TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1523-1538, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

A NEW EXPLICIT FORMULA FOR THE FUNDAMENTAL CLASS OF FUNCTIONS

Muharem Avdispahić and Lejla Smajlović

Abstract. Explicit formula for the fundamental class of functions $\left(Z, \widetilde{Z}, \Phi\right)$, introduced by J. Jorgenson and S. Lang, is given a new form valid for a more general fudge factor Φ. This is done for a larger class of test functions of generalized bounded variation.

1. Introduction

The representation of the Riemann zeta function in the form of the Euler product over all primes and the functional equation $\zeta(s) = \pi^{-s+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)} \zeta(1-s)$ that it satisfies are the basis for its application in the classical number theory. Generalization of these properties led J. Jorgenson and S. Lang to the definition of a fundamental class of functions, the class of triples $\left(Z, \widetilde{Z}, \Phi\right)$, where Z and $\widetilde{Z}$ are meromorphic functions of a finite order whose logarithmic derivative has an Euler sum representation and that satisfy a functional equation with Φ as a factor. In [8], J. Jorgenson and S. Lang proved an explicit formula for the fundamental class of functions implicitly assuming that the fudge factor Φ of the functional equation has finitely many zeros and poles in a certain vertical strip. The class of test functions was the class BV of functions of bounded Jordan variation.

In [3] we have proved that the explicit formula for the fundamental class of functions in the form given by J. Jorgenson and S. Lang is valid for a larger class of test functions of bounded generalized variation. We shall prove a new form of an explicit formula for the fundamental class of functions that is valid in the case when the fudge factor of the functional equation has infinitely many zeros and poles in any vertical strip. Test functions are of bounded generalized variation.

It can be shown that these two forms of an explicit formula are equvivalent in the case when Φ has finitely many zeros and poles in the strip $-a \leq Re(s) \leq \frac{\sigma_0}{2}$.

Received December 24, 2003; accepted August 24, 2005.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: 42A38, 26A45, 11M36.
Key words and phrases: Explicit formula, Weil's functional, $\phi-$variation.

2. Regularized Products and Series

Regularized products were introduced by J. Jorgenson and S. Lang in [7] as a generalization of a class of functions that have representation in a form of a Weierstrass product. We will be interested only in the spectral case when the regularized product is associated to the pair (L, A) of sequences of complex numbers λ_k and nonnegative integers a_k respectively. Here, λ_k correspond to zeros of the regularized product and a_k are their multiplicities. To the pair (L, A) is associated a theta function $\theta(t) = a_0 + \sum_k a_k e^{-\lambda_k t}, \quad t > 0$. In order that a regularized product and regularized harmonic series be well defined it is assumed that the theta function satisfies some asymptotic conditions at zero and infinity. Before stating these conditions, let us recall the definition of the asymptotic polynomial at zero.

To each number p in a sequence $\{p\} = \{p_j\}_{j\geq 0}$ of complex numbers with $Re(p_0) \leq Re(p_1) \leq ...$ increasing to infinity, is associated a polynomial B_p of degree n_p. Set $b_p(t) = B_p(\log t)$. An asymptotic polynomial at zero is

$$P_q(t) = \sum_{Re(p)<Re(q)} t^p b_p(t).$$

Set also $m(q) = \max \deg B_p$ for $Re(p) = Re(q)$, otherwise $m(q) = 0$.

Asymptotic conditions imposed on the theta function are the following:

(AS1.) Given $C > 0$ and $t_0 > 0$ there exists $N \in \mathbb{N}$ and $K > 0$ such that for all $t \geq t_0$ we have

$$\left| \theta(t) - \left(a_0 + \sum_{k=1}^{N-1} a_k e^{-\lambda_k t} \right) \right| \leq K e^{-Ct}.$$

(AS2.) For every $q \in \mathbb{C}$ there exists an asymptotic polynomial $P_q(t)$ at zero such that

$$\theta(t) - P_q(t) = O\left(t^{Re(q)} |\log t|^{m(q)} \right) (t \to 0).$$

(AS3.) Given $\delta > 0$ there exists $\alpha > 0$ and the constant $C > 0$ such that for all $N \in \mathbb{N}$ and $0 < t \leq \delta$ we have

$$|\theta(t) - Q_N(t)| \leq \frac{C}{t^\alpha}.$$

In what follows we will always assume that the theta function associated to the pair (L, A) satisfies conditions AS1-AS3. The order of a regularized product is defined to be (M, m), where M is an integer such that $-1 \leq M + Re p_0 < 0$ and

$m = m(p_0) + 1$ if there is a complex p such that $Re(p_0) = Re(p) < 0$, otherwise $m = m(p_0)$.

Regularized harmonic series $R(z)$ in the spectral case can be considered as a logarithmic derivative of a regularized product $D_L(z)$ associated to the pair (L, A). If $R(z) = \frac{D'}{D}(z)$, for some regularized product $D(z)$, the order of $R(z)$ is equal to the order of $D(z)$.

An important result on the representation of a regularized harmonic series is the following theorem.

Theorem 2.A. [7, Th.4.1., p. 49] *There is a polynomial $S_w(z)$ of a degree $\deg_z S_w < Re(p_0)$ such that for any $w \in \mathbb{C}$ with $Re(w) > 0$ and $Re(w) > \max_k(-Re(\lambda_k + z))$ we have*

$$R(z+w) = \int_0^\infty [\theta_z(t) - P_0\theta_z(t)]\, e^{-wz} dt + S_w(z) = I_w(z) + S_w(z),$$

where $\theta_z(t) = e^{-zt}\theta(t)$.

The factor $\eta(s) = \pi^{-s+\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}$ of the functional equation for the Riemann zeta function is not the regularized product or the regularized harmonic series since it is the ratio of two gamma functions. Knowing that the function $\frac{1}{\Gamma(s)}$ is an integral function of a finite order, we may consider $\eta(s)$ as the ratio of two regularized products.

The function $G(s)$ is the function of a regularized product type (RP type) if it can be represented as the product

$$G(z) = Q(z)\, e^{P(z)} \prod_{j=1}^{n} D_j(\alpha_j z + \beta_j)^{k_j},$$

where $Q(z)$ is a rational function, $P(z)$ is a polynomial, $D_j(z)$ is a regularized product with $\alpha_j,\beta_j \in \mathbb{C}$ and $k_j \in \mathbb{Z}$. Numbers α_j,β_j are restricted so that zeros and poles of $D_j(\alpha_j z + \beta_j)$ lie in the union of vertical strips and sectors

$\left\{z \in \mathbb{C} : -\frac{\pi}{2} + \epsilon < \arg(z) < \frac{\pi}{2} - \epsilon\right\}$,

$\left\{z \in \mathbb{C} : \frac{\pi}{2} + \varepsilon < \arg(z) < \frac{3\pi}{2} - \varepsilon\right\}$ for some $\epsilon, \varepsilon > 0$.

The function $R(z)$ is said to be of a regularized harmonic series type (RHS type) if it can be represented as the sum

$$R(z) = \sum c_j R_j(\alpha_j z + \beta_j) + P'(z) + Q'(z),$$

where R_j are regularized harmonic series, P' is a polynomial and Q' is a rational function.

The reduced order of a function that is of a RP type is defined as follows:

The reduced order of $G_0(z) = Q(z) e^{P(z)}$ is $(M-1, 0)$ where $M = \deg P$, or $(0,0)$ if $\deg P = 0$.

Let us denote by (M_j, m_j) the order of D_j. The reduced order of $G(z)$ is (M, m) where $M = \max_j (M_j)$ and $m = \max(m_j)$, the maximum being taken over all m_j such that $M = M_j$.

The logarithmic derivative of a function of a RP type is the function of RHS type of the same reduced order. The reduced order of the function is connected with its growth in a vertical strip, as illustrated by the following theorem:

Theorem 2.B. [8] *Let R be of a RHS type of a reduced order (M, m)*

(a) *Let $S = \{z \in \mathbb{C} \mid x_1 \leq Re(z) \leq x_2\}$ be a vertical strip that contains at most finitely many zeros and poles of R. Then uniformly for $x \in [x_1, x_2]$ we have the asymptotic relation*

$$R(x \pm iy) = O\left(|y|^M (\log|y|)^{m+1}\right) (|y| \to \infty).$$

(b) *Let $S = \{z \in \mathbb{C} \mid x_1 \leq Re(z) \leq x_2\}$ be a vertical strip that contains infinitely many zeros and poles of R. Then there is a sequence of real numbers $T_n \to \infty$ such that for all $x \in [x_1, x_2]$ we have the following uniform asymptotic relation*

$$R(x \pm iT_n) = O\left(T_n^M (\log T_n)^{m+1}\right) (T_n \to \infty).$$

3. A Fundamental Class of Functions

A fundamental class of functions is a generalization of a class of functions that have an Euler product representation and satisfy a functional equation whose fudge factor possesses a representation as a Weierstrass product.

Definition 3.A. [8, pp. 45-6] A triple $\left(Z, \widetilde{Z}, \Phi\right)$ is in the fundamental class of functions if the following conditions are satisfied:

(a) **Meromorphy.** Functions Z and $\widetilde{Z}$ are meromorphic functions of finite order;

(b) **Euler Sum.** There are sequences $\{q\}$ and $\{\widetilde{q}\}$ of real numbers greater than one that depend on Z and $\widetilde{Z}$ respectively such that:

- q and $\widetilde{q}$ converge to infinity
- there exist $\sigma_0' \geq 0$ and complex numbers $c(q)$ and $c(\widetilde{q})$ such that

$$\log Z(s) = \sum \frac{c(q)}{q^s}, \log \widetilde{Z}(s) = \sum \frac{c(\widetilde{q})}{\widetilde{q}^s},$$

for all s with $Re(s) > \sigma_0'$. These series are assumed to converge uniformly and absolutely in any half plane of the form $Re(s) \geq \sigma_0' + \varepsilon > \sigma_0'$.

(c) **Functional Equation.** There exist meromorphic functions G and $\widetilde{G}$, of a finite order and number σ_0, $0 \leq \sigma_0 \leq \sigma_0'$ such that

$$Z(s) G(s) = \widetilde{Z}(\sigma_0 - s) \widetilde{G}(\sigma_0 - s),$$

or

$$Z(s) \Phi(s) = \widetilde{Z}(\sigma_0 - s),$$

for $\Phi(s) = \frac{G(s)}{\widetilde{G}(\sigma_0 - s)}$.

Functions $G, \widetilde{G}$ and Φ are called fudge factors of the functional equation and are assumed to be of a regularized product type.

Fudge factors of the functional equation are closely related to the functions Z and $\widetilde{Z}$ in the sense of the following theorem:

Definition 3.A. [6, Th.1.5, p.391] Let Z and $\widetilde{Z}$ be meromorphic functions with an Euler product and functional equation. Assume that G and $\widetilde{G}$ are of regularized product type of reduced order M . Then Z and $\widetilde{Z}$ are of regularized product type of reduced order M.

4. Generalized Variation

The universal class of test functions in this paper is the class W of regulated functions [5, p. 145] i.e. functions possessing the one- sided limits at each point. For $f \in W$, we always suppose $2f(x) = f(x+0) + f(x-0)$. If I is an interval with endpoints a and b ($a < b$), we write $f(I) = f(b) - f(a)$.

Let ϕ be a continuous function defined on $[0, \infty)$ and strictly increasing from 0 to ∞. A function f is said to be of $\phi-$bounded variation on I if

$$V_\phi(f, I) = \sup \sum_n \phi(|f(I_n)|) < \infty,$$

where the supremum is taken over all systems $\{I_n\}$ of nonoverlapping subintervals of I.

Example. $\phi(u) = u$ gives us Jordan variation, and $\phi(u) = u^p$, $p > 1$, corresponds to Wiener $p-$variation.

In the latter case, $V_p(f)$ traditionally denotes the p-th root of $V_{u^p}(f)$.

It is customary to assume that the function ϕ satisfies some additional properties in order that the class ϕBV contains the class BV and that the space ϕBV of functions of bounded ϕ variation is linear. In what follows we will assume that the function ϕ is a convex function that satisfies conditions

(0_1) $\lim_{x\to 0}\frac{\phi(x)}{x}=0$

(∞_1) $\lim_{x\to\infty}\frac{\phi(x)}{x}=\infty$

(Δ_2) there exist positive constants x_0 and d ($d\geq 2)$ such that for $0\leq x\leq x_0$ the inequality $\phi(2x)\leq d\phi(x)$ holds.

The additional requirement we will impose on ϕ is related to existence and evaluation of the Stieltjes integral of functions of bounded generalized variation [11]. We assume that

$$\sum_n \phi^{-1}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{\frac{1}{p}}<\infty,$$

for some $p>1$.

Let $\Lambda=\{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\sum\frac{1}{\lambda_n}=\infty$. A function f is said to be of Λ-bounded variation on I $(f\in\Lambda BV(I))$ if

$$\sum\frac{|f(I_n)|}{\lambda_n}<\infty$$

for every choice of nonoverlapping intervals $I_n\subset I$. The supremum of these sums is called the $\Lambda-$ variation of f on I and denoted by $V_\Lambda(f,I)$. In the case $\Lambda=\mathbb{N}$, one speaks of harmonic bounded variation (HBV).

Perlman has shown that $W(I)$ is precisely the union and $BV(I)$ is the intersection of all $\Lambda BV(I)$. (See [1] for exact reference, as well as for the following remark.)

Remark 4.A. [1, p. 228] $\phi BV(I)\subset HBV(I)$, if $\sum\frac{1}{n}\phi^{-1}\left(\frac{1}{n}\right)<\infty$; $\phi BV(I)\subset\{n^\alpha\}BV(I)$, *if* $\sum\frac{1}{n^\alpha}\phi^{-1}\left(\frac{1}{n}\right)<\infty$, *for* $0<\alpha<1$.

Now, let f be an integrable function and $\widehat{f}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-itx}dx$ its Fourier transform. For such f and $A>0$, we define

$$f_A(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(y)\frac{\sin A(x-y)}{x-y}dy=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\widehat{f}(t)e^{itx}dx.$$

In [2] we have proved the following theorem.

Theorem 4.A. [2] *If* $f \in HBV\left(\mathbb{R}\right) \cap L^1\left(\mathbb{R}\right)$*, then* f_A *is bounded independently of* A*,* $f_A\left(x\right) \to f\left(x\right)\ \left(A \to \infty\right)$ *everywhere and convergence is uniform on compact sets of points of continuity of* f*.*

5. The Explicit Formula

Let f and F be functions defined on $\left(0, \infty\right)$ and $\left(-\infty, \infty\right)$ respectively, connected by the relation $f\left(x\right) = F\left(-\log x\right)$, $x \in \left(0, \infty\right)$. We will call both functions f and F test functions. We will formally denote by

$$Mf\left(s\right) = \int_0^\infty f\left(t\right) t^s \frac{dt}{t}$$

the Mellin transform of the function f, and by $M_u f\left(s\right)$ its translation by u.

Let $\left(Z, \widetilde{Z}, \Phi\right)$ be in the fundamental class of functions, with the fudge factor Φ of a reduced order $\left(M, m\right)$. Both functions Z and $\widetilde{Z}$ are of RP type of a reduced order $\left(M, m\right)$ or $\left(M, m+1\right)$. Let $a > 0$ be a real number such that $\sigma_0' < \sigma_0 + a$ and that functions Z, $\widetilde{Z}$ and Φ have no zeros or poles on lines $Re\left(s\right) = -a$ and $Re\left(s\right) = \sigma_0 + a$.

We will denote by:

- $R_{a,\sigma}$ the infinite rectangle bounded by the lines $Re\left(s\right) = -a$ and $Re\left(s\right) = \sigma + a$.
- $R_{a,\sigma_0}\left(T\right)$ the finite rectangle bounded by the lines $Re\left(s\right) = -a$, $Re\left(s\right) = \sigma_0 + a$ and horizontal lines $Im\left(s\right) = \pm T$.
- $\{\rho\}$ the set of zeros and poles of Z in the full strip R_{a,σ_0}.

If T is chosen such that functions $Z, \widetilde{Z}$ and Φ have no zeros or poles on the horizontal lines that border $R_{a,\sigma_0}\left(T\right)$, it is possible to form the sum

$$S_{Z,a}\left(f, T\right) = \sum_{\rho \in R_{a,\sigma_0}\left(T\right)} ord\left(\rho\right) \cdot M_{\frac{\sigma_0}{2}} f\left(\rho\right).$$

We will be interested in conditions on $\left(Z, \widetilde{Z}, \Phi\right)$ and f, or equivalently F, that ensure the convergence of the last sum when $T \to \infty$.

On an M-times differentiable test function F we will impose the following conditions:

- Ex I $\qquad F^{(j)}\left(x\right) e^{\left(a' + \frac{\sigma_0}{2}\right)\left|x\right|} \in \phi BV\left(\mathbb{R}\right) \cap L^1\left(\mathbb{R}\right),$

- Ex II $$F^{(j)}(x) = F^{(j)}(0) + O\left(|\log|x||^{-\alpha}\right) \quad (\alpha > M+2),$$

for some $a' > a > 0$ and $j \in \{0, 1, ...M\}$.

Let us note that the function ϕ is assumed to satisfy conditions given in the previous section.

In [3] we proved that the Mellin transform of a function that satisfies less restrictive properties than Ex I and Ex II has a polynomial decay in the vertical strip. Actually, we have proved that, for M times differentiable function F we have the estimate

$$M_{\frac{\sigma_0}{2}} f(s) = O\left(\left(\frac{1}{|t|}\right)^{M+1-\frac{1}{p}}\right),$$

uniformly in $-a \leq \sigma \leq \sigma_0 + a$.

The main result of the paper is a new explicit formula given by the following theorem.

Theorem 5.1. *Let* $\left(Z, \widetilde{Z}, \Phi\right)$ *be in the fundamental class of functions and assume that* Φ *is of a reduced order* (M, m). *Then for a test function* F *that satisfies conditions Ex I and Ex II functionals* $S_{Z,a}$ *and* $W_{\Phi,-a}$ *are well defined and the explicit formula, i.e. the formula*

$$S_{Z,a}(f) = \sum_q \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) + W_{\Phi,-a}(F_a)$$

holds, where $F_a(x) = F(x)\, e^{\left(\frac{\sigma_0}{2}+a\right)x}$ *and*

$$W_{\Phi,-a}(F_a) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{F}_a(t) \frac{\Phi'}{\Phi}(-a+it)\, dt,$$

denotes a generalized Weil functional.

The proof of the theorem consists of two major parts. In the first part we will evaluate sums over q and $\widetilde{q}$ using Euler sum representation and the functional equation. The first part of the proof does not differ from the proof of the explicit formula given in [3]. For the completeness we will give a short sketch of that part. The second part of the proof will be explained in details.

6. Evaluation of Sums

Let $B_a(T_n)$ denote the boundary of the rectangle $R_{a,\sigma_0}(T_n)$ defined above. By the residue theorem,

$$\sum_{\rho\in R_{a,\sigma_0}(T_n)} ord\,(\rho)\cdot M_{\frac{\sigma_0}{2}}f\,(\rho)=\frac{1}{2\pi i}\int_{B_a(T_n)} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{Z'}{Z}\,(s)\,ds$$
$$=\frac{1}{2\pi i}\left[\int_{-a-iT_n}^{\sigma_0+a-iT_n}+\int_{\sigma_0+a-iT_n}^{\sigma_0+a+iT_n}+\int_{\sigma_0+a+iT_n}^{-a+iT_n}+\int_{-a+iT_n}^{-a-iT_n}\right]$$
$$=I_1+I_2+I_3+I_4.$$

Since the function Z is of a reduced order at most $(M,m+1)$, we have that $M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{Z'}{Z}\,(s)\to 0,\ n\to\infty$ uniformly in s, for s on the lines $\sigma\pm iT_n$, $-a\leq\sigma\leq\sigma_0+a$. Therefore, $I_1\to 0$, and $I_3\to 0$ as $n\to\infty$. Hence,

$$\sum_{\rho\in R_{a,\sigma_0}(T_n)} ord\,(\rho)\cdot M_{\frac{\sigma_0}{2}}f\,(\rho)+o\,(1)=\frac{1}{2\pi i}\int_{\sigma_0+a-iT_n}^{\sigma_0+a+iT_n} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{Z'}{Z}\,(s)\,ds \tag{1}$$
$$+\frac{1}{2\pi i}\int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{Z'}{Z}\,(s)\,ds$$

Using the functional equation, the right - hand side of (1) becomes

$$\frac{1}{2\pi i}\int_{\sigma_0+a-iT_n}^{\sigma_0+a+iT_n} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{Z'}{Z}\,(s)\,ds-\frac{1}{2\pi i}\int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{\widetilde{Z}'}{\widetilde{Z}}\,(\sigma_0-s)\,ds$$
$$-\frac{1}{2\pi i}\int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}}f\,(s)\,\frac{\Phi'}{\Phi}\,(s)\,ds=J_1+J_2+J_3.$$

In this section we will deal with J_1 and J_2 and in the next section we will treat J_3. For simplicity, we will evaluate J_1. For $s=\sigma_0+a+it$, $-T_n\leq t\leq T$, it is possible to interchange the sum and the derivative in the Euler sum for $\log Z\,(s)$, since the sum converges uniformly for $Re\,(s)=\sigma_0+a>\sigma_0'$. We obtain

$$\frac{Z'}{Z}\,(s)=\sum_q\frac{c\,(q)\log q}{q^s}.$$

Making a change of variables $s=\sigma_0+a+it$, $-T_n\leq t\leq T$ in J_1, we get that

$$J_1=\frac{-1}{2\pi}\int_{-T_n}^{T_n}dt\left(\sum_q\int_{-\infty}^{\infty}F\,(x)\,e^{-\left(\frac{\sigma_0}{2}+a+it\right)x}\frac{c\,(q)\log q}{q^{\sigma_0+a+it}}dx\right),$$

and after a change of variables $x=y-\log q$, we obtain

$$J_1=\frac{-1}{2\pi}\int_{-T_n}^{T_n}dt\sum_q\int_{-\infty}^{\infty}F\,(y-\log q)\,e^{-\left(\frac{\sigma_0}{2}+a+it\right)y}q^{-\frac{\sigma_0}{2}}c\,(q)\log q dy.$$

Using the condition Ex I, it can be shown that the series

$$\sum_q F(y-\log q)\, e^{-\left(\frac{\sigma_0}{2}+a+it\right)y} q^{-\frac{\sigma_0}{2}} c(q) \log q = \sum_q B_q(y)\, e^{-ity}$$

converges uniformly in $y \in (-\infty, \infty)$. This enables us to interchange the integral and the sum in J_1, so that it becomes:

$$J_1 = -\frac{1}{2\pi} \int_{-T_n}^{T_n} dt \int_{-\infty}^{\infty} \left(\sum_q B_q(y) \right) e^{-ity} dy.$$

Let us put $B(y) = \sum_q B_q(y)$, where the series on the right converges uniformly in y. Using the condition Ex I it can be shown that $B_q \in L^1(\mathbb{R}) \cap \phi BV(\mathbb{R})$ for all q. Hence $B_q \in L^1(\mathbb{R}) \cap HBV(\mathbb{R})$, by the assumptions on ϕ.

Using the definition of B_q and the condition Ex I we obtain that its harmonic variation is $O\left(\frac{|c(q)\log q|}{q^{\sigma_0+a}}\right)$. The absolute convergence of the series $\sum \frac{c(q)\log q}{q^{\sigma_0+a}}$ implies that $B \in HBV(\mathbb{R})$

To the function B we can apply Theorem 4.A. to get

$$\lim_{n\to\infty} J_1 = \lim_{n\to\infty} \frac{-1}{\sqrt{2\pi}} \int_{-T_n}^{T_n} \widehat{B}(t)\, dt = -B(0) = -\sum_q \frac{c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q).$$

It is possible to carry out similar arguments for J_2, now putting

$$B_{\widetilde{q}}(y) = F(y + \log \widetilde{q})\, e^{\left(\frac{\sigma_0}{2}+a\right)y} \widetilde{q}^{-\frac{\sigma_0}{2}} c(\widetilde{q}) \log \widetilde{q}.$$

Finally we get that

$$\lim_{n\to\infty} J_2 = -\widetilde{B}(0) = -\sum_{\widetilde{q}} \frac{c(\widetilde{q})\log\widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right).$$

At this point, we have proved that

$$\begin{aligned} S_{Z,a}(f) &= \sum_q \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log\widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) \\ &\quad + \lim_{n\to\infty} \frac{1}{2\pi i} \int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}} f(s) \left(-\frac{\Phi'}{\Phi}(s)\right) ds. \end{aligned} \tag{2}$$

7. Generalized Weil's Functional

In this section we will prove that the limit on the right hand side of (2) exists and will evaluate it. The evaluation is based on two lemmas and the general Parseval formula proved in [3] and [2]. Let us first recall these lemmas and the theorem.

Lemma 7.A. [3] *Assume that* $g \in HBV(\mathbb{R}) \cap L^1(\mathbb{R})$. *Then*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \widehat{g}(t)\,\frac{1}{t+\alpha}dt = \begin{cases} -i\int_0^{+\infty} g(x)\,e^{i\alpha x}dx, & Im\alpha > 0 \\ i\int_0^{+\infty} g(-x)\,e^{-i\alpha x}dx, & Im\alpha < 0 \end{cases}.$$

Lemma 7.B. [3] *Assume that* g *satisfies condition:*

$$g^{(j)} \in HBV(\mathbb{R}) \cap L^1(\mathbb{R}),$$

for $j = \overline{0, M}$. *Then, for all* $n \in \{0, 1, ..., M\}$ *we have that*

$$\lim_{T\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\widehat{g}(t)\,(it)^n\,e^{itx}dt = g^{(n)}(x).$$

Before we state the general Parseval formula, let us recall the definition of a functional $H_{\mu,\varphi}$ that is a generalization of the scalar product. It depends on a pair (μ, φ) of a measure μ and a function φ. Functions μ and φ are assumed to satisfy the following conditions:

1° μ is a Borel measure on $\mathbb{R}^+$ such that $d\mu(x) = h(x)\,dx$, where h is some bounded measurable function.

2° φ is a measurable function on $\mathbb{R}^+$ having following two properties:

(a) there exists a complex polynomial at zero $P_0(t)$ such that

$\varphi(x) - P_0(x) = O(|\log x|^m)$, $x \downarrow 0$ for some integer $m > 0$;

(b) let M be the integer such that $-1 \le M + Re(p_0) < 0$. Then both $x^M P_0(x)$ and $\varphi(x)$ are in $L^1(|\mu|)$ outside a neighborhood of zero.

Given a special pair (μ, φ), a functional $H_{\mu,\varphi}$ on the Schwartz space $\mathcal{S}$ of test functions is defined by

$$H_{\mu,\varphi}(\beta) = \int_0^{\infty}\left(\varphi(x)\,\beta(x) - \sum_{k=0}^{M}(-1)^k\,u_k(x)\,\beta^{(k)}(0)\right)d\mu(x),\ \beta \in \mathcal{S},$$

where $u_k(x) = P_0(x)\,\frac{x^k}{k!}$. Its ”Fourier transform” as a distribution is the function $\widehat{H}_{\mu,\varphi}(t)$ such that

$$\begin{aligned}\widehat{H}_{\mu,\varphi}(t) &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\left(\varphi(x)\,e^{-itx} - \sum_{k=0}^{M}u_k(x)\,(-it)^k\right)d\mu(x) \\ &= \frac{1}{\sqrt{2\pi}}H_{\mu,\varphi}\left(e^{-itx}\right).\end{aligned}$$

The next theorem is a general Parseval formula.

Theorem 7.A. [2] *Let* f *be an* $M-$*times differentiable function such that* $f^{(j)} \in HBV(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R})$ *for* $j \in \{0, 1, ..., M-1\}$ and $f^{(M)} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$. *Assume that* $f^{(M)}(x) - f^{(M)}(0) = O\left(|\log|x||^{-\alpha}\right)$ *for some* $\alpha > M+2$. *Then, for a special pair* (μ, φ) *we have*

$$\lim_{A\to\infty}\int_{-A}^{A}\widehat{f}(t)\,\widehat{H}_{\mu,\varphi}(t)\,dt$$

$$= \int_0^\infty \left(\varphi(x) f(-x) - \sum_{k=0}^{M} u_k(x)(-1)^k f^{(k)}(0)\right) d\mu(x) = H_{\mu,\varphi}\left(f^-\right).$$

Now, let us show how these lemmas and the theorem apply to the evaluation of the generalized Weil functional.

After the change of variable, the integral in (2) can be written as

$$\frac{1}{2\pi i}\int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}} f(s)\left(-\frac{\Phi'}{\Phi}(s)\right) ds$$

(3)

$$= \frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}_a(t)\,\frac{\Phi'}{\Phi}(-a+it)\,dt = W_{\Phi,-a}(F_a).$$

The function F_a is an $M-$times differentiable function such that

$$F_a^{(k)}(x) = \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\left(a+\frac{\sigma_0}{2}\right)^j F^{(k-j)}(x)\, e^{\left(a'+\frac{\sigma_0}{2}\right)|x|} e^{\left(a+\frac{\sigma_0}{2}\right)x-\left(a'+\frac{\sigma_0}{2}\right)|x|}.$$

Since $F^{(k-j)}(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ and $e^{\left(a+\frac{\sigma_0}{2}\right)x-\left(a'+\frac{\sigma_0}{2}\right)|x|} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$, we get $F_a^{(k)}(x) \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ for all $k = 0, 1, ..., M$, because the class ϕBV is closed under multiplication. On the other hand,

$$F_a^{(M)}(x) - F_a^{(M)}(0)$$

$$= \sum_{j=0}^{M}\begin{pmatrix} M \\ j \end{pmatrix}\left(a+\frac{\sigma_0}{2}\right)^j\left(F^{(M-j)}(x)\,e^{\left(a+\frac{\sigma_0}{2}\right)x} - F^{(M-j)}(0)\right).$$

Since $e^{\left(a+\frac{\sigma_0}{2}\right)x} = 1 + O(|x|)$ $(x \to 0)$ the condition Ex II implies

$$F_a^{(M)}(x) - F_a^{(M)}(0) = O\left(|\log|x||^{-\alpha}\right)(x \to 0).$$

The inclusion $\phi BV(\mathbb{R}) \subseteq HBV(\mathbb{R})$ implies that the function F_a satisfies the assumptions of Lemma 7.A, Lemma 7.B and Theorem 7.A.

The function $\frac{\Phi'}{\Phi}$ is a sum of a rational function with no zeros and poles on the line $Re\,(s) = -a$, a polynomial of a degree less or equal M and a regularized harmonic series $\frac{D_j'}{D_j}$, where D_j is a regularized product. By the linearity of the integral in (3) we see that it is enough to consider the following three cases.

1^o $\Phi(z) = Q(z)$, for some rational function Q. We can express $\frac{\Phi'}{\Phi}$ as

$$\frac{\Phi'}{\Phi}(-a+it) = \sum_{\alpha=\sigma+i\tau} \frac{-iA_\alpha}{t + (-\tau + i(\sigma+a))},$$

where the sum on the left is taken over all zeros and poles α of the function $\Phi(z)$, where $\sigma + a \neq 0$. Application of Lemma 7.A. gives us

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}\pi} \int_{-T_n}^{T_n} \widehat{F}_a(t) \frac{\Phi'}{\Phi}(-a+it)\,dt$$
$$= \sum_{\substack{\alpha=\sigma+i\tau \\ \sigma>-a}} - A_\alpha \int_0^\infty F(x)\, e^{-\left(\sigma - \frac{\sigma_0}{2} + i\tau\right)x} dx$$
$$+ \sum_{\substack{\alpha=\sigma+i\tau \\ \sigma>-a}} A_\alpha \int_0^\infty F(-x)\, e^{\left(\sigma - \frac{\sigma_0}{2} + i\tau\right)x} dx.$$

This shows us how to evaluate $W_{\Phi,-a}(F_a)$ in the case $\Phi(z) = Q(z)$.

2^o Let $\Phi(z) = e^{P(z)}$, or $\frac{\Phi'}{\Phi}(z) = P'(z)$, for $\deg P'(z) \le M$. If $P'(-a+it) = \beta_M (it)^M + \ldots + \beta_1 (it) + \beta_0$, the application of Lemma 7.B. to the function F_a and polynomial P' gives us:

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}\pi} \int_{-T_n}^{T_n} \widehat{F}_a(t) \frac{\Phi'}{\Phi}(-a+it)\,dt$$
$$= \sum_{j=0}^{M} \beta_j \sum_{k=0}^{j} \binom{j}{k} \left(a + \frac{\sigma_0}{2}\right)^k F^{(j-k)}(0).$$

3^o Let $\Phi(z) = D(z)$, for some regularized product D of order less or equal M. Then, $D = D_L$ (up to the multiplicative constant), for some sequences $L = \{\lambda_k\}$ of complex numbers and $A = \{a_k\}$ of integers. It is known that conditions AS1-AS3 imply that $Re\,(\lambda_k) \to \infty$ when $k \to \infty$, so, for some $m \in \mathbb{N}$ and $L_m = \{\lambda_{m+1}, \ldots\}$ we have

$$\frac{D_L'}{D_L}(z) = \sum_{k=0}^{m} \frac{a_k}{z + \lambda_k} + \frac{D_{L_m}'}{D_{L_m}}(z),$$

where $-a > -Re(\lambda_k)$ for all $k \geq m+1$. In the first case we have evaluated the Weil functional of a rational function, so without loss of generality we may assume that the sequence L is such that $\max_{\lambda_k \in L}\{-Re(\lambda_k)\} < -a$. This enables us to apply Theorem 2.A. to the regularized harmonic series $R(z) = \frac{D'_L}{D_L}(z)$ and get, for arbitrary $\alpha > 0$ and $z = -(a+\alpha) + it$

$$\frac{D'_L}{D_L}(-a+it) = I_\alpha(-(a+\alpha)+it) + S_\alpha(-(a+\alpha)+it),$$

where $S_\alpha(z)$ is a polynomial of a degree less than or equal to M.

We will deal only with the first summand on the right-hand side of the last equality since the case when $\frac{\Phi'}{\Phi}$ is a polynomial was already treated. It is left to evaluate

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T_n}^{T_n} \widehat{F_a}(t)\, I_\alpha(-(a+\alpha)+it)\, dt$$

Let us write $I_\alpha(-(a+\alpha)+it)$ as

$$I_\alpha(-(a+\alpha)+it) = \int_0^\infty \left[\theta_{-(a+\alpha)}(x)\, e^{-itx} - \sum_{k+Re(p_0)<0} c_k(-(a+\alpha), x)(-it)^k\right] e^{-\alpha x} dx,$$

and show that $(\mu_\alpha, \theta_{-(a+\alpha)})$, for $d\mu_\alpha(x) = e^{-\alpha x}dx$ is a special pair.

The conditions required for the Borel measure μ_α are obviously satisfied.

The condition AS2 satisfied by the theta function implies that

$$\theta_{-(a+\alpha)}(x) - P_0\theta_{(-(a+\alpha)+it)}(x) = O(|\log x|^m),\ (x \downarrow 0).$$

Since $P_0\theta_{(-(a+\alpha)+it)}(x)$ is a polynomial in x, $x^M P_0\theta_{(-(a+\alpha)+it)}(x)$ is $\mu_\alpha-$integrable outside a neighborhood of zero.

To prove $\mu_\alpha-$integrability of the function $\theta_{-(a+\alpha)}(x) = \theta(x)\, e^{(a+\alpha)x}$ outside a neighborhood of zero, we will use condition AS1 which implies that for given $C > 0$ and $x_0 > 0$ there is a natural number N and a constant K such that

$$\left|\theta(x)\, e^{ax} - \sum_{k=0}^{N} a_k e^{-(\lambda_k - a)x}\right| \leq K e^{-(C-a)x},$$

for all $x \geq x_0$.

If we choose $C > a$, the condition $-a > -Re(\lambda_k)$ for all k implies that functions $e^{-(C-a)x}$ and $\sum_{k=0}^{N} a_k e^{-(\lambda_k - a)x}$ are integrable outside neighborhood of zero, and so is $\theta(x)\, e^{ax}$.

We see that $\left(\mu_\alpha, \theta_{-(a+\alpha)}\right)$ is a special pair, so

$$I_\alpha\left(-(a+\alpha)+it\right)=\sqrt{2\pi}\widehat{H}_{\mu_\alpha,\theta_{-(a+\alpha)}}\left(t\right).$$

Application of the Theorem 7.A. gives us that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F_a}\left(t\right)I_\alpha\left(-(a+\alpha)+it\right)dt &= H_{\mu_\alpha,\theta_{-(a+\alpha)}}\left(F_a^-\right)\\ &= \int_0^\infty\Bigg[\theta_{\left(\frac{\sigma_0}{2}-\alpha\right)}\left(x\right)F\left(-x\right)\\ -\sum_{k+Re(p_0)<0}\sum_{j=0}^{k}\begin{pmatrix}k\\ j\end{pmatrix}&\left(a+\frac{\sigma_0}{2}\right)^j c_k\left(-a-\alpha,x\right)F^{(k-j)}\left(0\right)\Bigg]e^{-\alpha x}dx.\end{aligned}$$

This completes the evaluation of the generalized Weil functional and finishes the proof of the new explicit formula.

Acknowledgment

This work is partially supported by a research grant of the Federal Ministry of Science and Education in Bosnia and Herzegovina.

References

1. M. Avdispahić, Concepts of generalized bounded variation and the theory of Fourier series, *Internat. J. Math. Math. Sci.*, **9** (1986), 223-244.
2. M. Avdispahić and L. Smajlović, $\phi-$variation and Barner-Weil formula, *Math. Balkanica*, **17** (2003), Fasc. 3-4, 267-289.
3. M. Avdispahić and L. Smajlović, An explicit formula for a fundamental class of functions, *Bull. Belg. Math. Soc. Simon Stevin*, **12(4)** (2005), 569-587.
4. K. Barner, On Weil's explicit formula, *J. Reine Angew. Math.*, **323** (1981), 139-152.
5. J. Dieudonne, *Treatise on Analysis*, Vol. I, (*Foundations of Modern Analysis*), Academic Press, New York and London, 1969.
6. J. Jorgenson and S. Lang, On Cramer's theorem for general Euler products with functional equation, *Math. Ann.*, **297** (1993), 383-416.
7. J. Jorgenson and S. Lang, *Basic analysis of regularized series and products*, Springer Lecture Notes in Mathematics, **1564** (1993).
8. J. Jorgenson and S. Lang, *Explicit Formulas for Regularized Products and Series*, Springer Lecture Notes in Mathematics **1593** (1994).

9. S. Lang, *Algebraic Numbers*, Adison-Wesley, Reading, Mass., 1964.

10. A. Weil, Sur les ”formules explicites” de la theorie des nombres premiers, *Comm. Sem. Math. Univ. Lund*, Suppl. (1952), 252-265.

11. L. C. Young, Generalized inequalities for Stieltjes integrals and the convergence of Fourier series, *Math. Ann.*, **115** (1938), 581-612.

Muharem Avdispahić and Lejla Smajlović
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35
71000 Sarajevo
Bosnia and Herzegovina
E-mail: mavdispa@pmf.unsa.ba
E-mail: lejlas@pmf.unsa.ba