpansive mappings T in a uniformly smooth and uniformly convex Banach space as follows: $x_0 \in C$ and

(1.2) $$x_{n+1} = \Pi_C J^{-1} \left(\alpha_n J x_n + (1 - \alpha_n) J T x_n\right), \quad n = 1, 2, \dots,$$

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where $\{\alpha_n\}$ is a sequence in $[0, 1]$, Π_C is a generalized projection of E onto C and J is the duality mapping on E; see [1] for generalized projections. They obtained that the sequence $\{x_n\}$ converges weakly to a fixed point of T under $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$.

Recently, Ibaraki and Takahashi [4] introduced a new nonexpansive mapping in a smooth Banach space: Let D be a nonempty closed convex subset of a smooth Banach space E. A mapping $R : D \to D$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and

$$V(Rx, y) \leq V(x, y) \tag{1.3}$$

for each $x \in D$ and $y \in F(R)$, where $V(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$ for all $u, v \in E$.

Our purpose in this paper is to prove a weak convergence theorem for finding a fixed point of a generalized nonexpansive mapping in a Banach space. Using this result, we first consider a proximal-type algorithm for finding a zero point of a maximal monotone operator in a Banach space. Next, we consider the feasibility problem of finding a common element of finite sets in a Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We write $x_n \rightharpoonup x_0$ to indicate that the sequence $\{x_n\}$ converges weakly to x_0. Similarly, $x_n \to x_0$ will symbolize the strong convergence. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \left\|\frac{x + y}{2}\right\| < 1.$$

Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \Rightarrow \left\|\frac{x + y}{2}\right\| \leq 1 - \delta.$$

The following result was proved by Xu [22].

Lemma 2.1. ([22]) *Let $r > 0$ and let E be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|) \tag{2.1}$$

for all $x, y \in B_r := \{z \in E : \|z\| \leq r\}$ and λ with $0 \leq \lambda \leq 1$.

A Banach space E is said to be smooth if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.2}$$

exists for each $x,y\in\{z\in E:\|z\|=1\}(=:S(E))$. In this case, the norm of E is said to be Gâteaux differentiable. The space E is said to have a uniformly Gâteaux differentiable norm if for each $y\in S(E)$, the limit (2.2) is attained uniformly for $x\in S(E)$. The norm of E is said to be Fréchet differentiable if for each $x\in S(E)$, the limit (2.2) is attained uniformly for $y\in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit (2.2) is attained uniformly for $x,y\in S(E)$.

An operator $T\subset E\times E^*$ with domain $D(T)=\{x\in E: Tx\neq\emptyset\}$ and range $R(T)=\cup\{Tx:x\in D(T)\}$ is said to be monotone if $\langle x-y,x^*-y^*\rangle\geq 0$ for any $(x,x^*),(y,y^*)\in T$. An operator T is said to be strictly monotone if $\langle x-y,x^*-y^*\rangle>0$ for any $(x,x^*),(y,y^*)\in T$ $(x\neq y)$. A monotone operator T is said to be maximal if its graph $G(T)=\{(x,x^*):x^*\in Tx\}$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the set $T^{-1}0=\{u\in E:0\in Tu\}$ is closed and convex. If E is reflexive and strictly convex, then a monotone operator T is maximal if and only if $R(J+\lambda T)=E^*$ for each $\lambda>0$. A monotone operator T is maximal if and only if there exists a $(p,p^*)\in E$ such that $\langle p-u,p^*-u^*\rangle\geq 0$ for each $(u,u^*)\in T$, then $(p,p^*)\in T$ (see [16, 19] for more details).

The normalized duality mapping J from E into E^* is defined by

$$J(x):=\left\{x^*\in E^*:\langle x,x^*\rangle=\|x\|^2=\|x^*\|^2\right\},\quad \forall x\in E.$$

We also know the following properties (see [18] for details):

(1) $Jx\neq\emptyset$ for each $x\in E$.

(2) J is a monotone operator.

(3) If E is strictly convex, then J is one to one, that is,
$x\neq y \;\Rightarrow\; Jx\cap Jy=\emptyset$.

(4) If E is reflexive, then J is a mapping of E onto E^*.

(5) If E is smooth, then the duality mapping J is single valued.

(6) If E has a Fréchet differentiable norm, then J is norm to norm continuous.

(7) E is strictly convex if and only if J is a strictly monotone operator.

(8) E is uniformly convex if and only if E^* is uniformly smooth.

Let E be a smooth Banach space and consider the following function studied in Alber [1] and Kamimura and Takahashi [6]:

$$V(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2$$

for each $x, y \in E$. It is obvious from the definition of V that

$$(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.3}$$

for each $x, y \in E$. We also know that

$$V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for each $x, y, z \in E$(see [6]). The following lemma is well-known.

Lemma 2.2. ([6]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty subset of a Banach space E and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of a mapping T [13] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Let D be a nonempty subset of E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \ \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is said to be a retraction if $Rx = x$, $\forall x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely decided (see [3,4]). Then, if E be a smooth and strictly convex, a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. Let D be a nonempty closed subset of a Banach space E. Then D is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto D (see [3,4] for more details). The set of fixed points of such a generalized nonexpansive retraction is D.

The following result was obtained in [3,4].

Lemma 2.3. ([3,4]) *Let D be a nonempty closed subset of a smooth and strictly convex Banach space E. Let R_D be a retraction of E onto D. Then R_D is sunny and generalized nonexpansive if and only if*

$$\langle x - R_D x, JR_D x - Jy\rangle \geq 0$$

for each $x \in E$ and $y \in D$, where J is the duality mapping of E.

Let E be a reflexive, strictly convex, and smooth Banach space with its dual E^*. If a monotone operator $B \subset E^* \times E$ is maximal, then $E = R(I + rBJ)$ for all

$r > 0$ (see Proposition 4.1 in [4]). So, for each $r > 0$ and $x \in E$, we can consider the set $J_r x = \{z \in E : x \in z + rBJz\}$. From [4], $J_r x$ consists of one point. We denote such a J_r by $(I + rBJ)^{-1}$. J_r is called a generalized resolvent of B (see [4] for more details).

The following two results were obtained in [4].

Lemma 2.4. ([4]) *Let E be a reflexive, strictly convex, and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) *$D(J_r) = E$ for each $r > 0$.*

(2) *$(BJ)^{-1}0 = F(J_r)$ for each $r > 0$.*

(3) *If E has a Fréchet differentiable norm, then $(BJ)^{-1}0$ is closed.*

(4) *J_r is generalized nonexpansive for each $r > 0$.*

(5) *For $r > 0$ and $x \in E$, $\frac{1}{r}(x - J_r x) \in BJJ_r x$.*

Theorem 2.5. ([4]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) *For each $x \in E$, $\lim_{r\to\infty} J_r x$ exists and belongs to $(BJ)^{-1}0$.*

(2) *If $Rx := \lim_{r\to\infty} J_r x$ for each $x \in E$, then R is a sunny generalized nonexpansive retraction of E onto $(BJ)^{-1}0$.*

3. Weak Convergence Theorem

In this section, we consider the weak convergence of (1.1). We can prove the following theorem for generalized nonexpansive mappings in Banach spaces.

Theorem 3.1. *Let E be a smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, let T be a generalized nonexpansive mapping from C into itself with $F(T) \neq \emptyset$, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_0 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n = 1, 2, \ldots.$$

If $F(T) = \hat{F}(T)$, then the sequence $\{x_n\}$ converges weakly to an element of $F(T)$.

Proof. Let $z \in F(T)$. From convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} V(x_{n+1}, z) &= V(\alpha_n x_n + (1-\alpha_n)Tx_n, z) \\ &\leq \alpha_n V(x_n, z) + (1-\alpha_n)V(Tx_n, z) \\ &\leq \alpha_n V(x_n, z) + (1-\alpha_n)V(x_n, z) \\ &= V(x_n, z) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} V(x_n, z)$ exists. So, we have from (2.3) that the sequence $\{x_n\}$ is bounded. This implies that $\{Tx_n\}$ is also bounded. Put $r := \sup_{n\in\mathbb{N}\cup\{0\}}\{\|x_n\|, \|Tx_n\|\}$. By Lemma 2.1, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ satisfying (2.1), where $B_r = \{x \in E \ : \|x\| \leq r\}$. Therefore we have

$$\begin{aligned} V(x_{n+1}, z) &= V(\alpha_n x_n + (1-\alpha_n)Tx_n, z) \\ &= \|\alpha_n x_n + (1-\alpha_n)Tx_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Tx_n, Jz\rangle + \|z\|^2 \\ &\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tx_n\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\ &\quad -2\alpha_n\langle x_n, Jz\rangle - 2(1-\alpha_n)\langle Tx_n, Jz\rangle + \|z\|^2 \\ &= \alpha_n\Big(\|x_n\|^2 - 2\langle x_n, Jz\rangle + \|z\|^2\Big) \\ &\quad +(1-\alpha_n)\Big(\|Tx_n\|^2 - 2\langle Tx_n, Jz\rangle + \|z\|^2\Big) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\ &= \alpha_n V(x_n, z) + (1-\alpha_n)V(Tx_n, z) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\ &\leq \alpha_n V(x_n, z) + (1-\alpha_n)V(x_n, z) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\ &= V(x_n, z) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \end{aligned}$$

and hence

$$\alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \leq V(x_n, z) - V(x_{n+1}, z).$$

Since $\{V(x_n, z)\}$ converges and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, it follows that

$$\lim_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

Then the properties of g yield that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

For a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in E$, by $F(T) = \hat{F}(T)$ we have that v is a fixed point of T.

Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. As above, we have $v_1, v_2 \in F(T)$. Put

$$a = \lim_{n\to\infty}\Big(V(x_n, v_1) - V(x_n, v_2)\Big).$$

Note that

$$V(x_n, v_1) - V(x_n, v_2) = 2\langle x_n, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2, \quad n = 1, 2, \ldots.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{3.1}$$

and

$$a = 2\langle v_2, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\langle v_1 - v_2, Jv_1 - Jv_2\rangle = 0.$$

Since J is strictly monotone, it follows that $v_1 = v_2$; see the property (7) of J. Therefore, $\{x_n\}$ converges weakly to an element of $F(T)$. ■

4. Proximal-type Algorithm

In this section, we first study a proximal-type algorithm for maximal monotone operators. We start with the following lemma.

Lemma 4.1. *Let E be a reflexive, strictly convex, and smooth Banach space, let $B \subset E^* \times E$ be a maximal monotone operator and let J_r be a generalized resolvent of B for all $r > 0$. Then, the following hold:*

(1) *If E has a Fréchet differentiable norm, then J_r is demiclosed;*

(2) *if the duality mapping J is weakly sequentially continuous, then $\hat{F}(J_r) = F(J_r)$.*

Proof. (1) Let $\{x_n\}$ be a sequence of E such that $x_n \rightharpoonup x_0$ and $J_r x_n \to y_0$. Let $(u^*, u) \in B$. Then, from monotonicity of B and Lemma 2.4 we have that

$$\left\langle \frac{x_n - J_r x_n}{r} - u, JJ_r x_n - u^* \right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get

$$\left\langle \frac{x_0 - y_0}{r} - u, Jy_0 - u^* \right\rangle \geq 0.$$

Since B is maximal monotone, we have $(x_0 - y_0)/r \in BJy_0$ and hence $x_0 \in y_0 + rBJy_0$. From definition of J_r, we get $y_0 = J_r x_0$.

(2) It is obvious that $F(J_r) \subset \hat{F}(J_r)$. Conversely, let $z \in \hat{F}(J_r)$. There exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup z$ and $x_n - J_r x_n \to 0$. Hence, we have $J_r x_n \rightharpoonup z$. Let $(u^*, u) \in B$. From the monotonicity of B and Lemma 2.4 that

$$\left\langle u - \frac{x_n - J_r x_n}{r}, u^* - JJ_r x_n \right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since J is weakly sequentially continuous, we get

$$\langle u, u^* - Jz \rangle \geq 0.$$

So, we have $0 \in BJz$. Therefore, we get $z \in (BJ)^{-1}0 = F(J_r)$. This implies that $\hat{F}(J_r) \subset F(J_r)$. So, we have $\hat{F}(J_r) = F(J_r)$. ∎

Using Theorem 3.1, Lemmas 2.4 and 4.1, we obtain the following result.

Theorem 4.2. *Let E be a smooth and uniformly convex Banach space, let $B \subset E^* \times E$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$, let J_r be a generalized resolvent of B for all $r > 0$, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_0 = x \in E$, and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)J_r x_n, \quad n = 1, 2, \ldots.$$

If the duality mapping J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to an element of $(BJ)^{-1}0$.

Proof. Since $B^{-1}0$ is nonempty, $(BJ)^{-1}0$ is nonempty(see [5]). From Lemma 2.4 and Lemma 4.1, the generalized resolvent J_r is generalized nonexpansive and $\hat{F}(J_r) = F(J_r) = (BJ)^{-1}0$. By Theorem 3.1, $\{x_n\}$ converges weakly to an element of $(BJ)^{-1}0$. ∎

Next, we apply Theorem 4.2 to solve the the convex minimization problem. As in [5], we can prove the following result.

Theorem 4.3. *Let E be a smooth and uniformly convex Banach space, let $f^* : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function with*

$(\partial f^*)^{-1}(0) \neq \emptyset$, let $r > 0$ and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_0 = x \in E$ and

$$y_n^* = \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2r}\|y^*\|^2 - \frac{1}{r}\langle x_n, y^*\rangle \right\},$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)J^{-1}y_n^*, \quad n = 1, 2, \ldots. \tag{4.1}$$

If the duality mapping J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to an element of $(\partial f^* J)^{-1}(0)$.

Proof. By Rockafellar's theorem [14, 15], the subdifferential mapping $\partial f^* \subset E^* \times E$ is maximal monotone. Fix $r > 0$ and $z \in E$. Let J_r be the generalized resolvent of ∂f^*. Then we have

$$z \in J_r z + r\partial f^* J J_r z$$

and hence,

$$0 \in \partial f^* J J_r z + \frac{1}{r} J^{-1} J J_r z - \frac{1}{r} z = \partial \left(f^* + \frac{1}{2r}\|\cdot\|^2 - \frac{1}{r}\langle z, \cdot\rangle \right) J J_r z.$$

Thus, we have

$$J J_r z = \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2r}\|y^*\|^2 - \frac{1}{r}\langle z, y^*\rangle \right\}.$$

Therefore, from (4.1) we have that $J^{-1}y_n^* = J^{-1}JJ_r x_n = J_r x_n$ for all $n \in \mathbb{N}$. By Theorem 4.2, $\{x_n\}$ converges weakly to an element of $(\partial f^* J)^{-1}0$. ∎

5. Feasibility Problem

In this section, we consider the feasibility problem. We know the W-mapping which was introduced by Takahashi and Shimoji [20]: Let C be a convex subset of a Banach space E. Let $T_1, T_2, \ldots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for each $i = 1, 2, \ldots, r$. Then, we define a mapping W of C into itself as follows:

$$\begin{aligned} U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\ U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\ &\vdots \\ U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I, \\ W = U_r &= \alpha_r T_r U_{r-1} + (1-\alpha_r)I. \end{aligned} \tag{5.1}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$.

To prove our result, we need the following lemmas.

Lemma 5.1. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_r$ be generalized nonexpansive mappings of C into itself such that $\cap_{i=1}^r F(T_i)$ is nonempty, and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i = 1, 2, \ldots, r-1$ and $0 < \alpha_r \leq 1$. Let W be a W-mapping of C into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then, $F(W) = \cap_{i=1}^r F(T_i)$.*

Proof. It is obvious that $\cap_{i=1}^r F(T_i) \subset F(W)$. Conversely, let $z \in F(W)$ and $u \in \cap_{i=1}^r F(T_i)$. Then, we have $z = Wz = \alpha_r T_r U_{r-1} z + (1 - \alpha_r) z$ and hence $T_r U_{r-1} z = z$. Further, we have

$$
\begin{aligned}
V(z,u) &= V(T_r U_{r-1} z, u) \\
&\leq V(U_{r-1} z, u) \\
&\leq \alpha_{r-1} V(T_{r-1} U_{r-2} z, u) + (1 - \alpha_{r-1}) V(z,u) \\
&\leq \alpha_{r-1} V(U_{r-2} z, u) + (1 - \alpha_{r-1}) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2} V(T_{r-2} U_{r-3} z, u) \\
&\quad + \alpha_{r-1}(1 - \alpha_{r-2}) V(z,u) + (1 - \alpha_{r-1}) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2} V(U_{r-3} z, u) + (1 - \alpha_{r-1}\alpha_{r-2}) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2}\alpha_{r-3} V(T_{r-3} U_{r-4} z, u) \\
&\quad + \alpha_{r-1}\alpha_{r-2}(1 - \alpha_{r-3}) V(z,u) + (1 - \alpha_{r-1}\alpha_{r-2}) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2}\alpha_{r-3} V(U_{r-4} z, u) + (1 - \alpha_{r-1}\alpha_{r-2}\alpha_{r-3}) V(z,u) \\
&\vdots \\
&\leq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 V(U_1 z, u) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1 V(T_1 z, u) \\
&\quad + \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 (1 - \alpha_1) V(z,u) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2) V(z,u) \\
&= \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1 V(T_1 z, u) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1) V(z,u) \\
&\leq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1 V(z,u) + (1 - \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1) V(z,u) \\
&= V(z,u)
\end{aligned}
$$

So, we have $V(z,u) = V(U_1z,u)$. Put $r := \max\{\|z\|, \|T_1z\|\}$. By Lemma 2.1, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ satisfying (2.1), where $B_r = \{x \in E : \|x\| \leq r\}$. We have

$$\begin{aligned}
V(U_1z,u) &= \|\alpha_1T_1z + (1-\alpha_1)z\|^2 - 2\langle \alpha_1T_1z + (1-\alpha_1)z, Ju\rangle + \|u\|^2 \\
&\leq \alpha_1\|T_1z\|^2 + (1-\alpha_1)\|z\|^2 - \alpha_1(1-\alpha_1)g(\|z-T_1z\|) \\
&\quad -2\alpha_1\langle T_1z, Ju\rangle - 2(1-\alpha_1)\langle z, Ju\rangle + \|u\|^2 \\
&= \alpha_1\Big(\|T_1z\|^2 - 2\langle T_1z, Ju\rangle + \|u\|^2\Big) + (1-\alpha_1)\Big(\|z\|^2 - 2\langle z, Ju\rangle + \|u\|^2\Big) \\
&\quad -\alpha_1(1-\alpha_1)g(\|z-T_1z\|) \\
&= \alpha_1V(T_1z,u) + (1-\alpha_1)V(z,u) - \alpha_1(1-\alpha_1)g(\|z-T_1z\|) \\
&\leq \alpha_1V(z,u) + (1-\alpha_1)V(z,u) - \alpha_1(1-\alpha_1)g(\|z-T_1z\|) \\
&= V(z,u) - \alpha_1(1-\alpha_1)g(\|z-T_1z\|)
\end{aligned}$$

Hence we have

$$g(\|z-T_1z\|) \leq \frac{1}{\alpha_1(1-\alpha_1)}\Big\{V(z,u) - V(U_1z,u)\Big\} = 0.$$

We get $z = T_1z$, and hence $z = U_1z$. Next, we also have that $V(z,u) = V(U_2z,u)$. From $U_1z = z$, we get

$$\begin{aligned}
V(U_2z,u) &= \|\alpha_2T_2U_1z + (1-\alpha_2)z\|^2 - 2\langle \alpha_2T_2U_1z + (1-\alpha_2)z, Ju\rangle + \|u\|^2 \\
&\leq \alpha_2\|T_2z\|^2 + (1-\alpha_2)\|z\|^2 - \alpha_2(1-\alpha_2)g(\|z-T_2z\|) \\
&\quad -2\alpha_2\langle T_2z, Ju\rangle - 2(1-\alpha_2)\langle z, Ju\rangle + \|u\|^2 \\
&\leq V(z,u) - \alpha_2(1-\alpha_2)g(\|z-T_2z\|).
\end{aligned}$$

So, we get $T_2z = z$ and hence $U_2z = z$. By such a method, we have $z = T_kz$ and $z = U_kz$ for each $k = 3, 4, \ldots, r-1$. Since $z = U_{r-1}z$ and $z = Wz$, we get $z = T_rU_{r-1}z = T_rz$. This implies $z \in \cap_{i=1}^r F(T_i)$. So, we have $F(W) \subset \cap_{i=1}^r F(T_i)$. Therefore, we have $F(W) = \cap_{i=1}^r F(T_i)$. ■

Lemma 5.2. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_r$ be generalized nonexpansive mappings of C into itself such that $\cap_{i=1}^r F(T_i)$ is nonempty, $F(T_i) = \hat{F}(T_i)$, and*

(5.2) $$V(x, T_ix) + V(T_ix, u) \leq V(x,u), \quad \forall x \in C,\ \forall u \in F(T_i)$$

for each $i = 1, 2, \ldots, r$. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_r$ *be real numbers such that* $0 < \alpha_i < 1$ *for each* $i = 1, 2, \ldots, r-1$ *and* $0 < \alpha_r \le 1$ *and let* W *be a* W*-mapping of* C *into itself generated by* $T_1, T_2, \ldots, T_r$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_r$. *Then,* $F(W) = \hat{F}(W)$

Proof. It is obvious that $F(W) \subset \hat{F}(W)$. Conversely, let $z \in \hat{F}(W)$. Then there exists a sequence $\{x_n\}$ such that $x_n \rightharpoonup z$ and $\|x_n - Wx_n\| \to 0$. From the definition of W, we have

$$\|T_r U_{r-1} x_n - x_n\| = \frac{1}{\alpha_r} \|W x_n - x_n\|$$

and hence $\|T_r U_{r-1} x_n - x_n\| \to 0$. From the definition of W, it is obvious that

$$V(U_j x, u) \le V(x, u), \ \forall x \in C, \ \forall u \in \bigcap_{i=1}^{r} F(T_i) \tag{5.3}$$

for each $j = 1, 2, \ldots, r$. Put $y_n = U_{r-1} x_n$ and let $u \in \cap_{i=1}^{r} F(T_i)$. Then, it follows from (5.2) and (5.3) that

$$\begin{aligned}
V(y_n, T_r y_n) &\le V(y_n, u) - V(T_r y_n, u) \\
&\le V(x_n, u) - V(T_r U_{r-1} x_n, u) \\
&= \|x_n\|^2 - \|T_r U_{r-1} x_n\|^2 - 2\langle x_n - T_r U_{r-1} x_n, Ju\rangle \\
&\le \Big(\|x_n\| + \|T_r U_{r-1} x_n\|\Big)\Big(\|x_n\| - \|T_r U_{r-1} x_n\|\Big) + 2\|x_n - T_r U_{r-1} x_n\|\|u\| \\
&\le \Big(\|x_n\| + \|T_r U_{r-1} x_n\|\Big)\|x_n - T_r U_{r-1} x_n\| + 2\|x_n - T_r U_{r-1} x_n\|\|u\|
\end{aligned}$$

and hence $V(y_n, T_r y_n) \to 0$. From Lemma 2.2, we get $\|y_n - T_r y_n\| \to 0$ and hence $\|y_n - x_n\| \to 0$. So, we have that $y_n \rightharpoonup z$. This implies that $z \in \hat{F}(T_r) = F(T_r)$. Moreover, we have

$$\begin{aligned}
\|x_n - U_{r-1} x_n\| &= \|x_n - T_r U_{r-1} x_n + T_r U_{r-1} x_n - U_{r-1} x_n\| \\
&\le \|x_n - T_r U_{r-1} x_n\| + \|T_r U_{r-1} x_n - U_{r-1} x_n\| \\
&= \|x_n - T_r U_{r-1} x_n\| + \|T_r y_n - y_n\|.
\end{aligned}$$

This implies that $\|x_n - U_{r-1} x_n\| \to 0$.

Similarly, from $\|T_{r-1} U_{r-2} x_n - x_n\| = \frac{1}{\alpha_{r-1}} \|U_{r-1} x_n - x_n\|$, we have $\|x_n - T_{r-1} U_{r-2} x_n\| \to 0$. As above, we get $z \in \hat{F}(T_{r-1})$ and $\|x_n - U_{r-2} x_n\| \to 0$. By such the method, we have $z \in \hat{F}(T_i)$ and $\|x_n - U_i x_n\| \to 0$ for each $i = r-3, r-4, \ldots, 2$. From the definition of T_1, we have $\|T_1 x_n - x_n\| = \frac{1}{\alpha_1} \|U_1 x_n - x_n\|$. Since

$\|T_1x_n - x_n\| \to 0$ and $x_n \rightharpoonup z$, we get $z \in \hat{F}(T_1)$. Hence we have $z \in \cap_{i=1}^r \hat{F}(T_i)$. From Lemma 5.1 and the assumption of T_i, then $F(W) = \cap_{i=1}^r F(T_i) = \cap_{i=1}^r \hat{F}(T_i)$. This implies that $z \in F(W)$. So, we have that $\hat{F}(W) = F(W)$. ■

Using Theorem 3.1, Lemmas 5.1 and 5.2, we can prove the following result.

Theorem 5.3. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \ldots, T_r$ be generalized nonexpansive mappings of C into itself such that $\cap_{i=1}^r F(T_i)$ is nonempty, $F(T_i) = \hat{F}(T_i)$, and*

$$V(x, T_ix) + V(T_ix, u) \le V(x, u), \quad \forall x \in C,\ \forall u \in F(T_i) \tag{5.4}$$

for each $i = 1, 2, \ldots, r$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i = 1, 2, \ldots, r-1$ and $0 < \alpha_r \le 1$ and let W be a W-mapping of C into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \le \beta_n \le 1$ for each $n = 1, 2, \ldots$, and $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_0 = x \in C$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Wx_n, \quad n = 1, 2, \ldots.$$

Then the sequence $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^r F(T_i)$.

Proof. From Lemma 5.2, we have $\hat{F}(W) = F(W) = \cap_{i=1}^r F(T_i)$ and hence, by the definition of W, it is obvious that $V(Wx, u) \le V(x, u)$ for each $x \in C$ and $u \in F(W)$. Therefore, by Theorem 3.1, $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^r F(T_i)$. ■

Next, we apply Theorem 5.3 to solve the feasibility problem. Before solving it, we prove the following lemmas.

Lemma 5.4. *Let D be a nonempty subset of a reflexive, strictly convex, and smooth Banach space E. If R is the sunny generalized nonexpansive retraction of E onto D, then*

$$V(x, Rx) + V(Rx, u) \le V(x, u) \tag{5.5}$$

for each $x \in E$ and $u \in D$.

Proof. Let $x \in E$ and $u \in D$. From (2.4) and Lemma 2.3, we have

$$\begin{aligned} V(x, u) &= V(x, Rx) + V(Rx, u) + 2\langle x - Rx, JRx - Ju\rangle \\ &\ge V(x, Rx) + V(Rx, u) \end{aligned}$$

for each $x \in E$ and $u \in D$. ■

Lemma 5.5. *Let E be a reflexive, strictly convex, and smooth Banach space and let D be a nonempty weakly closed subset of E. If R is the sunny generalized nonexpansive retraction of E onto D, then $\hat{F}(R) = F(R)$.*

Proof. It is obvious that $F(R) \subset \hat{F}(R)$. Conversely, let $z \in \hat{F}(R)$. There exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup z$ and $x_n - Rx_n \to 0$. Hence, we have $Rx_n \rightharpoonup z$. From $\{Rx_n\} \subset D$ and $Rx_n \rightharpoonup z$, we get $z \in D$. This implies that $\hat{F}(R) \subset D = F(R)$. So, we have that $\hat{F}(R) = F(R)$. ■

Finally, we prove the following result.

Theorem 5.6. *Let E be a smooth and uniformly convex Banach space, let $D_1, D_2, \ldots, D_r$ be nonempty weakly closed sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i$ is nonempty, and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for each $i = 1, 2, \ldots, r-1$ and $0 < \alpha_r \leq 1$. Let W be a W-mapping of E into itself generated by $R_1, R_2, \ldots, R_r$ and $\alpha_1, \alpha_2, \ldots, \alpha_r$, where each R_i is the sunny generalized nonexpansive retraction of E onto D_i. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 \leq \beta_n \leq 1$ for each $n = 1, 2, \ldots$, and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_0 = x \in E$ and*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Wx_n, \quad n = 1, 2, \ldots.$$

Then the sequence $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^r D_i$.

Proof. From Lemmas 5.4 and 5.5, we have $\hat{F}(R_i) = F(R_i)$ and

$$V(x, R_i x) + V(R_i x, u) \leq V(x, u) \quad \forall x \in E,\ \forall u \in D_i \tag{5.6}$$

for each $i = 1, 2, \ldots, r$. We recall that $F(R_i) = D_i$ for each $i = 1, 2, \ldots, r$. Using Theorem 5.3, $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^r D_i$. ■

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