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Schwarz Lemma at the Boundary for Holomorphic and Pluriharmonic Mappings Between p-unit Balls

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Abstract. We give Schwarz lemma at the boundary for holomorphic mappings between p-unit ball $B_p^n \subset \mathbb{C}^n$ and $B_p^N \subset \mathbb{C}^N$, where $p \geq 2$. When $p = 2$, this result reduces to that of Liu, Chen and Pan [21] between the Euclidean unit balls, and our method is new. By generalizing pluriharmonic Schwarz lemma of Chen and Gauthier [5] from $p = 2$ to $p \geq 2$, we obtain the boundary Schwarz lemma for pluriharmonic mappings between p-unit balls.

1. Introduction

Schwarz lemma is a fundamental result of complex analysis, which plays an important role in geometric function theory and complex geometry. The description of Schwarz lemma has attracted many mathematicians' attention, see [1, 4, 6, 7, 14, 23]. In recent years, many scholars have been interested in studying Schwarz lemma at the boundary of a domain. The following boundary Schwarz lemma in one complex variable is well-known.

Lemma 1.1. [8,15,24] *Let $\mathbb{U}$ be the unit disk of the complex plane $\mathbb{C}$. Suppose $f(z)\colon \mathbb{U} \to \mathbb{U}$ is a holomorphic function. If $f(z)$ is holomorphic at $z = 1$ (or more generally, if f is differentiable at $z = 1$), $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$. Moreover, $f'(1) = 1$ if and only if $f(z) \equiv z$.*

If the condition $f(0) = 0$ is removed, applying Lemma 1.1 to $g(z) = \frac{1-\overline{f(0)}}{1-f(0)} \frac{f(z)-f(0)}{1-\overline{f(0)}f(z)}$, then one has the following estimate instead:

$$f'(1) \geq \frac{|1-\overline{f(0)}|^2}{1-|f(0)|^2} \geq \frac{1-|f(0)|}{1+|f(0)|} > 0. \tag{1.1}$$

The boundary Schwarz lemma plays also important role in the classical complex analysis. For example, the Bieberbach conjecture and Bloch constant are the two most difficult

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problems in the geometric function theory of one complex variable. Since De Branges solved the Bieberbach conjecture in 1984, finding the exact value of the Bloch constant is still pending until now. However, applying to boundary Schwarz lemma, Bonk [3] improved the lower bound of the Bloch constant.

It is natural to generalize Lemma 1.1 to several complex variables. Liu et al. [20, Theorems 3.1 and 3.4] established the following boundary Schwarz lemma for holomorphic self-mappings defined on the Euclidean unit ball B^n in $\mathbb{C}^n$, which is a higher dimensional version of Lemma 1.1.

Theorem 1.2. [20] *Let $f\colon B^n \to B^n$ be a holomorphic mapping. If f is holomorphic at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in \partial B^n$, then the following statements hold.*

(i) $J_f(z_0)\mathcal{T}_{z_0}(\partial B^n) \subset \mathcal{T}_{w_0}(\partial B^n)$, *and* $J_f(z_0)\mathcal{T}^{1,0}_{z_0}(\partial B^n) \subset \mathcal{T}^{1,0}_{w_0}(\partial B^n)$.

(ii) *There is $\lambda \in \mathbb{R}$ such that $\overline{J_f(z_0)}^T w_0 = \lambda z_0$, where*

$$\lambda = \overline{w_0}^T J_f(z_0) z_0 \geq \frac{|1 - \overline{f(0)}^T w_0|}{1 - \|f(0)\|^2} > 0$$

and $\overline{J_f(z_0)}^T$ is the transpose of conjugate matrix $\overline{J_f(z_0)}$. If $f(0) = 0$, then $\lambda \geq 1$.

(iii) *For any μ_j, there exists $\alpha_j \in \partial B^n \cap T^{(1,0)}_{z_0}(\partial B^n)$ such that*

$$J_f(z_0)\alpha_j = \mu_j \alpha_j, \quad |\mu_j| \leq \sqrt{\lambda}, \quad j = 2, \ldots, n.$$

(iv) $|\det J_f(z_0)| \leq \lambda^{(n+1)/2}$, $|\operatorname{tr} J_f(z_0)| \leq \lambda + \sqrt{\lambda}(n-1)$.

Moreover, these inequalities (ii), (iii) and (iv) are all sharp.

Interestingly, boundary Schwarz lemma plays also an important tool to the study of geometric function theory in several complex variables, see [9, 11, 13, 16–20, 26, 27].

Recently, Liu et al. in [21, Theorem 1.1] gave a boundary Schwarz lemma for holomorphic mappings between the Euclidean unit balls in any dimensions as follows.

Theorem 1.3. [21] *Let $f\colon B^n \to B^N$ be a holomorphic mapping for $n, N \geq 1$. If f is $C^{1+\alpha}$ for some $\alpha \in (0,1)$ at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in \partial B^N$, then we have*

(i) $J_f(z_0)\mathcal{T}_{z_0}(\partial B^n) \subset \mathcal{T}_{w_0}(\partial B^N)$, *and* $J_f(z_0)\mathcal{T}^{1,0}_{z_0}(\partial B^n) \subset \mathcal{T}^{1,0}_{w_0}(\partial B^N)$.

(ii) *There exists $\lambda \in \mathbb{R}$ such that*

$$\overline{J_f(z_0)}^T w_0 = \lambda z_0,$$

where $\lambda \geq \frac{|1 - \overline{f(0)}^T w_0|}{1 - \|f(0)\|^2}$.

Furthermore, the following boundary Schwarz lemma for pluriharmonic mappings between unit balls was obtained by Liu, Dai and Pan in [22, Theorem 1.2].

Theorem 1.4. [22] *Let $f\colon B^n \to B^N$ be a pluriharmonic mapping for $n, N \geq 1$. If f is $C^{1+\alpha}$ for some $\alpha \in (0,1)$ at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in \partial B^N$, then we have*

(i) $Df(z_0')\mathcal{T}_{z_0}(\partial B^n) \subset \mathcal{T}_{w_0'}(\partial B^N)$.

(ii) *There exists $\lambda > 0$ such that*

$$\overline{Df(z_0')}^T w_0' = \lambda z_0',$$

where z_0' and w_0' are real version of vector z_0 and w_0 respectively, and $\lambda \geq \frac{1-\|f(0)\|}{2^{2n-1}} > 0$.

Hamada [10] gave an improvement to $\lambda \geq \frac{1-\|f(0)\|}{2} > 0$ under the weak condition that the pluriharmonic mapping f is of C^1 at $z_0 \in \partial B^n$.

In this paper, we consider Schwarz lemma at the boundary of the p-unit ball B_p^n defined by

$$B_p^n = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|_p = (|z_1|^p + \cdots + |z_n|^p)^{1/p} < 1\big\}, \quad p \geq 2.$$

It is clear to see that B_2^n is the Euclidean unit ball of $\mathbb{C}^n$, always written by B^n. If $2 \leq p < \infty$, the domain B_p^n is a class of important weak psuedoconvex domains in $\mathbb{C}^n$.

In view of the above results, the motivation for this paper can be summarized in terms of the following question.

Question 1.5. Let $p \geq 2$ and $n, N \geq 1$. Suppose that $f\colon B_p^n \to B_p^N$ is a holomorphic or pluriharmonic mapping. If f is of C^1 class at $z_0 \in \partial B_p^n$, $f(z_0) = w_0 \in \partial B_p^N$, then what conclusions can we obtain about $Df(z_0)$?

This paper is devoted to giving an affirmative answer to Question 1.5. Compared with the Euclidean unit ball for $p = 2$, the p-unit ball for $p \neq 2$ is not homogeneous. There appears some difficulty because we can not proceed in analogy with Liu and Pan's idea [21] of $p = 2$. Our way to overcome this obstacle is to introduce the slice holomorphic function and then apply the general boundary Schwarz lemma of the unit disk (i.e., the inequality (1.1)) to the slice function, see Theorem 4.1. As a consequence, we generalize Schwarz lemma at the boundary for holomorphic mappings between p-unit balls from $p = 2$ to $p \geq 2$. Also, our idea is simple and new. Next, by extending pluriharmonic Schwarz lemma of Chen and Gauthier [5] from $p = 2$ to $p \geq 2$, we then give a Schwarz lemma at the boundary for pluriharmonic mapping between p-unit balls, which is Theorem 4.3.

The paper is organized as follows. In Section 2 we introduce some notations and definitions. In Section 3, we introduce some lemmas which will be used to prove the main results. In Section 4, we prove our main results.

2. Preliminaries

To proceed further, let us first introduce some notations and definitions. Denote by $\mathbb{C}^n$ the n-dimensional complex Hilbert space with the inner product and the norm given by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j, \quad \|z\| = \langle z, z\rangle^{1/2},$$

where $z, w \in \mathbb{C}^n$. As real vectors in $\mathbb{R}^{2n}$, z and w are orthogonal if and only if $\mathrm{Re}\langle z, w\rangle = 0$.

Throughout this paper, we write a point $z \in \mathbb{C}^n$ as a column vector in $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Let $\Omega \subset \mathbb{C}^n$ be a domain. For a holomorphic mapping $f \colon \Omega \to \mathbb{C}^n$, we also write f as the $n \times 1$ matrix form

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

where f_j is a holomorphic function from Ω to $\mathbb{C}$, $j = 1, \dots, n$. The derivative of f at a point $z \in \Omega$ is the complex Jacobian matrix of f given by

$$J_f(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right)_{n\times n}.$$

Then $J_f(z)$ is a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. The symbol $J_f(x)^T$ stands for the transpose of the matrix $J_f(x)$.

A domain means a connected open set in $\mathbb{C}^n$ (or $\mathbb{R}^n$) and the symbol $J_f(z)^T$ stands for the transpose of the matrix $J_f(z)$. Denote by $\mathcal{C}^k(\Omega_1, \Omega_2)$ the set of k-times continuously differential mappings from $\Omega_1 \subset \mathbb{C}^n$ into $\Omega_2 \subset \mathbb{C}^n$. When $k = 0$, $\mathcal{C}^0(\Omega_1, \Omega_2) = \mathcal{C}(\Omega_1, \Omega_2)$ denotes the set of continuous mappings from Ω_1 into Ω_2.

The following definition is well-known.

Definition 2.1. A mapping $f \colon \Omega \to \mathbb{C}^n$ is said to be differentiable at $z \in \Omega$ if there exists a bounded real linear map Df such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0.$$

If f is differentiable at each point of Ω, then f is said to be differentiable on Ω. In this case, the mapping

$$Df\colon z \in \Omega \mapsto Df(z)$$

is called the derivative (or differential) of f on Ω. If f is continuous in a neighborhood of z, the mapping f is said to be of class C^1 at z.

Next, we need to introduce the following two definitions, see [15].

Definition 2.2. Let $z_0 \in \partial B_p^n$ with $p \geq 2$. The real tangent space $\mathcal{T}_{z_0}(\partial B_p^n)$ to ∂B_p^n at z_0 is defined by

$$\mathcal{T}_{z_0}(\partial B_p^n) = \left\{\alpha \in \mathbb{C}^n : \operatorname{Re}\langle \alpha, \nabla\rho(z_0)\rangle = \operatorname{Re}\left[\overline{\nabla\rho(z_0)}^T \alpha\right] = 0\right\}.$$

Definition 2.3. Let $z_0 \in \partial B_p^n$ with $p \geq 2$. The complex tangent space $\mathcal{T}^{1,0}_{z_0}(\partial B_p^n)$ to ∂B_p^n at z_0 is defined by

$$\mathcal{T}^{1,0}_{z_0}(\partial B_p^n) = \left\{\alpha \in \mathbb{C}^n : \langle \alpha, \nabla\rho(z_0)\rangle = \overline{\nabla\rho(z_0)}^T \alpha = 0\right\},$$

where T denotes the transpose of vectors and matrices.

Definition 2.4. A pluriharmonic mapping f is a continuous mapping and defined on a domain $\Omega \subset \mathbb{C}^n$ such that for fixed $z \in \Omega$ and $\eta \in \mathbb{C}^n$ with $\|\eta\| = 1$, the mapping $f(z+\xi\eta)$ is harmonic in the complex variable ξ, such that $|\xi|$ is smaller than the distance of z from $\partial\Omega$, see [5].

If Ω is a simply connected domain in $\mathbb{C}^n$, then $f\colon \Omega \to \mathbb{C}^N$ is pluriharmonic if and only if f has a representation $f = g + \overline{h}$, where g and h are holomorphic mappings from Ω into $\mathbb{C}^N$.

3. Some lemmas

In order to prove our main results, we exhibit several basic lemmas as follows. For $z \in \overline{B_p^n}$ with $p \geq 2$, the p-norm $\rho(z) = \|z\|_p$ of B_p^n is C^2 class except the origin. Let $\nabla\rho(z)$ be a gradient at a point $z \neq 0$. Then we have

$$\nabla\rho(z) = 2\frac{\partial\rho(z)}{\partial\overline{z}} = 2\left(\frac{\partial\rho(z)}{\partial\overline{z_1}}, \dots, \frac{\partial\rho(z_0)}{\partial\overline{z_n}}\right)^T = \left(\frac{\partial\rho}{\partial x_1} + i\frac{\partial\rho}{\partial y_1}, \dots, \frac{\partial\rho}{\partial x_n} + i\frac{\partial\rho}{\partial y_n}\right)^T,$$

where $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Moreover, if $z \in \mathbb{C}^n \setminus \{0\}$, then we get

$$\nabla\rho(z) = \frac{1}{\|z\|_p^{p-1}}\left(|z_1|^{p-2}z_1, \dots, |z_n|^{p-2}z_n\right)^T \quad \text{and} \quad \|\nabla\rho(z)\|_q = 1,$$

where $1/p + 1/q = 1$. Further, it is easy to see that $\rho(z)$ is a C^2 on the boundary ∂B_p^n for $p \in [2, \infty)$.

Lemma 3.1. *The p-norm $\rho(z)=\|z\|_p$ of B_p^n satisfies the following properties.*

(i) $\nabla\rho(z)=2\frac{\partial\rho}{\partial\bar z}(z)=2\big(\frac{\partial p}{\partial\bar z_1}(z),\ldots,\frac{\partial\rho}{\partial\bar z_n}(z)\big)$, $\forall\, z\in\mathbb{C}^n\setminus\{0\}$.

(ii) $\langle\nabla\rho(z_0),z_0\rangle=1$, $\forall\, z_0\in\partial B_p^n$.

(iii) $\langle\nabla\rho(z),z\rangle=2\operatorname{Re}\frac{\partial\rho}{\partial z}(z)z=\rho(z)$, $\forall\, z\in\mathbb{C}^n\setminus\{0\}$.

(iv) $\big|\langle\nabla\rho(z),w\rangle\big|\le\rho(w)$, $\forall\, z\in\mathbb{C}^n\setminus\{0\}$, $w\in\mathbb{C}^n$.

Proof. Firstly, the result (i) is clear. Since $\rho(z)=\big(|z_1|^p+\cdots+|z_n|^p\big)^{1/p}$, we have

$$\nabla\rho(z)=\frac{1}{\|z\|_p^{p-1}}\big(|z_1|^{p-2}z_1,\ldots,|z_n|^{p-2}z_n\big)^T,\quad z\neq 0.$$

Also, (ii) and (iii) hold by using the above equality.

By using Hölder inequality, we get

$$\begin{aligned}\big|\langle\nabla\rho(z),w\rangle\big|&=\frac{1}{\|z\|_p^{p-1}}\left|\sum_{k=1}^n|z_k|^{p-2}\overline{z_k}w_k\right|\le\frac{1}{\|z\|_p^{p-1}}\sum_{k=1}^n|z_k|^{p-1}|w_k|\\&\le\frac{1}{\|z\|_p^{p-1}}\left(\sum_{k=1}^n|z_k|^{(p-1)q}\right)^{1/q}\left(\sum_{k=1}^n|w_k|^p\right)^{1/p}=\sum_{k=1}^n(|w_k|^p)^{1/p}=\rho(w),\end{aligned}$$

which gives the proof of (iv). □

The following Schwarz lemma is due to Pavlović [25, Theorem 3.6.1].

Lemma 3.2. [25] *Let $f\colon\mathbb{U}\to\mathbb{U}$ be a harmonic mapping. Then*

$$\left|f(z)-\frac{1-|z|^2}{1+|z|^2}f(0)\right|\le\frac{4}{\pi}\arctan|z|,\quad\forall\, z\in\mathbb{U}.$$

The following Schwarz lemma is a generalization of [5, Theorem 4] from the Euclidean unit balls to p-unit balls, which is a high dimensional version of Lemma 3.2.

Lemma 3.3. [7] *Let $p\ge 2$ and let $f\colon B_p^n\to B_p^N$ be a pluriharmonic mapping. Then*

$$\left\|f(z)-\frac{1-\|z\|_p^2}{1+\|z\|_p^2}f(0)\right\|_p\le\frac{4}{\pi}\arctan\|z\|_p,\quad\forall\, z\in B_p^n.$$

When $f(0)=0$, it holds

$$\|f(z)\|_p\le\frac{4}{\pi}\arctan\|z\|_p,\quad\forall\, z\in B_p^n.$$

The following Harnack inequality can be found in [2, 12]. For the completeness, we give a self-contained proof.

Lemma 3.4. [12] *Let $\mathbb{B}^n$ be the real unit ball of $\mathbb{R}^n$. If $u\colon \mathbb{B}^n \to \mathbb{R}$ is a nonnegative harmonic function, then for any $x \in \mathbb{B}^n$, we have*

$$\frac{1-\|x\|}{(1+\|x\|)^{n-1}}u(0) \le u(x) \le \frac{1+\|x\|}{(1-\|x\|)^{n-1}}u(0).$$

In particular, when $-1 < x_1 < 1$, it holds

$$\frac{1-|x_1|}{(1+|x_1|)^{n-1}}u(0) \le u(x_1,0,\dots,0) \le \frac{1+|x_1|}{(1-|x_1|)^{n-1}}u(0).$$

Proof. Firstly, we suppose that u is harmonic on the closed unit ball $\overline{\mathbb{B}^n}$. According to Poisson's equation, we have

$$u(x) = \frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} \frac{1-\|x\|^2}{\|x-y\|^n}u(y)\,d\sigma(y)$$

holds for all $x \in \mathbb{B}^n$. Notice that

$$\frac{1-\|x\|^2}{(1+\|x\|)^n} \le \frac{1-\|x\|^2}{\|x-y\|^n} \le \frac{1-\|x\|^2}{(1-\|x\|)^n}, \quad u(0) = \frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}} u(y)\,d\sigma(y),$$

we get

$$\frac{1-\|x\|}{(1+\|x\|)^{n-1}}u(0) \le u(x) \le \frac{1+\|x\|}{(1-\|x\|)^{n-1}}u(0).$$

Secondly, for the general case, if $u(x)$ is a nonnegative harmonic function, then we apply the above result to the dilates function $u_r(x) = u(rx)$ $(0 < r < 1)$ and take the limit as $r \to 1^-$. □

4. Main results

In this section, we prove the main results of this article. Recall the following notations

$$\rho(z) = \left(\sum_{k=1}^{n}|z_k|^p\right)^{1/p} \quad \text{and} \quad \nabla\rho(z) = 2\frac{\partial\rho}{\partial\bar{z}}(z) = \frac{1}{\|z\|_p^{p-1}}\left(|z_1|^{p-2}\overline{z_1},\dots,|z_n|^{p-2}\overline{z_n}\right)^T.$$

Theorem 4.1. *Let $p \ge 2$ and $n, N \ge 1$. Suppose that $f\colon B_p^n \to B_p^N$ is a holomorphic mapping. If f is C^1 at $z_0 \in \partial B_p^n$, $f(z_0) = w_0 \in \partial B_p^N$, then the following two statements hold:*

(i) $J_f(z_0)\mathcal{T}_{z_0}(\partial B_p^n) \subset \mathcal{T}_{w_0}(\partial B_p^N)$ *and* $J_f(z_0)\mathcal{T}_{z_0}^{1,0}(\partial B_p^n) \subset \mathcal{T}_{w_0}^{1,0}(\partial B_p^N)$.

(ii) *There exists* $\lambda \ge \frac{1-\|f(0)\|_p}{1+\|f(0)\|_p} > 0$ *such that*

$$\overline{J_f(z_0)}^T\nabla\rho(w_0) = \lambda\nabla\rho(z_0), \quad \text{or} \quad \overline{\nabla\rho(w_0)}^T J_f(z_0) = \lambda\overline{\nabla\rho(z_0)}^T.$$

In particular, if $f(0)=0$*, then* $\overline{\nabla\rho(w_0)}^T J_f(z_0)\nabla\rho(z_0)=\lambda\geq 1$.

Proof. (i) For any unit vector $\alpha\in\mathcal{T}_{z_0}(\partial B_p^n)$, we can choose a smooth curve

$$\gamma\colon[-1,1]\setminus\{0\}\to B_p^n$$

such that

$$\gamma(0)=z_0,\quad \frac{d}{dt}\gamma(t)\Big|_{t=0}=\alpha.$$

Since f is C^1 at z_0, it yields that $f(\gamma[-1,1])\subset\overline{B_p^N}$. From $\rho(z)=\|z\|_p=\big(|z_1|^p+\cdots+|z_n|^p\big)^{1/p}$, we have

$$\max_{t\in(-1,1)}\rho(f(\gamma(t)))=\rho(f(\gamma(0)))=\rho(w_0)=1.$$

This implies that

$$\frac{d}{dt}\rho(f(\gamma(t)))\Big|_{t=0}=2\operatorname{Re}\left[\overline{\nabla\rho(z_0)}\right]^T J_f(z_0)\alpha=0,\quad \forall\,\alpha\in\mathcal{T}_{z_0}(\partial B_p^n).$$

Hence, $J_f(z_0)\alpha\in\mathcal{T}_{z_0}(\partial B_p^N)$. Notice that $J_f(z_0)$ is a $\mathbb{C}$-linear transformation, we get

$$J_f(z_0)\mathcal{T}_{z_0}^{1,0}(\partial B_p^n)\subset\mathcal{T}_{z_0}^{1,0}(\partial B_p^N).$$

The proof of (i) is completed.

(ii) The proof is divided into two steps.

Step 1. We claim that there exists $\lambda\in\mathbb{R}$ such that $\overline{J_f(z_0)}^T\nabla\rho(w_0)=\lambda\nabla\rho(z_0)$. In fact, suppose

$$\overline{J_f(z_0)}^T\nabla\rho(w_0)=\lambda\nabla\rho(z_0)+\beta$$

for some $\lambda\in\mathbb{R}$ and $\beta\in\mathcal{T}_{z_0}(\partial B_p^n)$. In terms of $J_f(z_0)\mathcal{T}_{z_0}(\partial B_p^n)\subset\mathcal{T}_{w_0}(\partial B_p^N)$ of result (i), we get

$$J_f(z_0)\beta\in\mathcal{T}_{w_0}(\partial B_p^N).$$

Namely, $\operatorname{Re}\langle J_f(z_0)\beta,\nabla\rho(w_0)\rangle=0$. Hence,

$$\|\beta\|^2=\operatorname{Re}\langle\beta,\lambda\nabla\rho(z_0)+\beta\rangle=\operatorname{Re}\langle\beta,\overline{J_f(z_0)}^T\nabla\rho(z_0)\rangle=\operatorname{Re}\langle J_f(z_0)\beta,\nabla\rho(z_0)\rangle=0.$$

This shows that $\beta=0$ and $\overline{J_f(z_0)}^T\nabla\rho(w_0)=\lambda\nabla\rho(z_0)$ for some $\lambda\in\mathbb{R}$. Consequently,

$$\overline{\nabla\rho(w_0)}^T J_f(z_0)=\lambda\overline{\nabla\rho(z_0)}^T. \tag{4.1}$$

Step 2. We prove the above real number $\lambda\geq\frac{1-\|f(0)\|_p}{1+\|f(0)\|_p}$. Notice that

$$\nabla\rho(w_0)=\big(|w_1|^{p-2}\overline{w_1},\ldots,|w_N|^{p-2}\overline{w_N}\big)\Big|_{w=w_0}^T.$$

Let

$$h(\zeta)=\langle f(\zeta z_0),\nabla\rho(w_0)\rangle=\overline{\nabla\rho(w_0)}^T f(\zeta z_0),\quad \zeta\in\mathbb{U}.$$

Then $h\colon \mathbb{U} \to \mathbb{U}$ is holomorphic, and

$$h(1) = \langle f(z_0), \nabla\rho(w_0)\rangle = \langle w_0, \nabla\rho(w_0)\rangle = 1.$$

Applying the equality (1.1) to $h(\zeta)$, this implies that

$$h'(1) = \lim_{t\to 1^-} \frac{h(1)-h(t)}{1-t} \ge \frac{1-|h(0)|}{1+|h(0)|} = \frac{1-|\overline{\nabla\rho(w_0)}^T f(0)|}{1+|\overline{\nabla\rho(w_0)}^T f(0)|}.$$

Since $\|\nabla\rho(w_0)\|_q = 1$, we get

$$|\overline{\nabla\rho(w_0)}^T f(0)| \le \|\nabla\rho(w_0)\|_q \|f(0)\|_p \le 1.$$

By using Lemma 3.1(iv), we get

$$|h(0)| = |\langle f(0), \nabla\rho(z_0)\rangle| \le \rho(f(0)) = \|f(0)\|_p,$$

which follows from Hölder inequality. Hence,

$$h'(1) \ge \frac{1-\|f(0)\|_p}{1+\|f(0)\|_p}.$$

In terms of the equality (4.1), we have

$$\lambda = \langle z_0, \overline{J_f(z_0)}^T \nabla\rho(w_0)\rangle = \langle J_f(z_0)z_0, \nabla\rho(w_0)\rangle.$$

By using Lemma 3.1 again, we get

$$h'(1) = \langle J_f(z_0)z_0, \nabla\rho(w_0)\rangle = \langle z_0, \overline{J_f(z_0)}^T \nabla\rho(w_0)\rangle = \lambda \ge \frac{1-\|f(0)\|_p}{1+\|f(0)\|_p}.$$

In particular, when $f(0) = 0$, then $\lambda \ge 1$. Hence, this completes the proof. □

Remark 4.2. Theorem 4.1 generalizes the boundary Schwarz lemma [21, Theorem 1.1] from the Euclidean unit ball of $p = 2$ to the p-unit ball of $p \ge 2$. Moreover, our method is quite simple.

For pluriharmonic mappings, we have the following boundary Schwarz lemma. Denote

$$\nabla\rho(z) = 2\left(\frac{\partial\rho(z)}{\partial\overline{z_1}}, \dots, \frac{\partial\rho(z)}{\partial\overline{z_n}}\right)^T = \left(\frac{\partial\rho(z)}{\partial x_1} + i\frac{\partial\rho(z)}{\partial y_1}, \dots, \frac{\partial\rho(z)}{\partial x_n} + i\frac{\partial\rho(z)}{\partial y_n}\right)^T,$$

where $z_j = x_j + iy_j$ for $j = 1, \dots, n$. For convenience to the following expression, we still use z standing for the real version of the complex vector z.

Theorem 4.3. *Let $p \ge 2$ and $n, N \ge 1$. Suppose that $f\colon B_p^n \to B_p^N$ is a pluriharmonic mapping. If f is C^1 at $z_0 \in \partial B_p^n$, $f(z_0) = w_0 \in \partial B_p^N$, then the following two statements hold:*

(i) $Df(z_0)\mathcal{T}_{z_0}(\partial B_p^n) \subset \mathcal{T}_{w_0}(\partial B_p^N)$.

(ii) *There exists* $\lambda \geq \max\left\{\frac{2}{\pi} - \|f(0)\|_p, \frac{1-\mathrm{Re}\langle f(0), \nabla\rho(w_0)\rangle}{2}\right\}$ *such that*

$$\overline{Df(z_0)}^T \nabla\rho(w_0) = \lambda\nabla\rho(z_0), \quad \text{or} \quad \overline{\nabla\rho(w_0)}^T Df(z_0) = \lambda\overline{\nabla\rho(z_0)}^T.$$

In particular, if $f(0) = 0$*, then* $\overline{\nabla\rho(w_0)}^T Df(z_0)\nabla\rho(z_0) = \lambda \geq 2/\pi$.

Proof. By using the arguments similar to Theorem 4.1, we have $Df(z_0)\mathcal{T}_{z_0}(\partial B_p^n) \subset \mathcal{T}_{w_0}(\partial B_p^N)$ and $\overline{Df(z_0)}^T \nabla\rho(w_0) = \lambda\nabla\rho(z_0)$ for some $\lambda \in \mathbb{R}$. Hence, we need to prove part (ii).

Firstly, we prove $\lambda \geq 2/\pi - \|f(0)\|_p$. By some calculations, we get

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{\|f(z_0)\|_p - \|f(z_0 - t\nabla\rho(z_0))\|_p}{t} \\
&= \lim_{t\to 0^+} \frac{\rho(f(z_0)) - \rho(f(z_0) - t\nabla\rho(z_0))}{t} = 2\,\mathrm{Re}\left\{\left(\frac{\partial\rho}{\partial w}(w_0)\right)^T J_f(z_0)\nabla\rho(z_0)\right\} \\
&= \mathrm{Re}\left\{\overline{\nabla\rho(w_0)}^T J_f(z_0)\nabla\rho(z_0)\right\} = \lambda\|\nabla\rho(z_0)\|^2.
\end{aligned}$$

According to Lemma 3.3, we have

$$\|f(z)\|_p \leq \frac{4}{\pi}\arctan\|z\|_p + \frac{1-\|z\|_p^2}{1+\|z\|_p^2}\|f(0)\|_p, \quad z \in B_p^n.$$

Taking $z = z_0 - t\nabla\rho(z_0)$ and letting $t \to 0^+$, we get

$$\|f(z_0 - t\nabla\rho(z_0))\|_p \leq \frac{4}{\pi}\arctan\|z_0 - t\nabla\rho(z_0)\|_p + \frac{1-\|z_0 - t\nabla\rho(z_0)\|_p^2}{1+\|z_0 - t\nabla\rho(z_0)\|_p^2}\|f(0)\|_p.$$

Then

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{1-\|f(z_0 - t\nabla\rho(z_0))\|_p}{t} \\
&\geq \lim_{t\to 0^+} \frac{1 - \frac{4}{\pi}\arctan\|z_0 - t\nabla\rho(z_0)\|_p - \frac{1-\|z_0-t\nabla\rho(z_0)\|_p^2}{1+\|z_0-t\nabla\rho(z_0)\|_p^2}\|f(0)\|_p}{t}
\end{aligned}$$

for $t \to 0^+$. Consequently,

$$\lambda\|\nabla\rho(z_0)\|^2 \geq \left(\frac{2}{\pi} - \|f(0)\|_p\right)\|\nabla\rho(z_0)\|^2.$$

This shows that

$$\lambda \geq \frac{2}{\pi} - \|f(0)\|_p. \tag{4.2}$$

Secondly, we prove $\lambda \geq \frac{\operatorname{Re}\langle f(0), \nabla\rho(w_0)\rangle}{2}$. Define

$$h(\xi) = 1 - \operatorname{Re}\langle f(\xi z_0), \nabla\rho(w_0)\rangle, \quad \xi \in \mathbb{U}.$$

Then h is harmonic on $\mathbb{U}$. Notice that

$$h(\xi) = 1 - \operatorname{Re}\langle f(\xi z_0), \nabla\rho(w_0)\rangle \geq 1 - \rho(f(\xi z_0)) \geq 0$$

follows from Lemma 3.1. Applying to Lemma 3.4 for the harmonic function $h(\xi)$ with $n = 2$, we get

$$\frac{1-r}{1+r} h(0) \leq h(\xi) \leq \frac{1+r}{1-r} h(0), \quad |\xi| = r < 1.$$

Since

$$h(1) = 1 - \operatorname{Re}\langle f(z_0), \nabla\rho(w_0)\rangle = 1 - \operatorname{Re}\langle w_0, \nabla\rho(w_0)\rangle = 0,$$

it yields that

$$\lim_{r\to 1^-} \frac{h(r) - h(1)}{1-r} = \lim_{r\to 1^-} \frac{h(r)}{1-r} \geq \lim_{r\to 1^-} \frac{h(0)}{1+r} = \frac{h(0)}{2}.$$

By using (i), we get

$$\lim_{r\to 1^-} \frac{h(r) - h(1)}{1-r} = \operatorname{Re}\langle Df(z_0) z_0, \nabla\rho(w_0)\rangle = \operatorname{Re}\langle z_0, \overline{Df(z_0)}^T \nabla\rho(w_0)\rangle = \lambda.$$

Hence,

$$\lambda \geq \frac{h(0)}{2} \geq \frac{1 - \operatorname{Re}\langle f(0), \nabla\rho(w_0)\rangle}{2}. \tag{4.3}$$

Putting (4.2) and (4.3) together, we get

$$\lambda \geq \max\left\{\frac{2}{\pi} - \|f(0)\|_p, \frac{1 - \operatorname{Re}\langle f(0), \nabla\rho(w_0)\rangle}{2}\right\}. \qquad \square$$

Remark 4.4. When $p = 2$, Theorem 4.3 reduces to [9, Proposition 1.8] for the case of finite dimensional Hilbert spaces.

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