FUNCTIONS OF BOUNDED MEAN OSCILLATION

Der-Chen Chang and Cora Sadosky

Abstract. BMO, the space of functions of bounded mean oscillation, was first introduced by F. John and L. Nirenberg in 1961. It became a focus of attention when C. Fefferman proved that BMO is the dual of the (real) Hardy space H^1 in 1971. In the past 30 years, this space was studied extensively by many mathematicians. With the help of BMO, many phenomena can be characterized clearly. In this review we discuss the connections between BMO functions, the sharp function operator, Carleson measures, atomic decompositions and commutator operators in $\mathbf{R}^n$. We strive to cover some of the main developments in the theory, including BMO in a bounded Lipschitz domain in $\mathbf{R}^n$ and in the product space $\mathbf{R} \times \mathbf{R}$.

1. The BMO Space and the Sharp Function Operator

The Lebesgue L^p spaces play an important role in Fourier analysis, as can be seen in many examples (see e.g., Stein's books [39] and [40]). However, many important classes of operators are not well behaved on the spaces L^1 and L^∞. In fact, many of these operators are unbounded on L^1. Therefore, L^1 is too large to be the domain of such operators. By the same token, the target space of many canonical operators exceeds L^∞. Hence L^∞ is too small to be the range of such operators. (These two deficiencies are dual in a certain sense.) The motivation to find substitutes for the spaces L^1 and L^∞ led to the Hardy space H^1 (derived from complex function theoretic considerations in the early part of last century), and

to the space BMO of functions of bounded mean oscillation (introduced by John and Nirenberg [29] in 1961, in the context of partial differential equations). These spaces turned out to be the "right" spaces to study instead of L^1, L^∞ respectively. In fact, many of the operators that we wish to study, and which are ill-behaved on L^1 or L^∞, are bounded on H^1 and on BMO; an example being the $T(1)$ theorem for Calderón-Zygmund operators by David and Journé [18]. These two spaces lead to deep insights concerning quasi-conformal mappings, Cauchy integrals on Lipschitz curves, probability theory, and partial differential equations.

In this paper, we just concentrate on the space BMO. We discuss the connections between BMO functions, the sharp function operator, Carleson measures, atomic decompositions, and commutator operators in $\mathbf{R}^n$. We also summarize some of the main developments in the BMO theory, including those on bounded Lipschitz domains in $\mathbf{R}^n$, and on the product space $\mathbf{R} \times \mathbf{R}$.

We start with two definitions. Here, as in the rest of the survey, Q stands for a cube with sides parallel to the axes.

Definition 1.1. Let f be a locally integrable function defined on $\mathbf{R}^n, n \geq 1$. Denote $f_Q^\sharp$ the mean oscillation of f in a cube Q,

$$f_Q^\sharp = \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx$$

where

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx.$$

The sharp function operator $\sharp : f \mapsto f^\sharp$ is defined as follows:

$$(1.1) \qquad f^\sharp(x) = \sup_{r>0} f_{Q(x;r)}^\sharp,$$

where $Q(x;r)$ is a cube centered at x and of diameter $\ell(Q) = r$.

Definition 1.2. f is a funcion of bounded mean oscillation, $f \in BMO$, if and only if $f^\sharp \in L^\infty(\mathbf{R}^n)$.

Denote $\|f\|_* = \|f^\sharp\|_{L^\infty}$ as the BMO norm of f. Note that

$$\|f\|_* = 0 \quad \text{if} \quad f \equiv \text{constant}.$$

Therefore, an element in BMO is in fact an equivalent class. More precisely,

$$f = g \quad \text{in } BMO \quad \Leftrightarrow \quad f - g = \text{constant}.$$

Obviously, $BMO(\mathbf{R}^n)$ is a normed space. Furthermore, we may replace f_Q, the mean of f over Q, by any other constant. More precisely, one can show that $f \in BMO(\mathbf{R}^n)$ if and only if for any cube $Q \subset \mathbf{R}^n$, there exists a constant C_Q (depending on Q) such that

$$\sup_Q \frac{1}{|Q|} \int_Q |f(x) - C_Q(x)| dx < \infty.$$

Denote

$$\|f\|_{**} = \sup_Q \inf_C \frac{1}{|Q|} \int_Q |f(x) - C| dx.$$

It can be shown that $\|f\|_{**}$ and $\|f\|_*$ are equivalent. We may use either $\|\cdot\|_*$ or $\|\cdot\|_{**}$ as the norm function on BMO to show that BMO is a Banach space.

It is easy to see that $L^\infty \subset BMO$, but $BMO \not\subset L^\infty$. A famous example is $\log|x| \in BMO(\mathbf{R}^n)\backslash L^\infty(\mathbf{R}^n)$. Another interesting phenomenon is that, in general, we cannot localize a BMO function. For example, $\chi_{[0,\infty)} \log|x| \notin BMO(\mathbf{R})$ although $\log|x| \in BMO(\mathbf{R})$.

From the definitions of $\|\cdot\|_*$ and $\|\cdot\|_{**}$, a deeper knowledge about $BMO(\mathbf{R}^n)$ can be obtained. In fact, from $\|\cdot\|_*$ (or $\|\cdot\|_{**}$), the statement

$$\text{``} f \in BMO(\mathbf{R}^n) \Leftrightarrow \int_Q |f(x) - f_Q|\, dx < \infty \text{''}$$

implicitly implies that

$$\text{``} f \in BMO(\mathbf{R}^n) \Leftrightarrow \int_Q |f(x) - f_Q|^p\, dx < \infty, \quad \text{for } 1 \le p < \infty. \text{''}$$

Furthermore, we can make a weaker assumption on $\int_Q |f(x) - f_Q| dx < \infty$ in order to characterize $BMO(\mathbf{R}^n)$.

Let us define

(1.2) $$\mu_Q(\alpha) = |\{x \in Q : |f(x) - f_Q| > \alpha\}|$$

where $|A|$ is the Lebesgue measure of the set A. Then we have

Proposition 1.3. *If there exist two constants B, β such that, for all cubes Q,*

(1.3) $$\mu_Q(\alpha) \le B \cdot |Q| \cdot e^{-\beta\alpha}$$

then $f \in BMO(\mathbf{R}^n)$.

In fact, property (1.3) characterizes BMO, as proved by F. John and L. Nirenberg [29]. Here we state a slightly more general version of this theorem. Let $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$ be a continuously increasing function satisfying $\Phi(\mathbf{R}^+) = \mathbf{R}^+$ and

$$\Phi(\alpha + \beta) \le C_\Phi(\Phi(\alpha) + \Phi(\beta) + 1), \quad \text{for all } \alpha, \beta \in \mathbf{R}^+,$$

for $C_\Phi \geq 1$ a universal constant. Denote

$$BMO_\Phi(\mathbf{R}^n) =$$

$$\left\{ f \in L_{loc}^\Phi(\mathbf{R}^n) : \|f\|_{*,\Phi} = \Phi^{-1}\left(\sup_Q \inf_{C_Q \in \mathbf{C}} \frac{1}{|Q|} \int_Q \Phi(|f(x) - C_Q|) dx \right) < \infty \right\}.$$

Theorem 1.4. (John-Nirenberg inequality). *For every $f \in BMO_\Phi$, every $Q \subset$ $\mathbf{R}^n$, and $\alpha > 0$, there exist two positive constants B, depending only on the dimension n, and b_f, such that*

(1.4) $$\frac{1}{|Q|}\mu_Q(\alpha) \leq B \cdot \exp\left(-\frac{\alpha}{b_f} \right)$$

where $\mu_Q(\alpha)$ is defined by (1.2).

As a consequence, one has

Corollary 1.5. *Suppose that f is a measurable function having the property that for all cubes Q there exists a constant C_Q such that*

$$\sup_Q \frac{1}{|Q|}|\{x \in Q : |f(x) - C_Q| > \alpha\}| = \psi(\alpha) \to 0, \qquad as\ \alpha \to \infty.$$

Then $f \in BMO(\mathbf{R}^n)$.

One of the applications of the John-Nirenberg inequality is to show that all $BMO_p(\mathbf{R}^n)$, i.e., BMO_Φ for $\Phi(x) = x^p$, are equivalent when $1 \leq p < \infty$. By definition, $BMO_p(\mathbf{R}^n)$ is the class of all measurable functions such that

$$\|f\|_{*,p}^p = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx < \infty$$

with $BMO_1 = BMO$.

Assume that $f \in BMO_p(\mathbf{R}^n)$. Since $\Phi(x) = x^p$, we have

$$\Phi(\alpha + \beta) \leq C_p(\Phi(\alpha) + \Phi(\beta)).$$

Furthermore, one may choose the constant B in (1.3) as e. Therefore, the constant b_f will be

$$(C_p 2^n e\|f\|_{*,p}^p + C_p\|f\|_{*,p}^p)^{\frac{1}{p}} = C_p'\|f\|_{*,p}.$$

Therefore, we can rewrite (1.4) as follows:

$$\frac{1}{|Q|}|\{x \in Q : |f(x) - c_Q| > \alpha\}| \leq e \cdot \exp\left(-\frac{\alpha}{C_p'\|f\|_{*,p}}\right).$$

It follows that

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - c_Q| dx$$

$$\leq e \cdot \int_0^\infty \exp\left(-\frac{\alpha}{C_p'\|f\|_{*,p}}\right) d\alpha \leq \tilde{C}_p \cdot \|f\|_{*,p}.$$

Similarly, we have

$$\|f\|_{*,p} = \sup_Q \left(\frac{1}{|Q|}\int_Q |f(x) - c_Q|^p dx\right)^{\frac{1}{p}}$$

$$\leq C_p' \left(\int_0^\infty \alpha^{p-1} \cdot e^{-\frac{\alpha}{C_p\|f\|_*}} d\alpha\right)^{\frac{1}{p}}$$

$$\leq \tilde{C}_p \cdot \|f\|_*.$$

Now we have the following corollary.

Corollary 1.6.   *Let $1 \leq p < \infty$, then $BMO_p(\mathbf{R}^n) = BMO(\mathbf{R}^n)$. More precisely,*

$$C_p^{-1}\|f\|_{*,p} \leq \|f\|_* \leq C_p\|f\|_{*,p}.$$

We now state the maximal theorem for the sharp function operator due to C. Fefferman and E.M. Stein [20]. This theorem will allow the interpolation between L^p and BMO spaces.

Theorem 1.7.   (Sharp function operator). *Let $1 < p < \infty$. For every $f \in L^p(\mathbf{R}^n)$ there exists a constant C_p, independent of f, depending only on n and p, such that*

(1.5) $$C_p^{-1}\|f\|_{L^p} \leq \|f^\sharp\|_{L^p(\mathbf{R}^n)} \leq C_p\|f\|_{L^p}$$

and thus

(1.6) $$f \in L^p(\mathbf{R}^n) \Leftrightarrow f^\sharp \in L^p(\mathbf{R}^n).$$

Remark. We know that if $f \in L^\infty \cap L^1$ then $f \in L^p$ for all $1 < p < \infty$. A similar result holds for BMO, namely that if $f \in BMO \cap L^1$ then $f \in L^p$, $1 < p < \infty$. Moreover,

$$\|f\|_{L^p} \leq C_p \cdot \|f\|_{L^1}^{\frac{1}{p}} \cdot \|f\|_*^{\frac{1}{p'}}.$$

This fact can be proved by using Theorem 1.7.

As we mentioned at the beginning of this section, many classical operators T are bounded from $L^p(\mathbf{R}^n)$ to itself for $1 < p < \infty$, but not for $p = 1$ or $p = \infty$ (see e.g., Sadosky [37], Stein-Weiss [42]). The substitute results for $p = 1$ and $p = \infty$ usually are $T : L^1 \to L^{1,\infty}$ and $T : L^{\infty,\infty} \to L^\infty$. Here $L^{p,\infty}$, $1 \leq p < \infty$, is the space of all measurable functions such that

$$\left| \left\{ x \in \mathbf{R}^n : T(f)(x) > \alpha \right\} \right| \leq \frac{C}{\alpha} \|f\|_{L^p}, \quad \text{for all } \alpha > 0.$$

One may ask for a substitute result for the action of T on L^∞. We shall see that this will be $T : L^\infty \to BMO$, since BMO plays the same role with respect to L^∞ as $L^{p,\infty}$ plays with respect to L^p, for $1 \leq p < \infty$. The next theorem of C. Fefferman and E.M. Stein [20] is, thus, an extension of the Marcinkiewicz interpolation theorem (see e.g., Sadosky [37], Stein-Weiss [42]) for the endpoint (∞, ∞).

Theorem 1.8. (Interpolation theorem between L^p and BMO). *Let $1 < p < \infty$ and T be a linear operator, continuous from L^p into itself and from L^∞ into BMO, i.e.,*

$$T : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n) \quad and \quad T : L^\infty(\mathbf{R}^n) \to BMO(\mathbf{R}^n)$$

continuously. Then $T : L^q(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ continuously for all $p < q < \infty$.

Let $1 < q \leq \infty$. A $(1, q)$-atom centered at the origin is defined as follows. Let a be a measurable function on $\mathbf{R}^n$, supported on a cube Q, and satisfying

- *Size condition:*
$$\|a\|_{L^q} \leq |Q|^{\frac{1}{q}-1};$$

- *Moment condition:*
$$\int_Q a(x)\,dx = 0.$$

An integrable function f is in the space $H^1(\mathbf{R}^n)$ if and only if

$$f = \sum_{k=1}^\infty \lambda_k a_k$$

where a_k are $(1, q)$-atoms with $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. We may define

$$\|f\|_{H^1} = \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| \right\}$$

where the infimum is taken over all possible atomic decompositions of f, as in Coifman [12]. Now we can state the famous duality theorem for the real Hardy space $H^1(\mathbf{R}^n)$ which was first proved by C. Fefferman [19].

Theorem 1.9. (C. Fefferman-E.M. Stein). *The dual space of the real Hardy space $H^1(\mathbf{R}^n)$ is $BMO(\mathbf{R}^n)$.*

Moreover, we have the following result as an application of Theorem 1.9.

Theorem 1.10. *A function $g \in BMO(\mathbf{R}^n)$ if and only if there exist functions $g_0, g_1, \ldots, g_n \in L^{\infty}(\mathbf{R}^n)$ such that*

$$g = g_0 + \sum_{j=1}^{n} \mathcal{R}_j(g_j),$$

for

$$\mathcal{R}_j(g_j)(x) = C_n \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} g_j(y) dy, \quad j = 1, \ldots, n,$$

where the $\mathcal{R}_j$, $j = 1, \ldots, n$, are the Riesz transforms on $\mathbf{R}^n$.

In the case $n = 1$ there is only one Riesz transform, i.e., the Hilbert transform,

$$\mathcal{H}(g)(x) = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(y)}{x - y} \, dy.$$

Then, a function $g \in BMO(\mathbf{R})$ if and only if

(1.7) $$g = g_0 + \mathcal{H}(g_1), \quad g_0, \, g_1 \in L^{\infty}(\mathbf{R}).$$

Observe that this is again equivalent to the duality in one dimension:

$$BMO(\mathbf{R}) = (H^1(\mathbf{R}))^* = (L^1(\mathbf{R}) \cap \mathcal{H} L^1(\mathbf{R}))^* = L^{\infty}(\mathbf{R}) + \mathcal{H} L^{\infty}(\mathbf{R}).$$

In 1976, Coifman, Rochberg and Weiss [14] introduced a new characterization of $BMO(\mathbf{R}^n)$ in terms of commutator operators bounded on $L^2(\mathbf{R}^n)$. Let M_f be the multiplication operator by a function f, and let K be a Calderón-Zygmund singular integral operator. Then $[M_f, K]$ is the commutator operator defined by $[M_f, K]\phi = f K(\phi) - K(f\phi)$.

Theorem 1.11. *Let $\mathcal{R}_j$, $j = 1, \ldots, n$, be the Riesz transforms, and let K be a Calderón-Zygmund singular integral operator in $\mathbf{R}^n$. Then, if $f \in BMO(\mathbf{R}^n)$, the commutator $[M_f, K]$ is a bounded operator on $L^p(\mathbf{R}^n)$, for $1 < p < \infty$. Conversely, if the commutators $[M_f, \mathcal{R}_j]$ for $j = 1, \ldots, n$, are bounded operators on $L^p(\mathbf{R}^n)$, for some p, $1 < p < \infty$, then $f \in BMO(\mathbf{R}^n)$.*

In the case when $n = 1$ this reduces to $f \in BMO(\mathbf{R})$ if and only if $[M_f, \mathcal{H}]$ is a bounded operator in $L^2(\mathbf{R})$.

Theorem 1.7 is in turn equivalent to the "weak factorization" property for the space real $H^1(\mathbf{R}^n)$:

For all $f \in H^1(\mathbf{R}^n)$ there exist two sequences (g_i), $(h_i) \subset H^2(\mathbf{R}^n)$ such that

$$f = \sum_{i=1}^{\infty} g_i \, h_i \quad \text{and} \quad \sum_{i=1}^{\infty} \|g_i\|_{L^2} \|h_i\|_{L^2} < \infty \,.$$

Remark.

Let $w(x) > 0$. We call w an A_p weight, and write $w \in A_p$, if

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q w(x)dx \Big) \Big(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(x)dx \Big)^{p-1} < \infty, \quad 1 < p < \infty \,.$$

For the maximal function operator, $\mathcal{M}(w)(x) \leq Cw(x)$ for almost every $x \in \mathbf{R}^n$, and for $p = 1$, $w \in A_1$ if and only if

$$w\big(\{x \in \mathbf{R}^n : \mathcal{M}(f)(x) > \lambda\}\big) \leq \frac{c}{\lambda} \int_{\mathbf{R}^n} |f(x)|dx$$

where

$$w(E) = \int_E w(x)dx \quad \text{for all} \quad E \subset \mathbf{R}^n \,.$$

In [33] Muckenhoupt showed that, for $1 < p < \infty$,

$$\int_{\mathbf{R}^n} \mathcal{M}(f)^p(x)w(x)dx \leq C_p \int_{\mathbf{R}^n} |f(x)|^p w(x)dx$$

if and only if $w \in A_p$. Hunt, Muckenhoupt and Wheeden proved similar (and much deeper) results for the Hilbert transform [28], later generalized (with simplified proofs) for Calderón-Zygmund operators in $\mathbf{R}^n$ by Coifman and C. Fefferman [13]. A close connection between the BMO functions and the A_p weights follows from the central role played by the Hilbert transform in BMO theory. More precisely, for $1 < p < \infty$, if $w \in A_p$ then $\log w \in BMO$; conversely, if $\log w \in BMO$, then there exists a positive number $\alpha > 0$ such that $w^\alpha \in A_p$. For how these facts, together with the Helson-Szegö characterization of the A_2 weights, provide a different proof of the duality $(H^1)^* = BMO$ in one variable, see e.g. [38].

2. THE BMO SPACE AND CARLESON MEASURES

The space of BMO is closely linked with the Carleson measures. Carleson measures were first introduced by the Swedish mathematician L. Carleson in the early 60's to solve the corona problem [1]. Since then it has become an increasingly important tool in Fourier analysis. In his celebrated paper quoted above, Carleson sought the characterization of all non-negative measures μ defined on $\mathbf{R}_+^2$ satisfying

$$(2.8) \qquad \int_{\mathbf{R}_+^2} \left| P_t * f(y) \right|^2 d\mu(y, t) \leq C \|f\|_{L^2(\mathbf{R})}^2$$

for all $f \in L^2(\mathbf{R})$. Here

$$P_t * f(y) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{t}{(t^2 + |y - z|^2)} f(z) dz$$

is the Poisson integral of f and C is a universal constant. Let us look at this problem more carefully, in the general case of $\mathbf{R}^n, n \geq 2$.

Let Q be any cube in $\mathbf{R}^n$ of diameter $\ell(Q)$. Define the *Carleson box* corresponding to Q as

$$S(Q) = \left\{ (y, t) \in \mathbf{R}_+^{n+1} : \; y \in Q \,, 0 < t < \ell(Q) \right\}.$$

Consider $f = \chi_{2Q}$. It is easy to show that $(P_t * f)(y) \geq c$, for a positive constant c whenever $(y, t) \in S(Q)$. Plugging this into (2.8),

$$c' \mu(S(Q)) \leq \int_{\mathbf{R}_+^{n+1}} \left| P_t * f(y) \right|^2 d\mu(y, t) \leq C' |Q|,$$

it is

$$\mu(S(Q)) \leq C |Q|.$$

Thus, we have the following definition.

Definition 2.1. A non-negative measure μ defined on $\mathbf{R}_+^{n+1}$ is called a Carleson measure if

$$(2.9) \qquad \mu(S(Q)) \leq C |Q|, \quad \text{for all } Q \subset \mathbf{R}^n,$$

where C is a universal constant. The smallest constant C satisfying (2.9) is called the "norm" of the Carleson measure μ and is denoted by $\|\mu\|_{\mathcal{C}}$.

Indeed, (2.8) holds for all Carleson measures μ. We start with the following theorem.

Theorem 2.2. *Let f be a continuous function defined on $\mathbf{R}_+^{n+1}$ and let μ be a Carleson measure. Then*

(2.10) $$\int_{\mathbf{R}_+^{n+1}} |f(y,t)|d\mu(y,t) \le C \int_{\mathbf{R}^n} f^*(x)dx,$$

where

$$f^*(x) = \sup_{|y-x|<t} |f(y,t)|$$

is the nontangential maximal function of f. Here C is a constant depending only on $\|\mu\|_{\mathcal{C}}$.

As a consequence, we have the following corollary.

Corollary 2.3. (Carleson Embedding Theorem). *Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, and let μ be a Carleson measure defined on $\mathbf{R}_+^{n+1}$. Then the Poisson integral $P_t * f(y)$ of f satisfies the following inequality*

$$\int_{\mathbf{R}_+^{n+1}} |P_t * f(y)|^p d\mu(y,t) \le C\|f\|_{L^p}^p.$$

The space BMO has deep connections with Carleson measures, as shown by the characterization given by C. Fefferman [19] of BMO functions through canonically associated Carleson measures. First we relate $BMO(\mathbf{R}^n)$ with the Poisson integrals. For details see Stein [39].

Proposition 2.4. *Let $\varepsilon > 0$ and let Q be an arbitrary cube centered at x_0 with $\ell(Q) = d$. Then there exists a constant $A = A_{\varepsilon,n}$ such that for all $f \in BMO(\mathbf{R}^n)$, then*

(2.11) $$\int_{\mathbf{R}^n} \frac{d^\varepsilon |f(x) - f_Q|}{d^{n+\varepsilon} + |x - x_0|^{n+\varepsilon}} dx \le A \cdot \|f\|_*.$$

The following corollary is a consequence of the above proposition in the case $\varepsilon = 1$.

Corollary 2.5. *Let $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then the Poisson integral of f, $P_t * f(x)$, exists for every $(x,t) \in \mathbf{R}_+^{n+1}$.*

Theorem 2.6. (C. Fefferman). *The function $g \in BMO(\mathbf{R}^n)$ if and only if the associated measure defined on $\mathbf{R}_+^{n+1}$ by*

$$d\mu_g(x,t) = t \left|\nabla u\right|^2 (x,t) \, dx \, dt,$$

where $u(x,t) = P_t * g(x)$ is the Poisson integral of g, is a Carleson measure.

Theorem 2.7. *Let $g \in BMO(\mathbf{R}^n)$ and let ψ be a radial function in $\mathcal{S}(\mathbf{R}^n)$ satisfying $\int_{\mathbf{R}^n} \psi(x)dx = 0$. Then*

$$\left| \psi_t * g(x) \right|^2 \frac{dxdt}{t}, \quad with \quad \psi_t(\cdot) = \frac{1}{t^n}\psi\left(\frac{\cdot}{t}\right)$$

is a Carleson measure defined on $\mathbf{R}_+^{n+1}$.

Theorem 2.2 has an interesting generalization, that eventually leads to the converse of Theorem 2.7. Let $f(y,t)$ and $g(y,t)$ be two functions defined on $\mathbf{R}_+^{n+1}$. Denote by

$$A_p(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^p \frac{dydt}{t^{n+1}} \right)^{\frac{1}{p}}, \qquad p < \infty,$$

$$A_\infty(f)(x) = \sup_{(y,t)\in\Gamma(x)} \left| f(y,t) \right|,$$

and by

$$C_p(g)(x) = \sup_{B, x\in B} \left(\frac{1}{|B|} \int_B |g(y,t)|^p \frac{dydt}{t} \right)^{\frac{1}{p}}, \qquad p < \infty,$$

where $\Gamma(x) = \{(y,t) : |x - y| < t\}$ is the cone with vertex at the point x. Then we have the following theorem of Coifman, Meyer and Stein [15].

Theorem 2.8. *For $\frac{1}{p} + \frac{1}{p'} = 1$, $1 < p \leq \infty$, we have*

(2.12)
$$\int\int_{\mathbf{R}_+^{n+1}} |f(y,t)g(y,t)| \frac{dydt}{t} \leq C \int_{\mathbf{R}^n} A_p(f)(x)C_{p'}(g)(x)dx,$$

where C is a constant independent of f and g.

Remark. In particular, when $p = \infty$, $p' = 1$ and $C_{p'}(g)(x) \in L^\infty(\mathbf{R}^n)$, i.e., when $|g(y,t)|\frac{dydt}{t}$ is a Carleson measure, inequality (2.12) becomes

$$\int\int_{\mathbf{R}_+^{n+1}} |f(y,t)g(y,t)| \frac{dydt}{t} \leq C \int_{\mathbf{R}^n} A_\infty(f)(x)dx,$$

which is (2.10).

By Theorem 2.8, one can give another proof of the result $(H^1)^* = BMO$. Let ψ be a radial function in the class of $\mathcal{S}(\mathbf{R}^n)$ such that $\text{supp}(\psi) \subset \{|x| < 1\}$ and $\int_{\mathbf{R}^n} \psi(x)dx = 0$. Assume further that

$$\int_0^\infty \frac{|\hat{\psi}(t)|^2}{t} dt = 1.$$

Let $g \in BMO(\mathbf{R}^n)$. Then by Theorem 2.7, $\left|\psi_t * g(x)\right|^2 \frac{dxdt}{t}$ is a Carleson measure. Moreover, $A_2(f)(x) = S(f)(x)$, the area function of f. For arbitrary $f \in H^1(\mathbf{R}^n)$, applying Theorem 2.8 to f and g with $p = p' = 2$, it is

$$
\begin{aligned}
\int_{\mathbf{R}^n} f(x)g(x)dx &= \int\int_{\mathbf{R}^{n+1}_+} \left(\psi_t * f(x)\right)\left(\psi_t * g(x)\right)\frac{dxdt}{t} \\
&\leq C \int_{\mathbf{R}^n} S(f)(x)C_2\left(\psi_t * g\right)(x)dx \\
&\leq C\|g\|_* \cdot \int_{\mathbf{R}^n} S(f)(x)dx \\
&\leq C\|g\|_* \cdot \|f\|_{H^1}.
\end{aligned}
$$

This tells us that g defines a continuous linear functional on $H^1(\mathbf{R}^n)$.

If we assume now that $(H^1)^* = BMO$, Theorem 2.8 can be applied for $f \in H^1(\mathbf{R}^n)$ with $p = p' = 2$, to prove the following result which can be considered as the converse of Theorem 2.7.

Theorem 2.9. *Let $\psi \in \mathcal{S}(\mathbf{R}^n)$ satisfying $\int_{\mathbf{R}^n} \psi(x)dx = 0$. We assume further that*

$$
\int_0^\infty \frac{|\hat{\psi}(\xi t)|^2}{t}dt = C \neq 0, \qquad \forall \xi \in \mathbf{R}^n.
$$

If $g \in L^1_{loc}(\mathbf{R}^n)$ and

$$
\left|\psi_t * g\right|^2 \frac{dxdt}{t} \in \mathcal{C}\left(\mathbf{R}^{n+1}_+\right),
$$

then $g \in BMO(\mathbf{R}^n)$.

Further Remarks.

1. If we replace the Carleson box $S(Q)$ by a *Carleson tent*, defined by

$$
\mathcal{T}(Q) = \left\{(y, t) \in \mathbf{R}^{n+1}_+ : \ y \in Q, \ B(y; t) \subset Q\right\},
$$

   where $B(y; t)$ is the ball centered at y with radius t, then the results in this section remain true. Furthermore, if we replace also the cubes Q by a balls B, the same still holds.

2. Varopoulos gave another characterization of the space $BMO(\mathbf{R}^n)$ providing applications to the $\bar{\partial}$-problem and the Corona problem. In [**43**], he showed that for a compactly supported BMO function f, there exists $F \in C^\infty(\mathbf{R}^{n+1}_+)$ such that

   - $\lim_{t \to 0} F(x, t) - f(x) \in L^\infty(\mathbf{R}^n)$

- The measure

$$|\nabla F|\,dxdt = \Big(\big|\frac{\partial F}{\partial t}\big| + \sum_{j=1}^{n} \big|\frac{\partial F}{\partial x_j}\big|\Big)dxdt \in \mathcal{C}(\mathbf{R}_+^{n+1})$$

- There exists a function $g \in L^1(\mathbf{R}^n)$ such that

$$\sup_{t>0} |F(x,t)| \le g(x)$$

- $|\nabla F(x,t)| = \mathcal{O}(t^{-1})$.

Conversely, if $F \in C^1(\mathbf{R}_+^{n+1})$ is such that $|\nabla F|\,dxdt$ is a Carleson measure and $\lim_{t\to 0} F(x,t) = f(x)$ for almost every $x \in \mathbf{R}^n$, then

- $f \in BMO(\mathbf{R}^n)$ when $n = 1$;
- If $|\nabla F(x,t)| = \mathcal{O}(t^{-1})$, then $f \in BMO(\mathbf{R}^n)$ when $n \ge 2$.

3. In 1976, Carleson gave a new decomposition theorem for BMO functions [**3**]. Let $\varphi \in C^1(\mathbf{R}^n)$ be a radial function satisfying

$$|\varphi(x)| + |\nabla\varphi(x)| \le C(1+|x|)^{-n-1}, \qquad \int_{\mathbf{R}^n} \varphi(x)dx = 1.$$

If $g(x)$ is a compactly supported BMO function, then there exists a sequence $\{g_j\}$ such that

$$\sum_{j=1}^{\infty} \|g_j\|_{L^\infty} \le C\|g\|_*,$$

where C is a constant depending only on n. Moreover, there exists a sequence $\{\lambda_j\}$ depending on x such that

$$g(x) = g_1(x) + \sum_{j=2}^{\infty} \int_{\mathbf{R}^n} \varphi_{\lambda_j}(x-y)g_j(y)dy + \text{constant}.$$

Conversely, if a function g has the above decomposition, then $g \in BMO(\mathbf{R}^n)$ and

$$\|g\|_* \le C \cdot \sum_{j=1}^{\infty} \|g_j\|_{L^\infty}.$$

As a consequence of the Carleson's result, one may obtain another proof of maximal function characterization of the space $H^1(\mathbf{R}^n)$.

4. Let $F(z)$ be an analytic function defined on the upper half plane $\mathbf{R}_+^2 = \{z = x + iy : y > 0\}$. Then F is an analytic BMO function if and only if $y|F'(z)|^2 dxdy$ is a Carleson measure defined on $\mathbf{R}_+^2$. Define $\|F\|_{BMO} = \|y|F'(z)|^2 dxdy\|_{\mathcal{C}}$. Let z_1, $z_2 \in \mathbf{R}_+^2$. Define the hyperbolic distance between z_1 and z_2 as follows:

$$d(z_1, z_2) \approx \log\left(1 + \frac{|x_1 - x_2|}{y_1 + y_2}\right) + \left|\log\frac{y_1}{y_2}\right|.$$

We call a sequence $\{z_k\} \subset \mathbf{R}_+^2$ is η-dense if for arbitrary $z \in \mathbf{R}_+^2$, there exists $z_j \in \{z_k\}$ such that $d(z, z_j) < \eta$. We call $\{z_k\}$ is η-separated if for any z_1, $z_2 \in \{z_k\}$, one has $d(z_j, z_k) \geq \eta$. Furthermore, we call $\{z_k\}$ is a η-lattice if the sequence is 5η-dense and $\frac{\eta}{5}$-separated. In [**34**], Rochberg and Semmes proved the following result. If f is an analytic BMO function, then for arbitrary constant $\alpha > 1$, there exists a η-lattice $\{z_k\}$ and a universal constant C such that

$$(2.9) \qquad F(z) = \sum_{k=1}^{\infty} \lambda_k \frac{y_k^\alpha}{(z - \bar{z}_k)^\alpha}$$

and

$$\left\|\sum_{k=1}^{\infty} |\lambda_k|^2 y_k \delta_{z_k}\right\|_{\mathcal{C}} \leq C\|F\|_{BMO}^2.$$

Here δ_{z_k} is the Dirac delta measure defined at the point z_k. Conversely, if (2.9) holds and $\sum_{k=1}^{\infty} |\lambda_k|^2 y_k \delta_{z_k}$ is a Carleson measure, then the series $\sum_{k=1}^{\infty} \lambda_k \frac{y_k^\alpha}{(z - \bar{z}_k)^\alpha}$ converges to the function $F(z)$ in the BMO norm. Moreover,

$$\|F\|_{BMO}^2 \leq C\left\|\sum_{k=1}^{\infty} |\lambda_k|^2 y_k \delta_{z_k}\right\|_{\mathcal{C}}.$$

Using this result, one may prove that the Hankel operator is bounded on L^2 if and only if the Bergman projection of its symbol belongs to BMO. For more detail, readers may consult the papers [34] and [35].

3. BMO Functions Defined on a Bounded Domain

The developments described in the last two sections are closely tied with the groups that act naturally on Euclidean space: translations, rotations, and dilations. Indeed the profound impact that these groups have on the analysis of Euclidean space is one of the principal themes of the books by Sadosky [37], Stein [39] and Stein-Weiss [42]. Since variable coefficient operators do not respect the action of

these groups, the theory of pseudodifferential operators was developed, at least in part, as an extension of the theory of singular and fractional integrals in this more general setting (see Stein [40]). However, the corresponding fine structure provided by the new function spaces in the constant coefficient theory (particularly the spaces H^p and BMO) has been slow to evolve in this more general context.

Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain. In the papers [6] and [7], we study the local Hardy spaces on such a domain. Using a constructive method to derive atomic decompositions for two different Hardy spaces: $h_d^p(\Omega)$ and $h_z^p(\Omega)$. Here

$$h_d^p(\Omega) = \left\{ f \in (C_d^\infty)'(\bar{\Omega}) : m_d(f) \in L^p \right\}$$

with

$$C_d^\infty(\bar{\Omega}) = \left\{ \phi \in C^\infty(\bar{\Omega}) : \phi|_{\partial\Omega} = 0 \right\} \quad \text{and} \quad m_d(f)(x) = \sup_{\phi \in C_d^\infty} \left| \langle f, \phi \rangle \right|.$$

The space $h_z^p(\Omega)$ is defined

$$h_z^p(\Omega) = \left\{ f \in h^p(\mathbf{R}^n) : f = 0 \text{ on } \mathbf{R}^n \setminus \bar{\Omega} \right\},$$

where

$$h^p(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \mathcal{M}_{loc}(f)(x) = \sup_{0 < \varepsilon < 1} \left| \phi_\varepsilon * f(x) \right| \in L^p(\mathbf{R}^n) \right\}$$

is the local Hardy space developed by Goldberg [24]. Intuitively, we may consider the space h_d^p as the restriction of $h^p(\mathbf{R}^n)$ to the domain Ω which can be considered the largest Hardy space defined on the domain Ω. An element $f \in h_z^p$ can be extended to a global h^p distribution by setting $f(x) = 0$ when $x \notin \bar{\Omega}$. This is the smallest Hardy space can be defined on the domain Ω. In order to make these notions more precise, we need the following definition.

Definition 3.1. Let Ω be a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary. A cube $Q \subset \Omega$ (with sides parallel to the axes) is of type (a) if $\ell(Q) < 1$ and $4Q \subset \Omega$. A cube Q is of type (b) if $\ell(Q) \geq 1$ or if $2Q \cap \Omega^c = \varnothing$ and $4Q \cap \Omega^c \neq \varnothing$.

Since we have two h^1 spaces, we also define two $bmo(\Omega)$ spaces.

Definition 3.2. Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. We say that a function g on $\mathbf{R}^n$ is in $bmo_z(\Omega)$ if g is locally integrable and

(3.14)
$$\sup_{\text{type}\,(a)\,\text{cubes}} \frac{1}{|Q|} \int_Q |g(x) - g_Q|\,dx + \sup_{\text{type}\,(b)\,\text{cubes}} \frac{1}{|Q|} \int_Q |g(x)|\,dx$$

$$\equiv \|g\|_{bmo_z(\Omega)} < \infty.$$

It can be shown that $bmo_z(\Omega) = \left\{ g \in bmo(\mathbf{R}^n) : \mathrm{supp}(g) \subset \bar{\Omega} \right\}$.

Definition 3.3. Let Ω be a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary. We say that a function g on Ω is in $bmo_r(\Omega)$ if g is locally integrable and

$$\sup_{0 < \ell(Q) < 1} \frac{1}{|Q|} \int_Q |g(x) - g_Q| dx + \sup_{\ell(Q) > 1} \frac{1}{|Q|} \int_Q |g(x)| dx \equiv \|g\|_{bmo_r(\Omega)} < \infty,$$

where the suprema are taken over all cubes $Q \subset \Omega$.

It can be shown that when $g \in bmo_r(\Omega)$ then there exists a function $G \in bmo(\mathbf{R}^n)$ such that

$$G\big|_\Omega = g \ \text{ and } \ \|G\|_{bmo} \le C_\Omega \cdot \|g\|_{bmo_r},$$

where C_Ω is a constant depending on Ω only.

Similar to C. Fefferman's duality theorem, we have the following theorem from [4].

Theorem 3.4. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$. Then we have*

- *If $g \in bmo_r(\Omega)$, then there exists a unique linear functional $\mathcal{L}$ in the dual space of $h_z^1(\bar{\Omega})$ such that*

  (3.15) $$\mathcal{L}(f) = \int_\Omega f(x)g(x)dx$$

  for all $f \in h_z^1(\bar{\Omega})$. Conversely, if $\mathcal{L}$ in the dual space of $h_z^1(\Omega)$, then there exists a unique $g \in bmo_r(\Omega)$ such that (3.15) holds. The correspondence $\mathcal{L} \leftrightarrow g$ given by (3.15) is a Banach space isomorphism between $bmo_r(\Omega)$ and the dual space of $h_z^1(\Omega)$.

- *If $g \in bmo_z(\Omega)$, then there exists a unique linear functional $\mathcal{L}$ in the dual space of $h_d^1(\bar{\Omega})$ such that (3.15) holds for all $f \in h_d^1(\bar{\Omega})$. Conversely, if $\mathcal{L}$ in the dual space of $h_r^1(\Omega)$, then there exists a unique $g \in bmo_z(\Omega)$ such that (3.15) holds. The correspondence $\mathcal{L} \leftrightarrow g$ given by (3.15) is a Banach space isomorphism between $bmo_z(\Omega)$ and the dual space of $h_r^1(\Omega)$.*

In [16], Coifman, Lions, Meyer, and Semmes showed that for $g \in BMO(\mathbf{R}^n)$,

$$\|g\|_{BMO} \approx \sup_{\vec{E}, \vec{F}} \int_{\mathbf{R}^n} g(x) \cdot \langle \vec{E}, \vec{F} \rangle dx,$$

where the supremum is taken over all vector fields $\vec{E}$, $\vec{F}$ in $L^2(\mathbf{R}^n, \mathbf{R}^n)$ satisfying $\nabla \cdot \vec{E} = 0$, $\nabla \times \vec{F} = \vec{0}$. Here $L^2(\mathbf{R}^n, \mathbf{R}^n)$ is the collection of all vector fields $\vec{E} = (E_1, \ldots, E_n) : \mathbf{R}^n \to \mathbf{R}^n$ such that $\|E_\ell\|_{L^2} \le 1$ for $\ell = 1, \ldots, n$. We

may obtain similar results for bmo_r and bmo_z. For detailed discussion, see the forthcoming paper [9]. More precisely, we have the following theorem.

Theorem 3.5. *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$.*

- *If $g \in bmo_r(\Omega)$ then*

$$\|g\|_{bmo_r} \approx \sup_{\vec{E},\vec{F}} \int_\Omega g(x) \cdot \langle \vec{E}, \vec{F} \rangle dx,$$

the supremum is taken over all vector fields $\vec{E}$ and $\vec{F}$ in $L^2(\Omega, \mathbf{R}^n)$ such that $\nabla \cdot \vec{E} = 0$, $\vec{E} \cdot \mathbf{n}\big|_{\partial\Omega} = 0$, $\nabla \times \vec{F} = \vec{0}$, $\vec{F} \times \mathbf{n}\big|_{\partial\Omega} = \vec{0}$. Here $L^2(\Omega, \mathbf{R}^n)$ is the collection of all vector fields $\vec{E} = (E_1, \ldots, E_n) : \Omega \to \mathbf{R}^n$ such that $\|E_\ell\|_{L^2(\Omega)} \leq 1$ for $\ell = 1, \ldots, n$.

- *If $g \in bmo_z(\Omega)$, then*

$$\|g\|_{bmo_z} \approx \sup_{\vec{e},\vec{f}} \int_\Omega g(x) \cdot \langle \vec{e}, \vec{f} \rangle dx,$$

the supremum is taken over all vector fields $\vec{e} = \vec{E}\big|_\Omega$ and $\vec{f} = \vec{F}\big|_\Omega$ where $\vec{E}, \vec{F} \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ satisfying $\nabla \cdot \vec{E} = 0$, $\nabla \times \vec{F} = \vec{0}$.

The above results give decompositions of the space $h_r^1(\Omega)$ and $h_z^1(\Omega)$. We have the following theorem.

Theorem 3.6. *Let Ω be a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary.*

- *For a function $f \in h_z^1(\Omega)$, there exist two sequences $\{\vec{e}_k\}$, $\{\vec{f}_k\} \subset L^2(\Omega, \mathbf{R}^n)$ satisfying $\nabla \cdot \vec{e}_k = 0$, $\mathbf{n} \cdot \vec{e}_k\big|_{\partial\Omega} = 0$, $\nabla \times \vec{f}_k = 0$, and $\mathbf{n} \times \vec{f}_k\big|_{\partial\Omega} = \vec{0}$ such that*

$$f(x) = \sum_{k=1}^\infty \lambda_k \langle \vec{e}_k, \vec{f}_k \rangle.$$

Moreover, one has $\sum_{k=1}^\infty |\lambda_k| < \infty$. Here $\mathbf{n}$ is the unit outward normal along $\partial\Omega$.

- *Similarly, for a function $f \in h_r^1(\Omega)$, there exist two sequences $\{\vec{E}_k\}$, $\{\vec{F}_k\} \subset L^2(\mathbf{R}^n, \mathbf{R}^n)$ satisfying $\nabla \cdot \vec{E}_k = 0$ and $\nabla \times \vec{F}_k = 0$ such that*

$$f(x) = \sum_{k=1}^\infty \lambda_k \langle \vec{e}_k, \vec{f}_k \rangle$$

where $\vec{e}_k = \vec{E}_k\big|_\Omega$ and $\vec{f}_k = \vec{F}_k\big|_\Omega$. Moreover, one has $\sum_{k=1}^\infty |\lambda_k| < \infty$.

Further Remarks.

1. The main purpose of the papers [6] and [7] is to study H^p regularity properties of boundary value problems for Laplacian on a smooth domains in $\mathbf{R}^n$. In order to achieve this goal, we need extra assumptions on the smoothness of the domain Ω. Indeed, one may prove that if $\mathbf{G}$ is the solution operator for the Dirichlet problem on a $C^{(1,1)}$ domain Ω, then for $1 \leq j, k \leq n$, $\frac{\partial^2 \mathbf{G}}{\partial x_j \partial x_k}$ is a bounded operator from $bmo_z(\Omega)$ to $bmo_z(\Omega)$ and from $bmo_r(\Omega)$ to $\tilde{bmo}_r(\Omega)$ (see Chang and Li [8]).

2. Similar result can be obtained for Neumann problem. Let $\tilde{\mathbf{G}}$ be the solution operator for the Neumann problem, then for $1 \leq j, k \leq n$, $\frac{\partial^2 \tilde{\mathbf{G}}}{\partial x_j \partial x_k}$ is a bounded operator from $bmo_r(\Omega)$ to $bmo_r(\Omega)$. Moreover, there exists counterexample to show that $\frac{\partial^2 \tilde{\mathbf{G}}}{\partial x_j \partial x_k}$ is not bounded from $bmo_z(\Omega)$ to $bmo_z(\Omega)$.

4. BMO and Carleson Measures on Product Spaces

Before we go further, let us look at the Hardy spaces in $\mathbf{R}^n$. The space $H^p(\mathbf{R}^n)$ is invariant under the automorphisms:

(4.16) $$x \in \mathbf{R}^n \mapsto \delta x, \qquad \delta > 0.$$

For example, denote $f_\delta(x) = \delta^{-n/p} f\left(\frac{x}{\delta}\right)$, then

$$\|f_\delta\|_{H^p} = \|f\|_{H^p}.$$

This is so because the maximal function which characterizes H^p space,

$$\phi^*(f)(x) = \sup_{|x-y|<t} \left|(\phi_t * f)(y)\right|$$

is defined by $\phi_t(\cdot) = t^{-n}\phi\left(\frac{\cdot}{t}\right)$ is invariant under the automorphism (4.16). Roughly speaking, we are using a single parameter to study H^p and BMO functions in $\mathbf{R}^n$. The story is totally different when we move to product spaces. Let us consider H^p and BMO spaces on the simplest product space, $\mathbf{R} \times \mathbf{R}$. The dilations defined on this product space are bi-parametric:

$$(x_1, x_2) \in \mathbf{R} \times \mathbf{R} \mapsto (\delta_1 x_1, \delta_2 x_2), \qquad \delta_1, \delta_2 > 0.$$

Harmonic analysis on multi-parameter and single-parameter spaces have significant differences. Let us look at a simple example. The Hardy-Littlewood maximal operator corresponding to the single-parameter harmonic analysis is:

$$\mathcal{M}(f)(x) = \sup_Q \frac{1}{|Q|} \int_Q |f(x+y)|dy,$$

where Q is a cube centered at the origin. It is well known that the operator $\mathcal{M}$ is bounded from L^p to itself for $1 < p \leq \infty$ and of weak type $(1,1)$. However, the corresponding operator in multi-parameter harmonic analysis is the *strong maximal operator*:

$$\mathcal{M}_s(f)(x_1, x_2) = \sup_{I \times J} \frac{1}{|I \times J|} \int_{I \times J} |f(x_1 + y_1, x_2 + y_2)| dy_1 dy_2,$$

where $I \times J$ is a rectangle containing the point (x_1, x_2). There is a big difference between the operators $\mathcal{M}$ and $\mathcal{M}_s$. While the operator $\mathcal{M}_s$ is still bounded from L^p to itself for $1 < p \leq \infty$, it is no longer of weak type $(1,1)$. The best result that can obtained is

$$\left| \{ (x_1, x_2) \in I \times J : \ \mathcal{M}_s(f)(x_1, x_2) > \lambda \} \right| \leq \frac{C}{\lambda} \|f\|_{L \log L(I \times J)},$$

the famous Jessen-Marcinkiewicz-Zygmund inequality. This simple example provides a very strong hint that rectangles are not enough to define H^p atoms on the product spaces. Indeed, Carleson proved in 1974 that on the product space $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ rectangles are not enough to characterize the support of H^p atoms and Carleson measures. In [2], he constructed his famous counterexample of a measure satisfying

$$\mu(S(R)) \leq C|R|, \quad \text{where } R = I \times J \subset \mathbf{R} \times \mathbf{R} \text{ is an arbitrary rectangle,}$$

but for which the Carleson Embedding Theorem

$$\int \int_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} P_{t_1 t_2}[f]^p(y_1, y_2) d\mu \leq C_p \int_{\mathbf{R} \times \mathbf{R}} |f(x_1, x_2)|^p dx_1 dx_2, \qquad p > 1$$

does not hold. By this example it follows that the space $BMO_{Rect}(\mathbf{R} \times \mathbf{R})$, introduced by R. Fefferman, of functions f satisfying

$$(4.17) \quad \sup_{I,J} \frac{1}{|I|} \frac{1}{|J|} \int_I \int_J |f(x_1, x_2) - f_J(x_1) - f_I(x_2) + f_{I \times J}|^2 dx_1 \, dx_2 < \infty$$

(where $f_J(x_1) = \frac{1}{|J|} \int_J f(x_1, x_2) dx_2$ and $f_I(x_2) = \frac{1}{|I|} \int_I f(x_1, x_2) dx_1$ are the mean values of $f(x_1, \cdot)$, $f(\cdot, x_2)$ over the intervals J and I, respectively, and $f_{I \times J}$ is the mean value of $f(x_1, x_2)$ over the rectangle $I \times J$) is not the dual of $H^1(\mathbf{R} \times \mathbf{R})$, since it is too large.

Let us remark here that the smaller space $bmo\,(\mathbf{R} \times \mathbf{R})$, of functions f of *bounded mean oscillation* in $\mathbf{R} \times \mathbf{R}$, i.e., satisfying

$$(4.18) \quad \sup_{I,J} \frac{1}{|I \times J|} \int_{I \times J} \left| f(x_1, x_2) - f_{I \times J} \right| dx_x \, dx_2 \ < \ \infty$$

is not the dual of $H^1(\mathbf{R} \times \mathbf{R})$ (see e.g., [40]).

The fact that rectangles were not enough to deal with Carleson measures created a fundamental difficulty to build up an H^p theory on product spaces. Using probability methods, Gundy and Stein [**25**] reached a breakthrough in 1979. In order to explain it, let us first recall some definitions. A function $u(x,t) = u(x_1, t_1, x_2, t_2)$ is called biharmonic on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ if u is harmonic in (x_1, t_1) and (x_2, t_2) separately. Denote $\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2)$ where $\Gamma(x_j) = \left\{ (y_j, t_j) \in \mathbf{R}_+^2 \ : \ |y_j - x_j| < t_j \right\}$ for $j = 1, 2$. In [**25**], Gundy and Stein showed that for a biharmonic function u,

$$\|u^*\|_{L^p} \approx \|S(u)\|_{L^p}, \qquad 0 < p < \infty.$$

Here

$$u^*(x) = u^*(x_1, x_2) = \sup_{(y,t) \in \Gamma(x)} \left| u(x,t) \right|$$

is the nontangential maximal function, and

$$S(u)(x) = S(u)(x_1, x_2) = \left(\int \int_{\Gamma(x)} \left| \nabla_1 \nabla_2 u(y,t) \right|^2 dy dt \right)^{1/2}$$

is the area integral of u. This allowed them to build up a real H^p theory on the product space $\mathbf{R}_+^2 \times \mathbf{R}_+^2$, i.e., a tempered distribution f,

$$(4.19) \qquad f \in H^p(\mathbf{R} \times \mathbf{R}) \Leftrightarrow \varphi^*(f)(x) = \sup_{(y,t) \in \Gamma(x)} \left| (f * \varphi_t)(y) \right| \in L^p(\mathbf{R}^2)$$

where $\varphi \in C_0^\infty(\mathbf{R}^2)$, $\int \varphi(x) dx \neq 0$, $\varphi_t(x) = \frac{1}{t_1 t_2} \varphi\left(\frac{x_1}{t_1}, \frac{x_2}{t_2} \right)$.

A main question in the theory in product space is: *What is the correct atomic decomposition for $H^p(\mathbf{R} \times \mathbf{R})$?* The answer is an atomic decomposition using all open subsets in $\mathbf{R} \times \mathbf{R}$, instead of just rectangles. This is a fundamental result due to S.-Y. A. Chang and R. Fefferman (see [10] and [11]), which we explain below.

Let Ω be a bounded open set in $\mathbf{R} \times \mathbf{R}$, and denote by $\mathcal{D}(\Omega)$ the set of all dyadic rectangles contained in Ω. We define $(p, 2)$-atoms as follows.

Definition 4.1. Let $0 < p \leq 1$. A function $a(x_1, x_2)$ is a $(p, 2)$-atom defined on $\mathbf{R} \times \mathbf{R}$ if it satisfies the following properties:

- supp$(a) \subset \Omega$, where Ω is a bounded open set in $\mathbf{R} \times \mathbf{R}$

- a has a further decomposition

$$a = \sum_{R \in \mathcal{D}(\Omega)} a_R$$

    where each a_R satisfies

(i) $\operatorname{supp}(a_R) \subset 3R$

(ii)

$$\int_{\mathbf{R}} a_R(x_1, x_2) x_1^{k_1} dx_1 = 0, \quad \text{for all } x_2 \text{ and } 0 \le k_1 \le \left[\frac{1}{p} - 1\right]$$

$$\int_{\mathbf{R}} a_R(x_1, x_2) x_2^{k_2} dx_2 = 0, \quad \text{for all } x_1 \text{ and } 0 \le k_2 \le \left[\frac{1}{p} - 1\right]$$

where k_1 and k_2 are two non-negative integers

- $\|a\|_{L^2}^2 \le |\Omega|^{1-\frac{2}{p}}$ and $\sum_{R \in \mathcal{D}(\Omega)} \|a_R\|_{L^2}^2 \le |\Omega|^{1-\frac{2}{p}}$.

Now we have the following theorem.

Theorem 4.2. *A tempered distribution $f \in H^p(\mathbf{R} \times \mathbf{R})$ if and only if*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

where a_j's are $(p, 2)$-atoms with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover,

$$\|f\|_{H^p} = \inf \Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}}$$

where the infimum is taking over all possible atomic decomposition of f.

The necessity of the above theorem was obtained by R. Fefferman in [21] and the sufficiency was proved by Han [26]. Theorem 4.2 shows that the atomic decomposition for $H^p(\mathbf{R} \times \mathbf{R})$ is much more complicated than that of Hardy spaces on $\mathbf{R}$. One interesting question is: What function space can be obtained if we just use functions which have atomic decompositions in terms of atoms supported on rectangles? Here we consider only the case $p = 1$.

Definition 4.3. We call $a(x_1, x_2)$ a rectangular atom defined on $\mathbf{R} \times \mathbf{R}$ if

- $\operatorname{supp}(a) \subset R = I \times J$ where I and J are intervals in $\mathbf{R}$;

-
$$\Big\| \frac{\partial^{k_1+k_2} a}{\partial x_1^{k_1} \partial x_2^{k_2}} \Big\|_{L^\infty} \le |I|^{-1-k_1} |J|^{-1-k_2}$$

with $0 \le k_1, k_2 \le 1$;

-

$$\int_{\mathbf{R}} a(x_1, x_2)dx_1 = 0 \qquad \text{for all } x_2 \in \mathbf{R};$$

$$\int_{\mathbf{R}} a(x_1, x_2)dx_2 = 0 \qquad \text{for all } x_1 \in \mathbf{R}.$$

Definition 4.4. Let $f \in L^1(\mathbf{R} \times \mathbf{R})$. We call

$$S_1(f)(x_1, x_2) = \int \int_{\Gamma(x_1, x_2)} \big| f * \Psi_{t_1 t_2}(y_1, y_2) \big| \frac{dy dt}{t_1^2 t_2^2}$$

the generalized S-function of f. Here $\Psi(x_1, x_2) = \psi(x_1)\psi(x_2)$ with $\psi \in \mathcal{S}(\mathbf{R})$ and $\int_{\mathbf{R}} \psi(x)dx = 0$.

Now we have

Theorem 4.5. *Let $f \in L^1(\mathbf{R} \times \mathbf{R})$. Then $S_1(f) \in L^1(\mathbf{R} \times \mathbf{R})$ if and only if f has rectangular atomic decomposition, i.e., $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where a_j's are rectangular atoms with $\sum_{j=1}^{\infty} |\lambda_j| < \infty$. Moreover,*

$$\|S_1(f)\|_{L^1} \approx \inf \Big\{ \sum_{j=1}^{\infty} |\lambda_j| : \text{for all possible } f = \sum_{j=1}^{\infty} \lambda_j a_j \Big\}.$$

In fact, the condition $S_1(f) \in L^1(\mathbf{R} \times \mathbf{R})$ implies that f is an element in the Besov space $B_1^{0,1}(\mathbf{R} \times \mathbf{R})$. This is the smallest Banach space which norm is invariant under the automorphisms

$$(x_1, x_2) \mapsto (x_1 + h_1, x_2 + h_2), \quad \text{for all} \quad h_1, h_2 \in \mathbf{R}$$

and

$$(x_1, x_2) \mapsto (\delta_1 x_1, \delta_2 x_2), \quad \text{for all} \quad \delta_1, \delta_2 > 0.$$

It follows that $B_1^{0,1}(\mathbf{R} \times \mathbf{R})$ has nicer properties than $H^1(\mathbf{R} \times \mathbf{R})$. The rectangular atomic decomposition is one of them. It is not hard to show that the dual of this space is $BMO_{Rect}(\mathbf{R} \times \mathbf{R})$, and that it properly contains the dual of $H^1(\mathbf{R} \times \mathbf{R})$.

Let us now turn to Carleson measures on the product space.

Definition 4.6. Let μ be a non-negative measure defined on the product space $\mathbf{R}_+^2 \times \mathbf{R}_+^2$. We call μ a Carleson measure if

$$\int \int_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} P_{t_1 t_2}[f]^p(y_1, y_2)d\mu \leq C_p \int_{\mathbf{R} \times \mathbf{R}} |f(x_1, x_2)|^p dx_1 dx_2, \quad 1 < p < \infty$$

for all $f \in L^p(\mathbf{R} \times \mathbf{R})$. Here C_p is a constant independent of f, and $P_{t_1 t_2}[f]$ is the Poisson integral of f.

Definition 4.7. Let $\Omega \subset \mathbf{R}^2$ be an open set. Denote $R(y, t)$ the rectangle centered at $y = (y_1, y_2)$ with dimensions $2t_1$ and $2t_2$. The Carleson box $S(\Omega)$ on Ω is defined as

$$S(\Omega) = \bigcup_{R(y,t) \subset \Omega} S(R(y, t)).$$

Now we have the following theorem.

Theorem 4.8. (S.-Y. A. Chang) *The necessary and sufficient condition characterizing a Carleson measure μ defined on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ is*

$$\mu\big(S(\Omega)\big) \leq C|\Omega|$$

for all open sets $\Omega \subset \mathbf{R}^2$.

We now define by duality

$$BMO(\mathbf{R} \times \mathbf{R}) = (H^1(\mathbf{R} \times \mathbf{R}))^*.$$

The following theorem was proved by S.-Y. A. Chang and R. Fefferman in [11].

Theorem 4.9. *The following statements are equivalent:*

(1) $g \in BMO(\mathbf{R} \times \mathbf{R})$;

(2) *There exists $g_j \in L^\infty(\mathbf{R}^2)$, $j = 0, 1, 2, 3$, such that*

$$g = g_0 + \mathcal{H}_1(g_1) + \mathcal{H}_2(g_2) + \mathcal{H}_1 \mathcal{H}_2(g_3)$$

where $\mathcal{H}_k$, $k = 1, 2$, is the Hilbert transform with respect to the variable x_k:

$$\mathcal{H}_k(f)(x_k) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y_k| > \varepsilon} \frac{f(x_k - y_k)}{y_k} dy_k$$

(3) $d\mu_g(y, t) = \big|\nabla_1 \nabla_2 u\big|^2 t_1 t_2 dy dt$ *is a Carleson measure defined on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$, where $u = P[g]$ is the Poisson integral of g*

(4) $\big|\Psi_t * g\big|^2 \frac{dy dt}{t}$ *is a Carleson measure defined on $\mathbf{R}_+^2 \times \mathbf{R}_+^2$, where $\Psi(x) = \psi(x_1)\psi(x_2)$ with $\psi \in \mathcal{S}(\mathbf{R})$ and $\int_{\mathbf{R}} \psi(x) dx = 0$.*

The characterization of $BMO(\mathbf{R} \times \mathbf{R})$ given in Theorem 4.9 (2) corresponds to its duality with $H^1(\mathbf{R} \times \mathbf{R}) = \{f \in L^1(\mathbf{R}^2) : \mathcal{H}_1(f) \in L^1(\mathbf{R}^2), \mathcal{H}_2(f) \in L^1(\mathbf{R}^2), \mathcal{H}_1 \mathcal{H}_2(f) \in L^1(\mathbf{R}^2) \}$, and gives an easy way to *construct* functions in $BMO(\mathbf{R} \times \mathbf{R})$ starting from bounded functions. Observe that this is the analog to (4.9) when $BMO(\mathbf{R})$ is substituted for $BMO(\mathbf{R} \times \mathbf{R})$.

The main problem impairing the BMO theory in product spaces for more than twenty years was the impossibility to check if a given function is or not in $BMO(\mathbf{R} \times \mathbf{R})$. On the other hand, it is easy to see if a function is in $BMO(\mathbf{R}^n)$ just by checking the bounded mean oscillation condition (1.1) itself. But, in product spaces, bounded mean oscillation defines the space $bmo(\mathbf{R} \times \mathbf{R})$, which is strictly contained in $BMO(\mathbf{R} \times \mathbf{R})$, as shown in [17].

An alternative way to bounded mean oscillation for checking if a given function is or not in $BMO(\mathbf{R})$ exists and was given in Section 1. In fact, Theorem 1.11 asserts that the boundedness in $L^2(\mathbf{R})$ of the commutator operator $[M_f, \mathcal{H}]$ is equivalent to $f \in BMO(\mathbf{R})$.

It was precisely through commutators–this time in terms of a *nested* commutator with two one-dimensional Hilbert transforms–that checking if a function is in $BMO(\mathbf{R} \times \mathbf{R})$ was finally achieved by Lacey and Ferguson in [22]. The conjecture of characterizing a BMO space through the L^2 boundedness of the nested commutator appeared first in work of Ferguson and Sadosky [23], and its relevance to the problem was confirmed in work of Pott and Sadosky [36]. The proof of [22] is based on a wavelet version of a Chang-Fefferman characterization in [11], norm estimates for the nested commutator from [23], the John-Nirenberg inequality from [10], and a new version of Journé's geometric lemma [31], which is by far its most technical and intricate part. The result follows.

Theorem 4.10. (Ferguson and Lacey). $f \in BMO(\mathbf{R} \times \mathbf{R})$ *if and only if* $[[M_f, \mathcal{H}_1], \mathcal{H}_2]$ *extends to a bounded operator in* $L^2(\mathbf{R}^2)$ *and*

$$\|[[M_f, \mathcal{H}_1], \mathcal{H}_2]\|_{L^2 \to L^2} \approx \|f\|_{BMO}.$$

This result is now equivalent to the characterizations of $BMO(\mathbf{R} \times \mathbf{R})$ given in Theorem 4.9. It follows directly from [23] that all such characterizations of $BMO(\mathbf{R} \times \mathbf{R})$ are equivalent to the *weak factorization* of its predual, $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$:

For all $f \in H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, there exist $\{g_j\}$, $\{h_j\} \subset H^2(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ such that

$$f = \sum_{j=1}^{\infty} g_j h_j \quad \text{with} \quad \sum_{j=1}^{\infty} \|g_j\|_{L^2} \|h_j\|_{L^2} < \infty.$$

Final Remarks.

1. It was mentioned above that the space $BMO_{Rect}(\mathbf{R} \times \mathbf{R})$, defined by (4.17), admits a rectangular atomic decomposition. It can also be defined through associated Carleson measures defined on rectangles. On the other hand, the space $bmo(\mathbf{R} \times \mathbf{R})$ of bounded mean oscillation, defined by (**??**), is closely linked to the class A_p^* of weights in product spaces, and appears naturally in the solution of multi-parametric harmonic analysis problems. Among its several characterizations in [17], [23], is one in terms of *double* commutators, i.e., $f \in bmo(\mathbf{R} \times \mathbf{R})$ if and only if $[M_f, \mathcal{H}_1 \mathcal{H}_2]$ is bounded in $L^2(\mathbf{R}^2)$.

   The theory of Hankel operators, of independent interest, has many applications, in particular to engineering systems. It may seem a curiosity that the main result of the theory, the Nehari theorem on bounded Hankel operators, can be written in terms of the space $BMO(\mathbb{T})$ on the circle. But more is true. The spaces $BMO_{Rect}(\mathbb{T} \times \mathbb{T})$ and $bmo(\mathbb{T} \times \mathbb{T})$, deemed for many years of no interest in the theory of product BMO, reappeared in the theory of the "small" and "big" Hankel operators on the torus. It is a striking fact that the study of the symbols of the small and big Hankel operators played a crucial role in the characterization of product BMO on the torus given in [22] (for details, see [38]).

2. Recently Lacey and Terwilleger [32] have obtained the analogue of the Ferguson-Lacey characterization for $BMO(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ in terms of third order commutators of the type $[[[M_f, \mathcal{H}_1], \mathcal{H}_2], \mathcal{H}_3]$, by extending to the tridisk the geometrical study done in [22], through a new and essential twist. This result suggests that much of the theory may be satisfactorily extended to $\mathbf{R} \times \cdots \times \mathbf{R}$ for product spaces of n factors when $n > 2$, since the geometrical hurdles may be more surmontable than previously thought. It is still to be seen which of the particular features of the theory of $BMO(\mathbf{R}^n)$ actually extend to BMO in product spaces of several factors.

3. Many results on $H^1(\mathbf{R} \times \mathbf{R})$ can be generalized to the space $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$, but not **all**. A classical result in $\mathbf{R}^n$ asserts that if the operator T is bounded on $L^2(\mathbf{R}^n)$ and if $\int_{(2Q)^c} |T(a)(x)| dx \leq C\|a\|_{L^1}$, whenever a is an atom supported in a cube Q, then T is of weak type $(1, 1)$ (see also Chang [5] and Han [27]). In 1986, R. Fefferman [21] extended the result to product space. Let T be an operator bounded in $L^2(\mathbf{R} \times \mathbf{R})$. If for each $H^1(\mathbf{R} \times \mathbf{R})$ rectangular atom a, there exists a $\delta > 0$ such that

   $$(4.20) \qquad \int_{(\gamma R)^c} |T(a)(x)| dx \leq C\gamma^{-\delta}, \quad \forall \gamma \geq 2$$

   then T is bounded from $H^1(\mathbf{R} \times \mathbf{R})$ to $L^1(\mathbf{R}^2)$. Here γR is a rectangular box which has the same center as R such that $\ell(\gamma R) = \gamma \ell(R)$. This result

shows that to consider just *rectangular atoms* is enough to ensure the desired boundedness of the operator in this context.

Let us now call a function $a(x_1, x_2, x_3)$ an $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ rectangular atom if

- supp$(a) \subset R = I_1 \times I_2 \times I_3$,   where I_j is an interval in $\mathbf{R}$;
- $\int_{\mathbf{R}} a(x_1, x_2, x_3)dx_1 = \int_{\mathbf{R}} a(x_1, x_2, x_3)dx_2 = \int_{\mathbf{R}} a(x_1, x_2, x_3)dx_3 = 0$;
- $\|a\|_{L^2} \leq |R|^{-1/2} = \prod_{j=1}^{3} |I_j|^{-1/2}$.

Let T be a bounded operator on $L^2(\mathbf{R}^3)$, and assume that the condition (4.20) is satisfied for $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ atoms. A natural question is whether T is bounded from $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ to $L^1(\mathbf{R}^3)$. The answer is **no** in general. In [30], Journé showed that

- There exists an operator T satisfying the above conditions but which is not bounded from $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ to $L^1(\mathbf{R}^3)$;
- If T is an convolution operator defined on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ and $\delta > \frac{1}{8}$ in (4.20), then T is bounded from $H^1(\mathbf{R} \times \mathbf{R} \times \mathbf{R})$ to $L^1(\mathbf{R}^3)$.

This phenomenon reflects a striking difference between product spaces $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, which, in this case, may be the result of the essential role the *rectangular* atoms play in the theorem. There is a long way to go in order to get the full picture for harmonic analysis on product domains. Readers may consult the commemorative paper by Stein [41].

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Der-Chen Chang,
Department of Mathematics,
Georgetown University,
Washington DC, 20057, USA
E-mail: chang@georgetown.edu

Cora Sadosky
Department of Mathematics,
Howard University,
Washington DC, 20059, USA
E-mail: csadosky@howard.edu