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ON HOLLOW-LIFTING MODULES

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Abstract. Let R be any ring and let M be any right R-module. M is called *hollow-lifting* if every submodule N of M such that M/N is hollow has a coessential submodule that is a direct summand of M. We prove that every amply supplemented hollow-lifting module with finite hollow dimension is lifting. It is also shown that a direct sum of two relatively projective hollow-lifting modules is hollow-lifting.

1. Introduction and Preliminaries

Throughout this paper R is a ring with identity and every R-module is a unitary right R-module. $A \leq M$ will mean A is a submodule of M.

Let M be a module and A a submodule of M. A is called a *small submodule* of M (denoted by $A \ll M$) if for any $X \leq M$, $M = A + X$ implies $X = M$. Dually, A is called *essential* in M if for any $X \leq M$, $A \cap X = 0$ implies $X = 0$. The module M is called *hollow* if every proper submodule is small in M. Dually, M is called *uniform* if every nonzero submodule is essential in M. For $A \leq B \leq M$, if A is essential in B, then B is called an *essential extension* of A in M. A submodule A is said to be *closed* in M, if A has no proper essential extension in M. Dually, for $A \leq B \leq M$, A is said to be a *coessential submodule* of B in M if $B/A \ll M/A$. A is said to be *coclosed* in M (denoted by $A \leq_{cc} M$), if A has no proper coessential submodule in M. Also, we will call A a *coclosure* of B in M, if A is a coesential submodule of B and A is coclosed in M.

Let M be a module. For $N, L \leq M$, N is a *supplement* of L in M if N is minimal with respect to $M = N + L$. Equivalently, $M = N + L$ with $N \cap L \ll N$. If $M = N + L$ with $N \cap L \ll M$, then N is called a *weak supplement* of L in M. A module M is called *(weakly) supplemented* if every submodule of M has a

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(weak) supplement in M. It is called *amply supplemented* if for every $A, B \leq M$, $M = A + B$ implies A has a supplement in M contained in B.

A module M is said to *have finite hollow dimension*, if there is an epimorphism $f : M \longrightarrow \oplus_{i=1}^{k} H_i$ with each H_i hollow and $Kerf \ll M$, and then we say that hollow dimension of M is k (denoted by h(M)=k). It is shown in [4] that $h(M) = k$ if and only if M contains a finite coindependent family of submodules $\{N_1, \ldots, N_k\}$ such that $\cap_{i=1}^{k} N_i \ll M$ and M/N_i is a hollow module for every $1 \leq i \leq k$.

A module M is said to be *extending* if for every submodule N of M there exists a direct summand K of M such that N is essential in K. Dually, M is called *lifting* or $satisfies (D_1)$, if for every submodule N of M there exists a direct summand K of M such that K is a coessential submodule of N in M. M is said to *have* (D_3), if for every direct summands A and B of M with $M = A + B$, $A \cap B$ is a direct summand of M. The module M is called *quasi-discrete* if it is lifting and has (D_3).

Let M be a module. M is called *uniform-extending* if every uniform submodule of M is essential in a direct summand of M. Dually, M is called *hollow-lifting* if every submodule N of M with M/N hollow has a coessential submodule in M that is a direct summand of M (cf. [13]). Clearly, if M is hollow-lifting, then every coclosed submodule K of M with M/K hollow is a direct summand of M. The converse is true if M is amply supplemented by [10, Proposition 1.5].

Let M_1 and M_2 be modules. The module M_1 is *small M_2-projective* (*nearly M_2-projective*) if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submodule of M_2 and Im$f \ll M_2/A$ (Im$f \neq M_2/A$), can be lifted to a homomorphism $\varphi : M_1 \longrightarrow M_2$. Clearly, if M_1 is nearly M_2-projective, then M_1 is small M_2-projective, and if M_2 is hollow, then small M_2-projectivity and nearly M_2-projectivity coincide. If M_1 is small (nearly) M_2-projective and M_2 is small (nearly) M_1-projective, then M_1 and M_2 are called *relatively small (nearly) projective*.

A decomposition $M = \oplus_{i \in I} M_i$ is said to *complement direct summands* if for any direct summand K of M there exists a subset $J \subseteq I$ such that $M = K \oplus (\oplus_{i \in J} M_i)$. Let M be any module. M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules $B_i \leq A_i$.

In Section 2 we introduce the notion of hollow-lifting modules. We begin by showing some general properties of hollow-lifting modules. We prove that for an indecomposable module M, the module M is hollow-lifting if and only if M is hollow, or else M has no hollow factor modules (Proposition 2.7). In Section 3 we will be concerned with hollow-lifting modules over commutative rings. In this way, it is shown that a finitely generated module over a commutative ring is hollow-lifting if and only if it is lifting (Corollary 3.4). In Section 4 we give some conditions

under which a direct sum of hollow modules is hollow-lifting. Let $M = \oplus_{i=1}^{n} H_i$ with all H_i hollow. If M has (D_3), then the following are equivalent:
(1) M is hollow-lifting;
(2) M is lifting;
(3) M is quasi-discrete;
(4) H_i is H_j-projective for all $i \neq j$ (Theorem 4.10).

Section 5 is devoted to the study of hollow-lifting modules whose every direct summand is hollow-lifting. It is shown that if M is a hollow-liting module, then M/U is hollow-lifting for every fully invariant submodule U of M (Lemma 5.5). In section 6 we give some sufficient conditions for a direct sum of two hollow-lifting modules to be hollow-lifting. We prove that if $M = M_1 \oplus M_2$ is a duo module, then M is hollow-lifting if and only if M_1 and M_2 are hollow-lifting (Theorem 6.3). It is also proved that any direct sum of two relatively projective hollow-lifting modules is hollow-lifting (Proposition 3.2).

2. Some Properties of Hollow-lifting Modules

It is clear that hollow modules and semisimple modules are hollow-lifting. The following result gives other examples of hollow-lifting modules.

Proposition 2.1. *Let H_1 and H_2 be hollow modules. The following are equivalent for the module $M = H_1 \oplus H_2$:*

(*i*) *M is hollow-lifting;*

(*ii*) *M is lifting.*

Proof. (i) $\Rightarrow$ (ii) Let $N \leq M$. Consider the projections $\pi_1 : M \longrightarrow H_1$ and $\pi_2 : M \longrightarrow H_2$. If $\pi_1(N) \neq H_1$ and $\pi_2(N) \neq H_2$, then $N \ll M$. Now, assume that $\pi_1(N) = H_1$. Then $M = N + H_2$. Therefore, M/N is hollow. Hence there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Thus M is lifting.

(ii) $\Rightarrow$ (i) Clear. ∎

Example 2.2. Let p be any prime integer. Since the module $\frac{\mathbb{Z}}{p^2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$ is lifting (see [12, Proposition A.7]), it is hollow-lifting. But the module $\frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$ is not hollow-lifting because it is not lifting (see [12, Proposition A.7]).

Let R be a ring and M an R-module. Let U and V be two submodules of M. We will say that V is a *strong supplement* of U in M if V is a supplement of U in M and $V \cap U$ is a direct summand of U (see [18]).

Proposition 2.3. *Let U be a submodule of a module M. The following are equivalent:*

(i) *U has a strong supplement in M;*

(ii) *U has a coessential submodule that is a direct summand of M.*

Proof. (i) $\Rightarrow$ (ii) Let V be a strong supplement of U in M and let $W \leq M$ such that $(U \cap V) \oplus W = U$. Then $M = W \oplus V$. Moreover, if $\frac{U}{W} + \frac{X}{W} = \frac{M}{W}$ then $U + X = M$ and $(U \cap V) + W + X = M$. Since $U \cap V \ll V$, we have $W + X = M$. Hence $X = M$. Therefore $\frac{U}{W} \ll \frac{M}{W}$ and the result is proved.

(ii) $\Rightarrow$ (i) Let A be a coessential submodule of U that is a direct summand of M. Let B be a submodule of M with $M = A \oplus B$. Thus $U = A \oplus (B \cap U)$ and $U + B = M$. If $(U \cap B) + X = B$ then $A + (U \cap B) + X = M$. Hence $U + X = M$ and $\frac{U}{A} + \frac{X+A}{A} = \frac{M}{A}$. Since $\frac{U}{A} \ll \frac{M}{A}$, we have $X + A = M$. But $X \leq B$, then $X = B$. Therefore $U \cap B$ is small in B. Consequently, B is a strong supplement of U in M. ■

Corollary 2.4. *Let M be any module. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *every submodule N of M such that $\frac{M}{N}$ is hollow has a strong supplement in M.*

Proposition 2.5. *Let M be an R-module. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *every submodule N of M such that $\frac{M}{N}$ is hollow can be written as $N = K \oplus L$ with K is a direct summand of M and L is a small submodule of M.*

Proof. (i) $\Rightarrow$ (ii) Let N be a submodule of M such that $\frac{M}{N}$ is hollow. Since M is hollow-lifting, there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$. Let F be a submodule of M with $M = K \oplus F$. So $N = K \oplus (F \cap N)$. Further, if $X \leq F$ with $(F \cap N) + X = F$, then $N + X = M$. Since $\frac{N}{K} \ll \frac{M}{K}$, we have $X + K = M$. Hence $X = F$ and $F \cap N \ll F$. It suffices to take $L = F \cap N$.

(ii) $\Rightarrow$ (i) Let N be a submodule of M such that $\frac{M}{N}$ is hollow. Then N can be written as $N = K \oplus L$ with K is a direct summand of M and L is small in M. Let X be a submodule of M such that $K \leq X$ and $\frac{N}{K} + \frac{X}{K} = \frac{M}{K}$. Thus $N + X = M$. So $K + L + X = M$ and $K + X = M$. But $K \leq X$. Then $X = M$ and $\frac{N}{K} \ll \frac{M}{K}$. Therefore M is hollow-lifting. ■

Remark 2.6. It is clear that every module having no hollow factor modules is a hollow-lifting module.

Proposition 2.7. *Let M be an indecomposable module. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *M is hollow, or else M has no hollow factor modules.*

Proof. (i)⇒(ii) Suppose that M has a hollow factor module. Then there exists a proper submodule N of M such that $\frac{M}{N}$ is hollow. Since M is hollow-lifting, there is K a direct summand of M such that $\frac{N}{K}$ is small in $\frac{M}{K}$. But M is indecomposable. Then $K = 0$ and N is small in M. Therefore M itself is a hollow module.
(ii)⇒(i) Clear. ■

Corollary 2.8. *Let M be a nonzero indecomposable module over a commutative noetherian ring R. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *M is lifting;*

(iii) *M is hollow.*

Proof. (ii) ⇔ (iii) By [12, Corollary 4.9].

(iii) ⇒ (i) Clear.

(i) ⇒ (iii) By [15, Proposition 2.24 and Theorem 4.30], M has an artinian factor module. Since every artinian module has finite hollow dimension, M has a hollow factor module. Then M is hollow by Proposition 2.7. ■

Proposition 2.9. *Let $M_1, \ldots, M_n$ be modules having no hollow factor modules. Then $M = M_1 \oplus \cdots \oplus M_n$ is hollow-lifting.*

Proof. Suppose that M has a submodule N such that $\frac{M}{N}$ is hollow. Since $\frac{M_1+N}{N} + \cdots + \frac{M_n+N}{N} = \frac{M}{N}$, there exists $i \in \{1, \ldots, n\}$ such that $\frac{M_i+N}{N} = \frac{M}{N}$ is hollow. So M_i has a hollow factor module, a contradiction. Therefore M is hollow-lifting. ■

Remark 2.10. Proposition 2.7 gives an idea to find an example of a hollow-lifting module that is not a lifting module. In fact, it is clear that every indecomposable module M which has no hollow factor module is hollow-lifting but it is not a lifting module. On the other hand, let N be any indecomposable module having no hollow factor module and let K be a semisimple module. If L is a submodule of $M = N \oplus K$ such that $\frac{M}{L}$ is hollow, then we have $N + L = M$ or $K + L = M$. Since N has no hollow factor modules and $\frac{N+L}{L} \cong \frac{N}{N \cap L}$, we have $K + L = M$. But K is semisimple. So there is a submodule E of K such that $K = E \oplus (K \cap L)$. Therefore $E \oplus L = M$. Hence L is a direct summand of M. Consequently, M is hollow-lifting. It is clear that M is not lifting (N is not hollow).

In the same manner as in the proof of [18, Lemma 1.1], we can show the following result:

Lemma 2.11. *Let M_0 be a direct summand of a module M such that M_0 has the finite exchange property. If $M_0 \leq U \leq M$ and U has a strong supplement in M, then $\frac{U}{M_0}$ has a strong supplement in $\frac{M}{M_0}$.*

Proposition 2.12. *Let M_0 be a direct summand of a module M such that M_0 has the finite exchange property. If M is hollow-lifting, then $\frac{M}{M_0}$ is also hollow-lifting.*

Proof. Let N be a submodule of M with $M_0 \leq N$ and $\frac{\frac{M}{M_0}}{\frac{N}{M_0}}$ is hollow. Thus $\frac{M}{N}$ is hollow. By Corollary 2.4, N has a strong supplement in M. By Lemma 2.11, $\frac{N}{M_0}$ has a strong supplement in $\frac{M}{M_0}$. Therefore $\frac{M}{M_0}$ is hollow-lifting by Corollary 2.4. ∎

Proposition 2.13. *Let M be a hollow-lifting module such that M has a non-small hollow submodule. Then M has a hollow direct summand.*

Proof. Let H be a non-small hollow submodule of M. Then there is a proper submodule N of M such that $M = H + N$. Since M is hollow-lifting, there is a direct summand L of M such that $N/L \ll M/L$. Clearly M/L is hollow. Now $M = L \oplus K$ for some submodule K of M. Therefore K is a hollow direct summand of M. ∎

Lemma 2.14. *Let M be a hollow-lifting module having a maximal submodule N. Then M has a local direct summand.*

Proof. Since M is hollow-lifting and $\frac{M}{N}$ is simple, there is a submodule K of M that is a strong supplement of N in M. Thus K is a direct summand of M, $M = N + K$ and $\frac{K}{N \cap K} \cong \frac{M}{N}$ is simple. So K is local because $N \cap K$ is small in K. ∎

Proposition 2.15. *Let R be a right noetherian ring and M a finitely generated hollow-lifting right R-module. Then M is a finite direct sum of local modules.*

Proof. By Lemma 2.14 M has a local direct summand H_1. By [3, Theorem 4.2], $End(H_1)$ is local. So H_1 has exchange property. Then $\frac{M}{H_1}$ is hollow-lifting by Proposition 2.12. Hence we can get by induction that M is a direct sum of local modules. ∎

Recall that a module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M.

Proposition 2.16. *Let M be a coatomic hollow-lifting module. Then M can be written as an irredundant sum of local direct summands of M.*

Proof. The same proof of [7, Proposition 3.2]. ■

Corollary 2.17. *Let M be a coatomic module with $Rad(M) = 0$. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *M is supplemented;*

(iii) *M is semisimple.*

Proof. (i)⇒ (iii) By Proposition 2.16, M is a sum of local direct summands of M. But if H is a local direct summand of M, then H will be a simple module because $Rad(H) = 0$. Thus M is semisimple.

(iii)⇒ (ii) Clear.

(ii)⇒ (i) Let L be any submodule of M such that $\frac{M}{L}$ is hollow. Since M is supplemented, there exists a submodule H of M such that H is a supplement of L in M. But $Rad(M) = 0$. Thus $M = L \oplus H$. Hence M is hollow-lifting. ■

Let M be an amply supplemented module with finite hollow dimension. Then M has a coclosed submodule K with M/K hollow. For, since M has finite hollow dimension, there exists a submodule N of M such that M/N is hollow. Since M is amply supplemented, there is a coclosed submodule K of M such that $K \leq N$ and $N/K \ll M/K$ by [10, Proposition 1.5]. Therefore $(M/K)/(N/K) \cong M/N$ implies that M/K is hollow.

Lemma 2.18. *Let M be an amply supplemented hollow-lifting module and $K \leq_{cc} M$ such that M/K has finite hollow dimension. Then K is a direct summand of M.*

Proof. We give the proof by induction on hollow dimension of M/K. If hollow dimension of M/K is 1, then K is a direct summand of M since M is hollow-lifting. Assume that hollow dimension of M/K is n and for every coclosed submodule T of M such that M/T has hollow dimension less than n, T is a direct summand of M.

Let H/K be coclosed in M/K such that $(M/K)/(H/K)$ is hollow. By [10, Lemma 1.4], H is coclosed in M. Hence $M = H \oplus H'$ for some submodule H' of M as M is hollow-lifting. Then $K = H \cap (K \oplus H')$ and $M/K = H/K \oplus (K \oplus H')/K$.

Therefore $(K \oplus H')/K$ is coclosed in M/K. Again, by [10, Lemma 1.4], $K \oplus H'$ is coclosed in M. By induction, $K \oplus H'$ is a direct summand of M, and so K is a direct summand of M. ■

Proposition 2.19. *An amply supplemented module with finite hollow dimension is lifting if and only if it is hollow-lifting.*

Proof. Suppose that M is hollow-lifting and let K be a coclosed submodule of M. Since M has finite hollow dimension, M/K has finite hollow dimension. Therefore by Lemma 2.18, K is a direct summand of M. Hence M is lifting by [10, Proposition 1.5]. ■

3. Hollow-lifting Modules over Commutative Rings

Let R denote a commutative ring. Let Ω be the set of all maximal ideal of R. If $m \in \Omega$, M an R-module, we denote as in [20, p. 53] by $K_m(M) = \{x \in M \mid x = 0$ or the only maximal ideal over $Ann_R(x)$ is $m\}$ as the m*-local component* of M. We call M m*-local* if $K_m(M) = M$. In this case M is an R_m-module by the following operation: $(\frac{r}{s})x = rx'$ with $x = sx'$ $(r \in R, s \in R - m)$. The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx$ is supplemented$\}$ it is easily seen that $K(M) = \{x \in M \mid \frac{R}{Ann_R(x)}$ is semiperfect$\}$, and we always have the decomposition $K(M) = \oplus_{m\in\Omega} K_m(M)$ (see [20, Satz 2.3].

The next result shows that in studying of hollow-lifting or lifting modules M with $M = K(M)$ over commutative rings, one may restrict to the case of modules over local rings.

Proposition 3.1. *Let M be an R-module over the commutative ring R. Then: $K(M)$ is(hollow-)lifting if and only if $K_m(M)$ is (hollow-)lifting for all $m \in \Omega$.*

Proof. It is an immediate consequence of the fact that for every submodule N of $K(M)$ we have $N = \oplus_{m\in\Omega} N \cap K_m(M)$. ■

Lemma 3.2. (see [18, Folgerung 3.3]) *Let M be a finitely generated module over a commutative local ring R. The following are equivalent:*

(i) *M is lifting;*

(ii) *every submodule U of M such that $\frac{M}{U}$ is cyclic has a strong supplement in M.*

Proposition 3.3. *Let M be a finitely generated module over a commutative local ring R. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *M is lifting.*

Proof. (ii) $\Rightarrow$ (i) Clear.

(i) $\Rightarrow$ (ii) Let U be a submodule of M such that $\frac{M}{U}$ is cyclic. Since R is local, $\frac{M}{U}$ is a local module. Then U has a strong supplement in M by Corollary 2.4. Hence M is lifting by Lemma 3.2. ■

Corollary 3.4. *Let M be a finitely generated module over a commutative ring R. The following are equivalent:*

(i) *M is hollow-lifting;*

(ii) *M is lifting.*

Proof. Assume M is hollow-lifting. By Proposition 2.16, M is a finite sum of local submodules. So M is supplemented ([21, Lemma 1.3(c)]). Hence $\oplus_{m\in\Omega}K_m(M) = K(M) = M$ by [20, Satz. 1.6]. The result follows from Proposition 3.1 and Proposition 3.3. ■

Theorem 3.5. *The following are equivalent for a commutative ring R with radical J:*

(1) *R is artinian serial and $J^2 = 0$;*

(2) *Every R-module is lifting;*

(3) *Every R-module is hollow-lifting;*

(4) *Every finitely generated R-module is lifting;*

(5) *Every finitely generated R-module is hollow-lifting.*

Proof. By [11, Theorem 3.15] and Corollary 3.4. ■

4. Some Conditions Under Which a Direct Sum of Hollow Modules is Hollow-lifting

Theorem 4.1. *Let $M = \oplus_{i\in I}M_i$, where all M_i are hollow and $\oplus_{i\in I}M_i$ complements direct summands. If M is hollow-lifting, then $\oplus_{i\neq j}M_i$ is nearly M_j-projective.*

Proof. Consider any proper submodule A of M_j, the homomorphism f : $\oplus_{i\neq j}M_i \longrightarrow M_j/A$ with $\mathrm{Im}f \neq M_j/A$ and the natural epimorphism $\pi : M_j \longrightarrow$

M_j/A. Define $B = \{x + y \mid x \in \oplus_{i\neq j} M_i, y \in M_j \text{and} f(x) = -\pi(y)\}$. Then $M = B + M_j$, $A \leq B$ and M/B is hollow. By hypothesis, there exists a direct summand D of M such that $D \leq B$ and $B/D \ll M/D$. Therefore M/D is hollow. Since the decomposition of M complements direct summands, $M = D \oplus M_k$ for some $k \in I$. As $B/D \ll M/D$, we have $M = D + M_j$. If $k \neq j$, then f is an epimorphism, a contradiction. Therefore $k = j$. So $M = D \oplus M_j$. Let $\alpha : M = D \oplus M_j \longrightarrow M_j$ be the projection and β the restriction of α to $\oplus_{i\neq j} M_i$. Clearly, f can be lifted to the homomorphism β. Therefore, $\oplus_{i\neq j} M_i$ is nearly M_j-projective. ■

Since the properties nearly and small projectivity are inherited by direct summands, we have the following fact.

Corollary 4.2. *Let $M = \oplus_{i\in I} M_i$, where all M_i are hollow and $\oplus_{i\in I} M_i$ complements direct summands. If M is hollow-lifting, then for all $i \neq j$, M_i is nearly (small) M_j-projective.*

Corollary 4.3. *Let M_1 and M_2 be hollow modules with local endomorphism rings. If $M_1 \oplus M_2$ is hollow-lifting, then M_1 and M_2 are relatively nearly (small)-projective.*

Proof. By [1, Corollary 12.7] and Theorem 4.1. ■

In [2], Baba and Harada define almost projective modules. Let M_1 and M_2 be two modules. M_1 is called *almost M_2–projective*, if for every epimorphism $f : M_2 \longrightarrow K$ and every homomorphism $g : M_1 \longrightarrow K$, either there exists $h : M_1 \longrightarrow M_2$ with $fh = g$ or there exists a nonzero direct summand N of M_2 and $\bar{h} : N \longrightarrow M_1$ with $g\bar{h} = f\mid_N$.

Lemma 4.4. *Let M_1 be a hollow module and M_2 an indecomposable module. Assume that there is no epimorphism from M_2 to M_1. Then M_1 is almost M_2–projective if and only if M_1 is M_2–projective.*

Proof. Assume that M_1 is almost M_2–projective. Let $f : M_1 \longrightarrow M_2/A$ be any homomorphism and let $\pi : M_2 \longrightarrow M_2/A$ be the natural epimorphism with $A \leq M_2$. If there exist a nonzero direct summand K of M_2 and a homomorphism $h : K \longrightarrow M_1$ with $fh = \pi\mid_K$, then $K = M_2$ since M_2 is indecomposable. Hence $h : M_2 \longrightarrow M_1$ is an epimorphism, because M_1 is hollow, a contradiction. Therefore there is a homomorphism $g : M_1 \longrightarrow M_2$ such that $\pi g = f$. The converse is clear. ■

Theorem 4.5. *Let M_1 and M_2 be hollow modules with local endomorphism rings. Assume that there is no epimorphism between M_1 and M_2. Then the following are equivalent for the module $M = M_1 \oplus M_2$:*

(1) M *is hollow-lifting;*

(2) M *is lifting;*

(3) M *is quasi–discrete;*

(4) M_1 *and* M_2 *are relatively projective;*

(5) M_1 *and* M_2 *are relatively almost projective.*

Proof. (1)⇔(2) Follows from Proposition 2.1. (4)⇔(5) Follows from Lemma 4.4. (5)⇔(2) By [2, Theorem 1]. (3)⇔(4) By [8, Corollary 13]. ■

Corollary 4.6. *Let* M_1 *and* M_2 *be hollow modules with local endomorphism rings. Assume that Rad*$(M_1) = M_1$ *and* M_2 *is local. Then the conditions (1)-(5) in Theorem 4.5 are equivalent for the module* $M = M_1 \oplus M_2$.

Lemma 4.7. *Let* $M = M_1 \oplus M_2$ *be a module. Assume that for every proper submodule* N *of* M *if* $M = N + M_2$ *then* $M \neq N + M_1$. *Then there is no epimorphism from* M_1 *to* M_2.

Proof. Assume that there is an epimorphism $f : M_1 \longrightarrow M_2$. Define $N = \{m_1 - f(m_1) \mid m_1 \in M_1\}$. Then $M = M_2 \oplus N$. Since f is epic, $M = M_1 + N$, this is a contradiction. Therefore there is no epimorphism from M_1 to M_2. ■

Proposition 4.8. *Let* M_1 *and* M_2 *be hollow modules with local endomorphism rings. Assume that* $M = M_1 \oplus M_2$ *and for every proper submodule* N *of* M, *if* $M = N + M_i$, *then* $M \neq N + M_j$ $(i \neq j)$. *Then the following are equivalent:*

(1) M *is hollow-lifting;*

(2) M *is lifting;*

(3) M_1 *and* M_2 *are relatively projective;*

(4) M_1 *and* M_2 *are relatively small projective;*

(5) M_1 *and* M_2 *are relatively nearly projective;*

(6) M_1 *and* M_2 *are relatively almost projective;*

(7) M *is quasi-discrete.*

Proof. (1)⇔(2)⇔(3)⇔(6)⇔(7) By Theorem 4.5 and Lemma 4.7. (3)⇔(4)⇔(5) Clear by [9, Theorem 2.6]. ■

Proposition 4.9. *Let* $M = \oplus_{i \in I} H_i$ *with all* H_i *hollow. If* M *is hollow-lifting and has* (D_3), *then for every* $i \in I$, $\oplus_{i \neq j} H_j$ *is* H_i*-projective.*

Proof. Let N be a proper submodule of M with $M = N + H_i$. Since $M/N \cong H_i/N \cap H_i$ and $H_i/N \cap H_i$ is hollow, M/N is hollow. As M is hollow-lifting, there exists a direct summand N^* of M such that $N^* \leq N$ and $N/N^* \ll M/N^*$. Therefore, $M = H_i + N^*$. Since M has (D_3), $M = H_i \oplus N^*$. Hence $\oplus_{i \neq j} H_j$ is H_i-projective by [17, 41.14]. ■

Theorem 4.10. *Let $M = \oplus_{i=1}^n H_i$ with all H_i hollow. If M has (D_3), then the following are equivalent:*

(1) *M is hollow-lifting;*

(2) *M is lifting;*

(3) *M is quasi-discrete;*

(4) *H_i is H_j-projective for all $i \neq j$.*

Proof. (1)⇒(4) It is clear by Proposition 4.9. (3)⇔(4) By [10, Corollary 2.15]. (3)⇒(2)⇒(1) By definitions. ■

Proposition 4.11. *Let $M = \oplus_{i \in I} H_i$ be a direct sum of hollow modules H_i such that the decomposition $\oplus_{i \in I} H_i$ complements direct summands and there is no epimorphism between H_i and H_j $(i \neq j)$. If M is hollow-lifting, then $\oplus_{j \neq i} H_j$ is H_i-projective for each $i \in I$.*

Proof. Let X be a proper submodule of M such that $M = X + H_i$. It is clear that $\frac{M}{X}$ is hollow. Thus there exists a direct summand Y of M such that $Y \leq X$ and $\frac{X}{Y}$ is small in $\frac{M}{Y}$. Hence $M = Y + H_i$. Since the decomposition $\oplus_{i \in I} H_i$ complements direct summands and there is no epimorphism between H_i and H_j $(i \neq j)$, we have $M = Y \oplus H_i$. The result is proved by [17, 41.14]. ■

Proposition 4.12. *Let $M = \oplus_{i \in I} H_i$ be a direct sum of local modules H_i such that $Rad(M)$ is small in M and there is no epimorphism between H_i and H_j $(i \neq j)$. The following are equivalent:*

(i) *M is quasi-discrete;*

(ii) *M is hollow-lifting and the decomposition $\oplus_{i \in I} H_i$ complements direct summands;*

(iii) *H_i is H_j-projective for each i and j with $(i \neq j)$.*

Proof. (i)⇔(iii) By [12, Theorem 4.48, Proposition 4.31 and Corollary 4.51].
(i)⇒(ii) By [12, Theorem 4.48].
(ii)⇒(iii) By Proposition 4.11 and [12, Proposition 4.32]. ■

Corollary 4.13. *Let $M = H_1 \oplus \cdots \oplus H_n$ be a direct sum of local modules H_i such that $End(H_i)$ is local and there is no epimorphism between H_i and H_j $(i \neq j)$. The following are equivalent:*

(i) M *is quasi-discrete;*
(ii) M *is hollow-lifting;*
(iii) H_i *is* H_j*-projective for each* i *and* j *with* $(i \neq j)$.

Proof. By Proposition 4.12 and [1, Corollary 12.7]. ■

5. Completely Hollow-lifting Modules

It would be desirable to find a hollow-lifting module which has a direct summand that is not hollow-lifting but we have not been able to do this. This is remained open in this paper. In this vein we will say that a module M is *completely hollow-lifting* if every direct summand of M is hollow-lifting.

In [5], L. Ganesan and N. Vanaja introduced the UCC-modules. A module M is a *UCC-module* if every submodule of M has a unique coclosure in M.

Proposition 5.1. *Let* M *be a weakly supplemented UCC–module. If* M *is hollow-lifting, then* M *is completely hollow-lifting.*

Proof. Let $M = N \oplus N'$. Let $A \leq N$ with N/A hollow. By [5, Corollary 3.6], M is amply supplemented. Hence N is amply supplemented. Then there exists a coclosed submodule A' of N such that $A' \leq A$ and $A/A' \ll N/A'$. Since N is coclosed in M, then A' is coclosed in M by [5, Lemma 2.6]. Since N' is coclosed in M, $A' \oplus N'$ is coclosed in M by [5, Theorem 3.16]. On the other hand, $(N/A')/(A/A') \cong N/A$ implies that N/A' is hollow. Therefore $M/(N' \oplus A')$ is hollow. Since M is hollow-lifting, $N' \oplus A'$ is a direct summand of M and hence A' is a direct summand of N. ■

Proposition 5.2. *Let* M *be a hollow-lifting module having* (D_3). *Then* M *is completely hollow-lifting.*

Proof. Let N be a direct summand of M. Then $M = N \oplus N'$ for some submodule N' of M. Let $K \leq N$ such that N/K is hollow. Since $M/K = N/K \oplus (N' \oplus K)/K$, $M/(N' \oplus K)$ is hollow. By assumption, there exists a direct summand A of M such that $A \leq N' \oplus K$ and $(N' \oplus K)/A \ll M/A$. Then $M = A + N$. By [10, Lemma 1.3], $[N \cap (N' \oplus K)]/(A \cap N) \ll M/(A \cap N)$. So $K/(A \cap N) \ll M/(A \cap N)$. Since M has (D_3), $A \cap N$ is a direct summand of M and so $A \cap N$ is a direct summand of N. Since $K/(A \cap N) \leq N/(A \cap N)$ and $N/(A \cap N)$ is a direct summand of $M/(A \cap N)$, we have $K/(A \cap N) \ll N/(A \cap N)$. Thus N is hollow-lifting. ■

Theorem 5.3. *Let* M *be a hollow-lifting module having* (D_3). *If* M *has finite hollow dimension, then* M *is lifting and it is a finite direct sum of hollow modules.*

Proof. Let M be a hollow-lifting module having (D_3). We first show that M is a finite direct sum of hollow modules. We use induction on $h(M)$. If $h(M) = 1$, M is hollow. Assume now $1 < n$ and assume that for every hollow-lifting module N with $(D3)$ such that $h(N) < n$, N is a finite direct sum of hollow modules. Let M be a hollow-lifting module with $h(M) = n$. Suppose that M is indecomposable. Since M has finite hollow dimension, there exists a proper submodule A of M such that M/A is hollow. As M is hollow-lifting, there exists a direct summand B of M such that $B \leq A$ and $A/B \ll M/B$. Then clearly M is hollow, a contradiction. Therefore we can assume that M is not indecomposable. So M has a decomposition $M = N \oplus L$ with N and L are nonzero submodules of M. Since $h(M) = h(N) + h(L)$, $h(N)$ and $h(L)$ are less than n. Further, by Proposition 5.2, N and L are hollow-lifting modules. By hypothesis they are finite direct sum of hollow modules and so is M. By Theorem 4.10, M is lifting. ■

Lemma 5.4. *Let M be a module. If $M = M_1 \oplus M_2$, then $M/A = (A + M_1)/A \oplus (A + M_2)/A$ for every fully invariant submodule A of M.*

Proof. Let A be a fully invariant submodule of M. Then $A = (A \cap M_1) \oplus (A \cap M_2)$. Hence $(A + M_1) \cap (A + M_2) \leq (M_1 + M_2 + A) \cap A + (M_1 + A + A) \cap M_2 = A + [M_1 + (A \cap M_1) \oplus (A \cap M_2)] \cap M_2 = A$. Therefore $M/A = (A + M_1)/A \oplus (A + M_2)/A$. ■

Lemma 5.5. *Let M be a module. If M is hollow-lifting, then M/U is hollow-lifting for every fully invariant submodule U of M.*

Proof. Let A/U be a submodule of M/U with $(M/U)/(A/U) \cong M/A$ hollow. Since M is hollow-lifting, there exists a direct summand B of M such that $B \leq A$, $A/B \ll M/B$ and $M = B \oplus B'$ for some submodule B' of M. By [12, Lemma 4.2(3)], $(A+U)/(B+U) = A/(B+U) \ll M/(B+U)$. Now it suffices to prove that $(B + U)/U$ is a direct summand of M/U. Since $M = B \oplus B'$, $M/U = (B + U)/U \oplus (B' + U)/U$ by Lemma 5.4. This completes the proof. ■

A module M is called a *duo-module*, if every submodule of M is fully invariant.

Corollary 5.6. *Let M be a duo hollow-lifting module. Then M is completely hollow-lifting.*

Proposition 5.7. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of modules M_i such that $End(M_i)$ is local and the decomposition $\oplus_{i \in I} M_i$ complements direct summands. If M is hollow-lifting, then M is completely hollow-lifting.*

Proof. By [12, Theorem 2.25 and Lemma 3.20], every direct summand of M has the exchange property. The result is proved by Proposition 2.12. ■

Corollary 5.8. *Let $M = \oplus_{i=1}^{n} M_i$ be a direct sum of modules M_i such that $End(M_i)$ is local. If M is hollow-lifting, then M is completely hollow-lifting.*

Proof. By Proposition 5.7 and [1, Corollary 12.7]. ■

6. Sufficient Conditions for a Direct Sum of Two Hollow-lifting Modules to be Hollow-lifting

Direct sum of two hollow-lifting modules need not be a hollow-lifting module as we see in the following example.

Example 6.1.

(i) Let M be the $\mathbb{Z}$–module $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$ are hollow, they are hollow-lifting. But M is not hollow-lifting (see Example 2.2). Note that $\mathbb{Z}/2\mathbb{Z}$ is not $\mathbb{Z}/8\mathbb{Z}$-projective.

(ii) Let R be a discrete valuation ring with field of fractions K, let P be the unique maximal ideal of R. Let M be the R-module $K/R \oplus R/P$. Since K/R and R/P are hollow, they are hollow-lifting. By [12, Proposition A.7], M is not lifting. Therefore by Proposition 2.1, M is not hollow-lifting.

Proposition 6.2. *Let M be an R-module. Suppose that $M = N \oplus K$, N and K both are hollow-lifting and N and K are relatively projective. Then M is hollow-lifting.*

Proof. Let L be a submodule of M such that $\frac{M}{L}$ is hollow. Then $M = N + L$ or $M = K + L$. Suppose that $M = N + L$ (the case M=K+L being analogous). Hence $\frac{N}{L\cap N}$ is hollow. Since K is N-projective, there exists a direct summand G of L such that $M = N \oplus G$ ([17, 41.14]). Then $L = (N \cap L) \oplus G$. Since N is hollow-lifting, there exists a direct summand X of N such that $X \leq L \cap N$ and $\frac{L\cap N}{X} \ll \frac{N}{X}$. Thus $X \oplus G$ is a direct summand of M and $X \oplus G \leq (L \cap N) \oplus G$. Let F be a submodule of M with $X \oplus G \leq F$ and $\frac{(L\cap N)\oplus G}{X\oplus G} + \frac{F}{X\oplus G} = \frac{M}{X\oplus G}$. Then $(L \cap N) + G + F = M$. So $(L \cap N) + F = M$. Hence $F = M$ (because $\frac{L\cap N}{X} \ll \frac{N}{X}$). Thus $X \oplus G$ is a coessential submodule of $(L \cap N) \oplus G = L$ in M. ■

Theorem 6.3. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is hollow-lifting if and only if M_1 and M_2 are hollow-lifting.*

Proof. (⇒:) It is clear by Corollary 5.6.
(⇐:) Let A be submodule of M with M/A hollow. By Lemma 5.4, $M/A = (A + M_1)/A \oplus (A + M_2)/A$. Since M/A is hollow, we can assume that $(A +$

$M_1)/A = M/A$. Then $M_2 \leq A$. Since $(A+M_1)/A \cong M_1/(A \cap M_1)$ and M_1 is hollow-lifting, there exists a direct summand B_1 of M_1 such that $B_1 \leq A \cap M_1$ and $(A \cap M_1)/B_1 \ll M_1/B_1$. Since $A = (A \cap M_1) \oplus (A \cap M_2)$, we get $A/(B_1 \oplus M_2) \ll M/(B_1 \oplus M_2)$. Moreover, it is easily seen that $B_1 \oplus M_2$ is a direct summand of M. Thus M is hollow-lifting. ■

Corollary 6.4. *Let $M = M_1 \oplus \cdots \oplus M_n$ be a duo module. Then M is hollow-lifting if and only if M_i is hollow-lifting for all $i = 1, 2, \ldots, n$.*

Proof. The proof is by induction on n and it is based on the fact that any direct summand of a duo module is duo. ■

The following example shows that in Theorem 6.3, Duo is essential:

Example 6.5. Consider the $\mathbb{Z}$–module M in Example 6.1(i). Then M is not duo. For, let $f : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ be the homomorphism defined by $f(\overline{p}, \overline{q}) = (\overline{p+q}, \overline{2q})$. Then $f(0 \oplus \mathbb{Z}/8\mathbb{Z}) \nsubseteq 0 \oplus \mathbb{Z}/8\mathbb{Z}$ $(f(\overline{0}, \overline{1}) = (\overline{1}, \overline{2}))$.

Let M_1 and M_2 be modules. The module M_1 is called *h-small* M_2-projective if every homomorphism $f : M_1 \longrightarrow M_2/A$, where $A \leq M_2$, M_2/A is hollow and $\mathrm{Im} f \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \longrightarrow M_2$. Obviously, if M_1 is small M_2-projective, then M_1 is h-small M_2-projective.

Lemma 6.6. *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. The following are equivalent:*

(i) *M_1 is h-small M_2-projective;*

(ii) *For every submodule N of M such that M/N is hollow and $M \neq M_1 + N$, there exists a submodule N' of N such that $M = N' \oplus M_2$.*

Proof. By the same proof of [10, Lemma 2.4]. ■

Lemma 6.7. *Let $M_1 = K \oplus L$ and M_2 be two modules. If M_1 is h-small M_2-projective, then K is h-small M_2-projective.*

Proof. Let X be a submodule of M_2 such that $\frac{M_2}{X}$ is hollow and let $f : K \longrightarrow \frac{M_2}{X}$ be an homomorphism with $Imf \ll \frac{M_2}{X}$. Let $\pi : M_2 \longrightarrow \frac{M_2}{X}$ be the natural epimorphism and let $g : M_1 \longrightarrow K$ be the canonical projection. Then $fg : M_1 \longrightarrow \frac{M_2}{X}$ is a homomorphism with $Im(fg) = Imf \ll \frac{M_2}{X}$. Since M_1 is h-small M_2-projective, there exists a homomorphism $\varphi : M_1 \longrightarrow M_2$ such that $\pi\varphi = fg$. It is clear that $\pi(\varphi_{/K}) = f$ where $\varphi_{/K}$ is the restriction of φ to K. Therefore K is h-small M_2-projective. ■

Lemma 6.8. *If M is h-small $(M_1 \oplus M_2)$-projective, then M is h-small M_i-projective for $i = 1, 2$.*

Proof. Let N_1 be a submodule of M_1 such that $\frac{M_1}{N_1}$ is hollow. Let $f : M \longrightarrow \frac{M_1}{N_1}$ be a homomorphism such that $Imf \ll \frac{M_1}{N_1}$. Let $h : \frac{M_1}{N_1} \longrightarrow \frac{M_1 \oplus M_2}{N_1 \oplus M_2}$ be the natural isomorphism defined by $h(m_1 + N_1) = m_1 + (N_1 \oplus M_2)$. Since M is h-small $(M_1 \oplus M_2)$-projective and $Imhf \ll \frac{M_1 \oplus M_2}{N_1 \oplus M_2}$, there is a homomorphism $\varphi : M \longrightarrow M_1 \oplus M_2$ such that $\pi\varphi = hf$ where $\pi : M_1 \oplus M_2 \longrightarrow \frac{M_1 \oplus M_2}{N_1 \oplus M_2}$ is the canonical epimorphism. Let $\alpha : M_1 \longrightarrow \frac{M_1}{N_1}$ and $\beta : M_1 \oplus M_2 \longrightarrow M_1$ be the canonical epimorphisms. It is clear that $\alpha\beta\varphi = f$ and $\beta\varphi$ is a homomorphism from M to M_1. Therefore M is h-small M_1-projective. ∎

Proposition 6.9. *Let M be any module and let H_1 and H_2 be two hollow modules. Assume that M is small H_1-projective and small H_2-projective.Then M is small $(H_1 \oplus H_2)$-projective.*

Proof. Let $f : M \longrightarrow B$ be any homomorphism and $g : H_1 \oplus H_2 \longrightarrow B$ be any epimorphism, where B is any module. Assume $Imf \ll B$. Since g is epic, $g(H_1) + g(H_2) = B$. It is easy to see that f can be lifted to a homomorphism from M to $H_1 \oplus H_2$ if $g(H_1) = B$ or $g(H_2) = B$. Now assume $g(H_1) \neq B$ and $g(H_2) \neq B$. Let $\pi : B \longrightarrow B/g(H_2)$ be the natural epimorphism and $\bar{g} : H_1 \longrightarrow B/g(H_2)$ be the epimorphism defined by $\bar{g}(h_1) = g(h_1) + g(H_2)$. Clearly, $(\pi f)(M) \ll B/g(H_2)$. Since M is small H_1-projective, there exists a homomorphism $h_1 : M \longrightarrow H_1$ such that $\bar{g}h_1 = \pi f$. Since for every $x \in M$ $f(x) - gh_1(x) \in g(H_2)$, consider the homomorphism $\bar{f} = f - gh_1 : M \longrightarrow g(H_2)$. Now we prove that $Im\bar{f} \ll g(H_2)$. Since H_2 is hollow, $g(H_2)$ is hollow. Therefore it is sufficient to show that $Im\bar{f} \neq g(H_2)$. If $Im\bar{f} = g(H_2)$, then $g(H_1) = B$ since $f(M) \ll B$, this is a contradiction. Thus $Im\bar{f} \neq g(H_2)$. Since M is small H_2-projective there exists a homomorphism $h_2 : M \longrightarrow H_2$ such that $gh_2 = \bar{f}$. Now let h be the homomorphism defined by $h = h_1 + h_2 : M \longrightarrow H_1 \oplus H_2$. Clearly f lifts to the homomorphism h. ∎

Lemma 6.10. *Let M_1 be any module and $M_2 = \oplus_{i=1}^{n} H_i$ be a finite direct sum of hollow modules. Then M_1 is h-small M_2-projective if and only if it is small M_2-projective.*

Proof. (⇐:)Clear.
(⇒:) Assume M_1 is h-small M_2-projective. Let $i \in \{1, 2, ..., n\}$. Clearly, M_1 is h-small H_i-projective (see Lemma 6.8) and hence M_1 is small H_i-projective. Therefore M_1 is small M_2-projective by Proposition 6.9. ∎

Lemma 6.11. *Let M_1 be any module, M_2 a hollow-lifting module and let $M = M_1 \oplus M_2$. If M_1 is h-small M_2-projective, then every coclosed submodule K of M such that M/K is hollow and $(K + M_1)/K \ll M/K$ is a direct summand of M.*

Proof. Let K be a coclosed submodule of M such that M/K is hollow and $(K + M_1)/K \ll M/K$. By Lemma 6.6, there exists a submodule N' of K such that $M = N' \oplus M_2$. Now M/N' is hollow-lifting, K/N' is coclosed in M/N' and $\frac{\frac{M}{N'}}{\frac{K}{N'}}$ is hollow. Therefore K/N' is a direct summand of M/N'. Hence K is a direct summand of M. ■

Theorem 6.12. *Let M_1 and M_2 be hollow-lifting modules and let $M = M_1 \oplus M_2$ be an amply supplemented module. If one of the following conditions holds, then M is hollow-lifting.*

- (i) *M_1 is h-small M_2-projective and every coclosed submodule K of M with M/K hollow and $M = K + M_1$ is a direct summand of M;*
- (ii) *M_1 and M_2 are relatively h-small projective and every coclosed submodule K of M with M/K hollow and $M = K + M_1 = K + M_2$ is a direct summand of M;*
- (iii) *M_2 is M_1-projective and M_1 is h-small M_2-projective;*
- (iv) *M_1 is semisimple and h-small M_2-projective.*

Proof. (i) and (ii) By Lemma 6.11 and [10, Proposition 1.5].

(iii) Let K be a coclosed submodule of M with M/K hollow and $M = K+M_1$. Since M_2 is M_1-projective, there exists a direct summand K' of K such that $M = K' \oplus M_1$ by [17, 41.14]. By the same proof of Lemma 6.11, we conclude that K is a direct summand of M. The result follows from (i).

(iv) Follows from (iii). ■

Lemma 6.13. *Let M_1 and M_2 be two modules. Suppose that $M = M_1 \oplus M_2$ is a hollow-lifting module having (D_3). Then M_1 and M_2 are relatively h-small projective.*

Proof. It suffices to prove that M_1 is h-small M_2-projective. Let N be a submodule of M with M/N hollow and $M \neq N + M_1$. Then $M = N + M_2$. As M is hollow-lifting, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Then $M = K + M_2$. Since M has $(D3)$, $K \cap M_2$ is a direct summand of M. Let L be a submodule of K with $K = (K \cap M_2) \oplus L$. Hence $M = L + M_2$. But $L \cap M_2 = L \cap K \cap M_2 = 0$. Then $M = L \oplus M_2$. By Lemma 6.6, M_1 is h-small M_2-projective. ■

Theorem 6.14. *Let $M = M_1 \oplus M_2$ be an amply supplemented module having (D_3) such that M_1 is semisimple and M_2 is a lifting module that is a finite direct sum of hollow modules. Then following are equivalent:*

(i) M *is lifting;*

(ii) M *is hollow-lifting;*

(iii) M_1 *is h-small* M_2*-projective;*

(iv) M_1 *is small* M_2*-projective.*

Proof. (i) $\Rightarrow$ (ii) It is clear. (iii)$\Leftrightarrow$ (iv) By Lemma 6.10. (ii)$\Rightarrow$ (iii) By Lemma 6.13. (iv)$\Rightarrow$ (i) By [10, Theorem 2.8]. ■

Let R be any ring and let M be an R-module. M is called a *radical* module if $\mathrm{Rad}(M) = M$. By $P(M)$ we denote the sum of all radical submodules of M. If $P(M) = 0$, M is called *reduced*. It is easy to see that $P(M)$ is a fully invariant submodule of M and it is always radical. So by Lemma 5.5, if M is hollow-lifting, then $\frac{M}{P(M)}$ is hollow-lifting. On the other hand, if M is a supplemented module, [21, Lemma 1.5 (c)] shows that $\frac{M}{P(M)}$ is coatomic.

Proposition 6.15. *Let M be an R-module. Suppose that M is hollow-lifting. If $P(M)$ is a direct summand of M, then $P(M)$ and $\frac{M}{P(M)}$ both are hollow-lifting.*

Proof. We only need to show that $P(M)$ is hollow-lifting. Let N be a submodule of M with $M = P(M) \oplus N$. Let L be a submodule of $P(M)$ such that $\frac{P(M)}{L}$ is hollow. Thus $\frac{M}{L\oplus N}$ is hollow. Since M is hollow-lifting, there exists a submodule X of M such that X is a strong supplement of $L \oplus N$ in M. Hence $(L \oplus N) + X = M$ and $\frac{P(M)}{L} \cong \frac{M}{L\oplus N} \cong \frac{X}{(L\oplus N)\cap X}$. Therefore X is hollow radical and $X \leq P(M)$. Hence $P(M) = L + X$. Since $X \cap L = (L \oplus N) \cap X$, $X \cap L$ is a direct summand of L. So X is a strong supplement of L in $P(M)$. Consequently, $P(M)$ is hollow-lifting. ■

Proposition 6.16. *Let M be an R-module. Suppose that $M = N \oplus K$, N is radical, K is coatomic, N and K both are hollow-lifting and N and K are relatively h-small projective. Then M is hollow-lifting.*

Proof. Let L be a submodule of M such that $\frac{M}{L}$ is hollow. Then $M = N + L$ or $M = K + L$. If $M = K + L = N + L$, then $\frac{M}{L} \cong \frac{N}{N\cap L} \cong \frac{K}{K\cap L}$. Hence $\frac{K}{K\cap L}$ is coatomic and radical. Thus $\frac{K}{K\cap L} = 0$ and $L = M$, a contradiction. Therefore we have $M = N + L$ and $M \neq K + L$ or $M \neq N + L$ and $M = K + L$. Since N and K are relatively h-small projective, the rest of the proof is the same as the proof of Proposition 6.2 (See Lemma 6.6). ■

Proposition 6.17. *Let M be an R-module. If $M = P(M) \oplus K$ for some coatomic submodule K of M and M is hollow-lifting, then $P(M)$ and K are relatively h-small projective.*

Proof. Let L be a submodule of M with $M \neq P(M) + L$ and $\frac{M}{L}$ hollow. Then $M = K + L$. Hence $\frac{M}{L} \cong \frac{K}{L \cap K}$ and $\frac{M}{L}$ is local. Since M is hollow-lifting, there exists a direct summand E of M such that $E \leq L$ and $\frac{L}{E} \ll \frac{M}{E}$. Then $M = E + K$ and $\frac{M}{E}$ is local. Let F be a submodule of M with $M = E \oplus F$. So $P(M) = P(E) \oplus P(F)$. Thus $P(F)$ is a radical direct summand of F. Therefore F is reduced because F is local. This gives $P(M) = P(E) \leq E$. Hence $P(M) \leq L$. Since $M = P(M) \oplus K$, $P(M)$ is h-small K-projective.

Now, let L be a submodule of M with $M \neq K + L$ and $\frac{M}{L}$ is hollow. Then $M = P(M) + L$, $\frac{M}{L} \cong \frac{P(M)}{L \cap P(M)}$ and $\frac{M}{L}$ is radical. Since M is hollow-lifting, there exists a direct summand B of M such that $B \leq L$ and $\frac{L}{B} \ll \frac{M}{B}$. It is clear that $\frac{M}{B}$ is hollow radical. Let A be a submodule of M with $M = A \oplus B$. Then $P(M) = P(A) \oplus P(B)$ and $P(M) = A \oplus P(B)$ because A is radical. Since $P(B)$ is a direct summand of B, there exists a submodule C of B such that $B = P(B) \oplus C$. Then $M = A \oplus B = A \oplus P(B) \oplus C$. Thus $M = P(M) \oplus C$. Since $C \leq L$, K is h-small $P(M)$-projective. ■

Theorem 6.18. *Let M be an R-module. Suppose that $M = P(M) \oplus K$ for some coatomic submodule K of M. Then M is hollow-lifting if and only if $P(M)$ and K are relatively h-small projective and $P(M)$ and K both are hollow-lifting.*

Proof. By Proposition 6.17, Proposition 6.16 and Proposition 6.15. ■

Corollary 6.19. *Let $M = M_1 \oplus M_2$ be an R-module such that M_1 is radical and M_2 is semisimple. Then M is hollow-lifting if and only if M_1 is hollow-lifting and M_2 is h-small M_1-projective.*

Proof. It is clear that $P(M) = M_1$ and M_2 is coatomic. The result follows from Theorem 6.18. ■

Example 6.20. Consider the module in Example 6.1(ii). Since M is not hollow-lifting, Corollary 6.19 shows that R/P is not h-small K/R-projective.

Lemma 6.21. *Let M be a hollow-lifting module such that $M = P(M) \oplus K$ for some submodule K of M and $P(M) = H$ is hollow. Then K is hollow-lifting and K is H-projective.*

Proof. By Proposition 6.15, K is hollow-lifting. Let N be a proper submodule of M such that $M = H + N$. Then $\frac{M}{N}$ is hollow. Since M is hollow-lifting, there is a direct summand X of M such that $X \leq N$ and $\frac{N}{X} \ll \frac{M}{X}$. Thus $M = H + X$. Therefore $\frac{M}{X}$ is radical. Let Y be a radical submodule of M such that $M = X \oplus Y$. It is clear that $Y = H$. Hence $M = X \oplus H$. Consequently, K is H-projective. ■

Lemma 6.22. *Let M be a hollow-lifting module such that $M = P(M) \oplus K$ for some local submodule K of M. Then $P(M)$ is hollow-lifting and it is K-projective.*

Proof. By Proposition 6.15, $P(M)$ is hollow-lifting. Let N be a proper submodule of M such that $M = N + K$. Then $\frac{M}{N}$ is hollow. Since M is hollow-lifting, there is a direct summand X of M such that $X \leq N$ and $\frac{N}{X} \ll \frac{M}{X}$. Thus $M = K + X$. Since $\frac{M}{K} \cong P(M)$ is radical, $\frac{X}{X \cap K}$ is also radical. But K is hollow. Then $X \cap K \ll X$ and X is radical. Therefore $X \leq P(M)$ and $M = X \oplus K$. So $P(M)$ is K-projective. ■

Proposition 6.23. *Let R be a commutative noetherian ring and let M be a supplemented R-module. Suppose that $P(M) = H$ is hollow and $M = H \oplus K$ for some submodule K of M. Then M is hollow-lifting if and only if K is hollow-lifting and H and K are relatively projective.*

Proof. Suppose that M is hollow-lifting. Then K is hollow-lifting by Proposition 6.15. By [14, Proposition 4.6], M is amply supplemented. Let N be a submodule of M with $M = N + K$. There is a submodule L of N such that L is a supplement of K in M. Since $\frac{M}{K}$ is radical, L is also radical. Thus $L \leq P(M)$. Hence $M = L \oplus K$. So H is K-projective. By Lemma 6.21, K is H-projective. For the converse we use Proposition 6.2. ■

Proposition 6.24. *Let R be a commutative noetherian ring and let M be a supplemented R-module. If M is completely hollow-lifting, then $M = P(M) \oplus N$ for some (coatomic) submodule N of M and $P(M)$ is a direct sum of hollow modules.*

Proof. By [20, Satz 2.3 and Satz 2.5], $M = \oplus_{m \in \Omega} K_m(M)$. So it suffices to prove the result over a local ring. Suppose that R is local. By [14, Corollary 2.5], $M = P(M) + X$ with a coatomic submodule X of M and $P(M)$ is supplemented and it is a sum of finitely many hollow modules. Then $P(M)$ has a finite hollow dimension. Let $P(M) = H_1 + \cdots + H_n$ with H_i is hollow for all i. Then $\frac{M}{H_2 + \cdots + H_n + X}$ is hollow radical. Since M is hollow-lifting, M has a hollow radical direct summand K_1. Let X_1 be a direct summand of M such that $M = K_1 \oplus X_1$. Since X_1 is a completely hollow-lifting supplemented module and $P(M)$ has finite hollow dimension, we have $M = P(M) \oplus N$ with a coatomic submodule N of M and $P(M)$ is a finite direct sum of hollow radical modules. ■

Recall that any module M has *finite Goldie dimension* if M does not contain an infinite direct sum of nonzero submodules.

Proposition 6.25. *Let R be a commutative noetherian ring and let M be a supplemented R-module having finite hollow dimension or finite Goldie dimension. If M is hollow-lifting, then $M = P(M) \oplus N$ such that N is a finite direct sum of local modules.*

Proof. By Proposition 3.1 and [20, Satz 2.3 and Satz 2.5], it suffices to prove the result over a local ring. Suppose that R is local. By [14, Corollary 2.5], $M = P(M) + X$ with a coatomic submodule X of M. Since M is hollow-lifting and X is a sum of local modules, M has a local direct summand (see Proposition 2.13). Let $M = K_1 \oplus X_1$ with K_1 local. Since X_1 is supplemented, we have $X_1 = P(X_1) + X_2$ with X_2 coatomic. Since K_1 has local endomorphism ring ([3, Theorem 4.1]), it has the exchange property. So X_1 is hollow-lifting by Proposition 2.12. If X_1 is not radical, then $X_2 \not\ll X_1$ and X_1 has a local direct summand. But M has finite dimension. Then we obtain that $M = Y \oplus K_1 \oplus \cdots \oplus K_n$ with Y radical and $K_i (i = 1, \ldots, n)$ are local modules. Since K_i are reduced, $Y = P(M)$. ■

The following example shows that Propositions 6.24 and 6.25 are not true in general if the ring R is not noetherian.

Example 6.26. Let K be a field and let R be the ring of polynomials in countably many commuting variables $x_1, x_2, \ldots,$ over K, subject to the relations $x_1^2 = 0$ and $x_n^2 = x_{n-1}$ for $n \geq 2$. By [16, Example 2.11], R is a local ring with maximal ideal J generated by the x_i. Further, J is nil but not nilpotent (in fact $J^2 = J$). So R is not noetherian. Let $L = R$ considered as an R-module. It is clear that L is a local module. Thus L is a completely hollow-lifting supplemented module. On the other hand, we have $P(L) = J$. Hence $P(L)$ is not a direct summand of L.

Recall that a module M is called *socle-free* if $\mathrm{Soc}(M) = 0$.

Corollary 6.27. *Let R be a commutative noetherian ring and let M be a supplemented socle-free R-module. If M is hollow-lifting, then $M = P(M) \oplus N$ for some (coatomic) submodule N of M and $P(M)$ and N both are direct sum of hollow modules.*

Proof. By Proposition 3.1 and [20, Satz 2.3 and Satz 2.5], it suffices to prove the result over a local ring. Suppose that R is local. By [14, Corollary 2.5], $M = P(M) + X$ with a coatomic submodule X of M and $P(M)$ is a finite sum of hollow radical submodules. Then M has a hollow radical direct summand H_1 (see

Proposition 2.13). By [14, Theorem 1.3] and [12, Proposition 5.10 and Corollary 5.5], every hollow radical direct summand of M has local endomorphism ring. Hence $\frac{M}{H_1}$ is hollow-lifting by Proposition 2.12. Since $\frac{M}{H_1}$ is supplemented and $P(M)$ has finite hollow dimension, we have $M = P(M) \oplus N$ with N coatomic, $P(M)$ is a finite direct sum of hollow modules and N is hollow-lifting. By [19, Folgerung 1 p. 225], N has finite Goldie dimension. By Proposition 6.25, N is a finite direct sum of local modules. ∎

Corollary 6.28. *Let R be a commutative local noetherian ring and let M be a supplemented socle-free R-module. Then M is hollow-lifting if and only if $M = P(M) \oplus N$ for some (coatomic) submodule N of M and $P(M)$ and N are relatively projective hollow-lifting modules.*

Proof. Suppose that M is hollow-lifting. By the proof of Corollary 6.27, $M = P(M) \oplus N$, $P(M) = \oplus_{i=1}^{n} H_i$ is a finite direct sum of hollow radical modules H_i and $N = \oplus_{j=1}^{m} K_j$ is a finite direct sum of local modules K_j. By Theorem 6.18, $P(M)$ and N are hollow-lifting relatively h-small projective modules. By Lemma 6.8 and Lemma 6.7, $P(M)$ and K_j are relatively h-small projective for all $j = 1, \ldots, m$ and N and H_i are relatively h-small projective for all $i = 1, \ldots, n$. By Theorem 6.18 and Proposition 6.15, $P(M) \oplus K_j$ and $H_i \oplus N$ are hollow-lifting for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By Lemma 6.21 and Lemma 6.22, $P(M)$ is K_j-projective and N is H_i-projective for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By [12, Proposition 4.33], $P(M)$ is N-projective and N is $P(M)$-projective. The converse follows from Proposition 6.2. ∎

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