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On Total Coloring of Some Classes of Regular Graphs

Shantharam Prajnanaswaroopa, Jayabalan Geetha, Kanagasabapathi Somasundaram*,
Hung-Lin Fu and Narayanan Narayanan

Abstract. In this paper, we have obtained upper bounds for the total chromatic number of some classes of Cayley graphs, odd graphs and mock threshold graphs.

1. Introduction

All the graphs considered here are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$, respectively. A *total coloring* of G is a mapping $f\colon V(G) \cup E(G) \to C$, where C is the set of colors and f satisfies

(a) $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$,

(b) $f(e) \neq f(e')$ for any two incident edges $e, e' \in E(G)$ and

(c) $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to v.

The *total chromatic number* of a graph G, denoted by $\chi''(G)$, is the minimum number of colors that are used in a total coloring. It is clear that $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Behzad [2] and Vizing [16] independently conjectured (Total Coloring Conjecture (TCC)) that for every graph G, $\chi''(G) \leq \Delta(G) + 2$. The graphs that can be totally colored with $\Delta(G) + 1$ colors are said to be type-I graphs, and those with total chromatic number $\Delta(G) + 2$ are said to be type-II. The total coloring conjecture is a long-standing conjecture and has defied several attempts for a proof. It has been shown that the decision algorithm for total coloring is NP-complete even for cubic bipartite graphs [14]. Still, a lot of progress has been made towards proving the TCC. It is easily seen that TCC is true for complete graphs, bipartite, and complete multipartite graphs. It was shown to be true for all graphs having maximum degree $\Delta(G) \leq 5$, $\Delta(G) \geq 3n/4$ and $\Delta(G) \geq n - 5$, where n is the number of vertices; using techniques like enlarge-matching

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*Corresponding author.

argument and fan recoloring process [17]. For planar graphs, TCC is confirmed for all $\Delta(G) \neq 6$ using the method of discharging. The total coloring conjecture has also been confirmed for several other classes of graphs. Some of the surveys of techniques and other results on total coloring include Yap [17], Borodin [4] and Geetha et al. [11].

2. Total coloring of Cayley graphs

Let Γ be a group with identity 1. For $S \subseteq \Gamma$, $1 \notin S$ and $S^{-1} = \{s^{-1} : s \in S\} = S$ the *Cayley graph* $X = \mathrm{Cay}(\Gamma, S)$ is the undirected graph with vertex set $V(X) = \Gamma$ and edge set $E(X) = \{(a, b) : ab^{-1} \in S\}$. Cayley graphs are vertex transitive; that is, there exists an automorphism that maps any vertex of the graph to any other vertex.

Cayley graphs associated with $\Gamma = \mathbb{Z}_n$, the group of integers modulo n under addition, are called *circulant graphs*. In other words, given a sequence of positive integers $1 \leq d_1 < d_2 < \cdots < d_l \leq \lfloor n/2 \rfloor$, the circulant graph $G = C_n(d_1, d_2, \ldots, d_l)$ has the vertex set $V = Z_n = \{0, 1, 2, \ldots, n-1\}$, and two vertices x and y are adjacent if and only if $x = (y \pm d_i) \pmod{n}$ for some i, $1 \leq i \leq l$. For positive integers n and k, $1 \leq k < \lfloor n/2 \rfloor$, the k^{th} power of the cycle C_n, denoted by C_n^k, has $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(C_n^k) = E^1 \cup E^2 \cup \cdots \cup E^k$, where $E^i = \{e_0^i, e_1^i, \ldots, e_{n-1}^i\}$ and $e_j^i = (v_j, v_{(j+i) \pmod{n}})$, $0 \leq j \leq n-1$ and $1 \leq i \leq k$. Note that the k^{th} power of the cycle graph C_n is the circulant graph over $\mathbb{Z}_n$ with the generating set $S = \{1, 2, \ldots, k, n-k, \ldots, n-2, n-1\}$.

Campos and de Mello [6] proved that C_n^2, $n \neq 7$, is type-I and C_7^2 is type-II. They [5] verified the TCC for C_n^k when $2 < k < n/2$ and n is even. They also showed that one can obtain a $(\Delta(C_n^k) + 2)$-total coloring for these graphs in polynomial time. Further, they proved that C_n^k with $n \equiv 0 \pmod{\Delta(C_n^k) + 1}$ is type-I, and proposed the following conjecture.

Conjecture 2.1. *Let $G = C_n^k$ with $2 \leq k < \lfloor n/2 \rfloor$. Then*

$$\chi''(G) = \begin{cases} \Delta(G) + 2 & \text{if } k > n/3 - 1 \text{ and } n \text{ is odd}, \\ \Delta(G) + 1 & \text{otherwise}. \end{cases}$$

A Latin square is an $n \times n$ array consisting of n entries of numbers (or symbols) with each row and column consisting of only one instance of each element. This means that the rows and columns are permutations of one single n-vector with distinct entries. A Latin square is said to be commutative if it is symmetric. A Latin square of numbers is said to be idempotent if each diagonal element consists of the number equal to its row index. In addition, if the rows of the Latin square are just cyclic permutations (one-shift of the elements to the right) of the previous row, then the Latin square is said to be dfcirculant (if the cyclic permutations are actually left shifts, it is called anti-circulant). The circulant (anti-circulant) Latin square can be generated from a single row vector. The Latin square of order $(2k+1)$ shown in Table 2.1 is anti-circulant, commutative and idempotent.

1	$k+2$	2	$k+3$	$\cdots$	$2k+1$	$k+1$
$k+2$	2	$k+3$	3	$\cdots$	$k+1$	1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$k+1$	1	$k+2$	2	$\cdots$	k	$2k+1$

Table 2.1

The entries of the Latin square above are as follows:

$$L = (l_{ij}) = \begin{cases} m & \text{if } i+j = 2m, \\ (k+m) \pmod{2k+1} + 1 & \text{if } i+j = 2m+1. \end{cases}$$

From the above description, it can be easily verified that the Latin square corresponding to the matrix L is commutative, idempotent and anti-circulant.

The following lemma is due to Stong [15, Corollary 2.3.1].

Lemma 2.2. *If Γ is an even order abelian group having generating set S, then* $\mathrm{Cay}(\Gamma, S)$ *is 1-factorizable.*

Lemma 2.3. *The graph $C_n^{(n-2)/4}$, where $(n-2)/4$ is an integer, is type-I.*

Proof. We know that $\Delta(C_n^{(n-2)/4}) = (n-2)/2$. To obtain a $((n-2)/2+1) = n/2$-total coloring of $C_n^{(n-2)/4}$, we form the color matrix (a matrix whose diagonal entries represent the color of the vertices and the other non-zero entries represent edge colors). We first fill the non-zero entries and diagonal entries in the first $n/2 \times n$ sub-matrix of the color matrix with the corresponding entries of the first $n/2$ rows of the Latin square which is constructed as commutative, idempotent and anti-circulant. The first non-zero entry of the $(n/2+1)$-th row of the color matrix is determined by the $(n/2+1)$-th entry of the $((n-2)/4+2)$-th row of the color matrix (as the color matrix is symmetric). The next non-zero entries of the $(n/2+1)$-th row are determined by the cyclic order of the previous row. Similarly, we determine the non-zero entries of the remaining rows (the first entry is determined by the symmetry of the color matrix and the next entries are determined by the cyclic order of the previous rows). Thus continuing, we can fill all the entries of the color matrix. It is easily verified that it satisfies the total coloring conditions, giving us a $(n/2)$-total coloring. In other words, $C_n^{(n-2)/4}$ is type-I. □

Example 2.4. Let us consider the graph C_{10}^2, whose adjacency matrix is given in Table 2.2.

0	1	1	0	0	0	0	0	1	1
1	0	1	1	0	0	0	0	0	1
1	1	0	1	1	0	0	0	0	0
0	1	1	0	1	1	0	0	0	0
0	0	1	1	0	1	1	0	0	0
0	0	0	1	1	0	1	1	0	0
0	0	0	0	1	1	0	1	1	0
0	0	0	0	0	1	1	0	1	1
1	0	0	0	0	0	1	1	0	1
1	1	0	0	0	0	0	1	1	0

Table 2.2: Adjacency matrix of C_{10}^2.

We use the (5×5)-anti-circulant, commutative and idempotent Latin square presented in Table 2.3. The filled up color matrix for C_{10}^2 (where $n = 10 = 2(2(2)+1) = 2(2(n-2)/4+1))$ is given in Table 2.4, which is a 5-total coloring of C_{10}^2. Note that the entries filled with 0's in the color matrix do not represent colors.

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Table 2.3: Anti-circulant, idempotent Latin square.

1	4	2	0	0	0	0	0	5	3
4	2	5	3	0	0	0	0	0	1
2	5	3	1	4	0	0	0	0	0
0	3	1	4	2	5	0	0	0	0
0	0	4	2	5	3	1	0	0	0
0	0	0	5	3	1	4	2	0	0
0	0	0	0	1	4	2	5	3	0
0	0	0	0	0	2	5	3	1	4
5	0	0	0	0	0	3	1	4	2
3	1	0	0	0	0	0	4	2	5

Table 2.4: Color matrix of C_{10}^2.

Theorem 2.5. *Let $n \equiv 2 \pmod 4$ and $(n-2)/4 < k < n/2$. If $S = \{x \in \mathbb{Z}_n \mid (n-2)/4 < x \leq k\}$ generates Z_n, then C_n^k is type-I.*

Proof. Consider a circulant graph, say H, on $\mathbb{Z}_n$ with the generating set $S = \{x \in \mathbb{Z}_n \mid (n-2)/4 < x \leq k\} = \{(n-2)/4+1, (n-2)/4+2, \ldots, k-1, k, n-k, n-k+1, \ldots, n-(n-2)/4-2, n-(n-2)/4-1\}$. Since the generating set of C_n^k is $\{1, 2, \ldots, k, n-k, n-k+1, \ldots, n-1\}$ and since $k > (n-2)/4$, we see that the edges of C_n^k are a disjoint union of the edges of $C_n^{(n-2)/4}$ and the edges of H. Because S generates $\mathbb{Z}_n$, it follows from Lemma 2.2 that H is 1-factorizable. Thus, H requires only $\Delta(H)$ colors for its edge coloring.

For constructing the color matrix of C_n^k, we start with the type-I total coloring of $C_n^{(n-2)/4}$ given in Lemma 2.3, which colors all the vertices and some edges of C_n^k. It remains to give a coloring to the edges of the (added) circulant graph H, which require $\Delta(H)$ extra colors. Thus, the total coloring of the graph C_n^k requires at most $2((n-2)/4)+1+\Delta(H) = 2((n-2)/4) + \big(2k - 2((n-2)/4)\big) + 1 = 2k+1$ colors, as required. $\square$

Theorem 2.6. *Assume that $n = s(2m+1)$, s is even, $k/2 \leq m \leq k$. Then, C_n^k is type-I.*

Proof. From the assumptions, $n \equiv 0 \pmod{2m+1}$. We know that there exists a commutative idempotent Latin square of order $2m+1$, that we denote by C'. Let e_{ij} denote the $(ij)^{\text{th}}$ entry in C'. For $m > k/2$, we define an upper triangular Tableau D of order $2m-k$ as follows:

$$D = (d_{ij}) = \begin{cases} e_{ij} & \text{if } i = 1, 2, \ldots, 2m-k \text{ and } j = k+i+1, k+i+2, \ldots, 2m+1, \\ \text{empty} & \text{otherwise.} \end{cases}$$

The structure of Tableau D is presented in Table 2.5.

$e_{1,k+2}$	$e_{1,k+3}$	$\cdots$	$\cdots$	$e_{1,2m+1}$
	$e_{2,k+3}$	$e_{2,k+4}$	$\cdots$	$e_{2,2m+1}$
		$\cdots$	$\cdots$	$e_{3,2m+1}$
			$\cdots$	$\vdots$
				$e_{2m-k,2m+1}$

Table 2.5: Tableau D.

We first define three Tableaux A, B and C of respective order k, k and $2m+1$ using portions of C', D and additional colors. The Tableau A is upper triangular whose entries in the main diagonal are $2k+1$ and the subsequent sub-diagonals are filled respectively with $2k$, $2k-1$ down to $2m+2$. Similarly, Tableau B is lower triangular with main diagonal entries being $2m+2$ and subsequent sub-diagonals increase values up to $2k+1$. We consider the following cases:

(i) If $m = k/2$, the Tableaux A and B are presented in Table 2.6. We let Tableau $C = C'$. In this case, D is undefined.

$2k+1$	$2k$	$2k-1$	$\cdots$	$2m+2$
	$2k+1$	$2k$	$\cdots$	$2m+3$
		$2k+1$	$2k$	$\cdots$
			$2k+1$	$2k$
				$2k+1$

$2m+2$				
$2m+3$	$2m+2$			
$2m+4$	$2m+3$	$2m+2$		
$\cdots$	$\cdots$	$\cdots$	$2m+2$	
$2k+1$	$\cdots$	$\cdots$	$\cdots$	$2m+2$

Table 2.6: Left-Tableau A and Right-Tableau B when $m = k/2$.

(ii) If $k/2 < m < k$, there will be $2m-k$ unfilled upper sub-diagonals in Tableau A. We fill these unfilled upper sub-diagonals of A using the entries from D as shown in Table 2.7. Similarly, there will be $2m-k$ unfilled lower sub-diagonals in Tableau B and these sub-diagonals are filled using D^T. The Tableau C is obtained by deleting D and D^T from C'.

$2k+1$	$2k$	$\cdots$	$2m+2$	$e_{1,k+2}$	$\cdots$	$e_{1,2m}$	$e_{1,2m+1}$
	$2k+1$	$2k$	$\cdots$	$2m+2$	$\cdots$		$e_{2,2m+1}$
		$2k+1$	$2k$	$\cdots$	$2m+2$		$\vdots$
			$2k+1$	$2k$	$\cdots$	$2m+2$	$e_{k+2,2m+1}$
				$2k+1$	$2k$		$2m+2$
					$2k+1$	$2k$	$\vdots$
						$2k+1$	$2k$
							$2k+1$

Table 2.7: Tableaux A for $k/2 < m < k$.

(iii) If $m = k$, then $A = D$ and $B = D^T$. Again the Tableau C is obtained by deleting D and D^T from C'.

We now arrange the three Tableaux as given below to form the $n \times n$ color matrix. We place s copies of Tableau C (of order $2m+1$) along the main diagonal as depicted in Table 2.8. Each cell represents a sub-matrix of order $2m+1$. In the cell $(i, i+1)$, $0 < i < s$, a copy of B (i odd) or A^T (i even) is placed bottom-left justified. Similarly, in the cell $(i+1, i)$ a copy of B^T (i odd) or A (i even) is placed top-right justified. A copy of A and A^T are respectively placed at cells $(1, s)$ and $(s, 1)$.

C	B				A
B^T	C	A^T			
	A	C	B		
		B^T	C	A^T	
			A	C	B
A^T				B^T	C

Table 2.8: Color matrix structure.

Observe that if we place Tableau B (justified bottom-left) and A (top-right) in a color matrix of order n, the first entry of B starts at row $2m - k + 2$ of the coloring matrix. Thus, the rows of A and B are placed with a shift of $2m - k + 1$, strictly more than the order of Tableau D. This ensures that entries in the rows (columns) of A and B do not clash.

From the above observation, the entries in rows (columns) of Tableau A placed at cell $(1, s)$ and Tableau B at $(1, 2)$ (at $(s - 1, s)$), respectively do not clash. By symmetry, the colors in B^T at $(2, 1)$ (at $(s, s - 1)$) and A^T at $(s, 1)$ won't clash either.

Hence, we obtain a proper total coloring of C_n^k, which is a type-I total coloring. $\square$

Example 2.7. For the color matrix of the graph C_{20}^4, since $20 = 4(4+1) = 4(2m+1)$ for $m = 2$, we take Tableaux C', A, B to be as shown in Table 2.9. The color matrix is given in Table 2.10.

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

9	8	7	6
	9	8	7
		9	8
			9

6			
7	6		
8	7	6	
9	8	7	6

Table 2.9: Tableau C' (left), Tableau A and Tableau B (right) for C_{20}^4.

1	4	2	5	3												9	8	7	6
4	2	5	3	1	6												9	8	7
2	5	3	1	4	7	6												9	8
5	3	1	4	2	8	7	6												9
3	1	4	2	5	9	8	7	6											
	6	7	8	9	1	4	2	5	3										
		6	7	8	4	2	5	3	1	9									
			6	7	2	5	3	1	4	8	9								
				6	5	3	1	4	2	7	8	9							
					3	1	4	2	5	6	7	8	9						
						9	8	7	6	1	4	2	5	3					
							9	8	7	4	2	5	3	1	6				
								9	8	2	5	3	1	4	7	6			
									9	5	3	1	4	2	8	7	6		
										3	1	4	2	5	9	8	7	6	
											6	7	8	9	1	4	2	5	3
9												6	7	8	4	2	5	3	1
8	9												6	7	2	5	3	1	4
7	8	9												6	5	3	1	4	2
6	7	8	9												3	1	4	2	5

Table 2.10: Color matrix of C_{20}^4.

1	6	2	7	3	8	4	9	5
6	2	7	3	8	4	9	5	1
2	7	3	8	4	9	5	1	6
7	3	8	4	9	5	1	6	2
3	8	4	9	5	1	6	2	7
8	4	9	5	1	6	2	7	3
4	9	5	1	6	2	7	3	8
9	5	1	6	2	7	3	8	4
5	1	6	2	7	3	8	4	9

11	10	4	9	5
	11	10	5	1
		11	10	6
			11	10
				11

10				
11	10			
4	11	10		
9	5	11	10	
5	1	6	11	10

Table 2.11: Tableau C' (left), A, B (right) for C_{18}^5.

Example 2.8. For the case of the color matrix of the graph C_{18}^5, since $18 = 2(8+1) = 2(2m+1)$ for $m = 4$, we have Tableaux C', A, B to be as shown in Table 2.11. The color matrix is shown in Table 2.12.

1	6	2	7	3	8								11	10	4	9	5
6	2	7	3	8	4	9								11	10	5	1
2	7	3	8	4	9	5	1								11	10	6
7	3	8	4	9	5	1	6	2								11	10
3	8	4	9	5	1	6	2	7	10								11
8	4	9	5	1	6	2	7	3	11	10							
	9	5	1	6	2	7	3	8	4	11	10						
		1	6	2	7	3	8	4	9	5	11	10					
			2	7	3	8	4	9	5	1	6	11	10				
				10	11	4	9	5	1	6	2	7	3	8			
					10	11	5	1	6	2	7	3	8	4	9		
						10	11	6	2	7	3	8	4	9	5	1	
							10	11	7	3	8	4	9	5	1	6	2
11								10	3	8	4	9	5	1	6	2	7
10	11								8	4	9	5	1	6	2	7	3
4	10	11								9	5	1	6	2	7	3	8
9	5	10	11								1	6	2	7	3	8	4
5	1	6	10	11								2	7	3	8	4	9

Table 2.12: Color matrix for C_{18}^5.

Theorem 2.9. *Let n, s, m and k be positive integers with $n = s(2m+1) - 1$, s is even and $k/2 \le m \le k$. Then C_n^k satisfies TCC.*

Proof. From the previous theorem, it follows that C_{n+1}^k is $(2k+1)$-total colorable. It remains to show that this coloring could be modified to a $(2k+2)$-total coloring for C_n^k. To this end, we first delete the $(n+1)^{\text{st}}$ row and $(n+1)^{\text{st}}$ column of the color matrix of C_{n+1}^k. Then, we fill the entries of the lower and upper $(n-k+1)^{\text{st}}$ sub-diagonal of the color matrix with the new color $2k+2$. It is easy to verify that this is the required total coloring. Note that, here we are adding a new color in the color matrix. So there is no clash of colors in the rows and columns. As an example, see the color matrix for C_{14}^3 in Table 2.13 and that of C_{13}^3, obtained from it by deleting the last row and last column and then using color $2k+2 = 8$ in the $n-k+1 = 11^{\text{th}}$ sub-diagonal (3-rd from the right) as shown in Table 2.14.

1	5	2	6								3	7	4
5	2	6	3	7								4	1
2	6	3	7	4	1								5
6	3	7	4	1	5	2							
	7	4	1	5	2	6	3						
		1	5	2	6	3	7	4					
			2	6	3	7	4	1	5				
				3	7	4	1	5	2	6			
					4	1	5	2	6	3	7		
						5	2	6	3	7	4	1	
							6	3	7	4	1	5	2
3								7	4	1	5	2	6
7	4								1	5	2	6	3
4	1	5								2	6	3	7

Table 2.13: Color matrix for C_{14}^3.

1	5	2	6							8	3	7
5	2	6	3	7							8	4
2	6	3	7	4	1							8
6	3	7	4	1	5	2						
	7	4	1	5	2	6	3					
		1	5	2	6	3	7	4				
			2	6	3	7	4	1	5			
				3	7	4	1	5	2	6		
					4	1	5	2	6	3	7	
						5	2	6	3	7	4	1
8							6	3	7	4	1	5
3	8							7	4	1	5	2
7	4	8							1	5	2	6

Table 2.14: Color matrix of C_{13}^3, with new color 8 circled.

□

We observe the following:

The graph C_n^k satisfies TCC if $n = t(2k+1)+1$, where n, t, k are positive integers. The proof is as follows:

If t is odd then $n = t(2k+1)+1$ is even and we know that C_n^k satisfy TCC [5].

Suppose that t is even. As before, by using Theorem 2.6, C_{n-1}^k is $(2k+1)$-total colorable. We show that this coloring could be modified to a $(2k+2)$-total coloring for C_n^k. Here, we add a row and a column at the end of the color matrix of C_{n-1}^k. Further, delete the first k colors (from top) of the lower and upper $(k+1)^{\text{st}}$ and $(n-k)^{\text{th}}$ sub-diagonals of the original color matrix of C_{n-1}^k. Assign these colors to the n^{th}-row and n^{th}-column of the color matrix of C_n^k. Assign the new color $2k+2$ to the new vertex and also to the lower and upper $(k+1)^{\text{st}}$ sub-diagonals of the color matrix of C_n^k.

The entries of the lower $(k+1)^{\text{st}}$ and upper $(n-k)^{\text{th}}$ sub-diagonals of the color matrix are projected vertically in the corresponding cells of the last row. Similarly, the entries of the upper $(k+1)^{\text{st}}$ and lower $(n-k)^{\text{th}}$ sub-diagonals of the color matrix are projected horizontally in the corresponding cells of the last column.

1	5	2	8								○	7	4	6
5	2	6	3	8								○	1	7
2	6	3	7	4	8								○	1
8	3	7	4	1	5	2								
	8	4	1	5	2	6	3							
		8	5	2	6	3	7	4						
			2	6	3	7	4	1	5					
				3	7	4	1	5	2	6				
					4	1	5	2	6	3	7			
						5	2	6	3	7	4	1		
							6	3	7	4	1	5	2	
○								7	4	1	5	2	6	3
7	○								1	5	2	6	3	4
4	1	○								2	6	3	7	5
6	7	1									3	4	5	8

Table 2.15: Color matrix for C_{15}^3.

The idempotent, anti-circulant, commutative Latin square that we have chosen has a special property. Diagonal of the matrix obtained by cyclic left shifts of the columns of this Latin square has distinct colors (all $2k+1$) appearing on its main diagonal. We observe that by our construction, the k colors deleted in the lower and upper $(k+1)^{\text{st}}$ and

$(n-k)^{\text{th}}$ sub-diagonals are actually entries of the diagonal of a matrix after k left shifts of the columns. Hence there is no clash in the colors in the last row and last column of the color matrix C_n^k. The color matrix of C_{14}^3 in Table 2.13 can be modified by adding one row and column to give the color matrix of C_{15}^3 as shown in Table 2.15.

In Table 2.15, the circled numbers show the changes from the color matrix of C_{14}^3.

Theorem 2.10. *Let G be a Cayley graph on a nilpotent group Γ of even order n with maximum degree $\Delta(G) \geq n/2$ and the generating set S not containing any element of order two. If G is total colorable with x colors and if G' is also a Cayley graph on the same group with maximum degree $\Delta(G')$, $\Delta(G) \leq \Delta(G') \leq n-2$ formed with generating set $S' = S \cup S''$ such that S'' generates the whole group, then the graph G' is total colorable with $x + (\Delta(G') - \Delta(G))$ colors. In particular, if G is type-I (type-II, respectively), then G' is also type-I (satisfies TCC, respectively).*

Proof. Let s be an element of order two in Γ (which is guaranteed by Cauchy's theorem). Since the vertices of the graph G, say g_i, $i \in \{1, 2, \ldots, n\}$, can be arranged in $n/2$ independent color classes as $\{g_1, g_1 s\}, \{g_2, g_2 s\}, \ldots, \{g_{n/2}, g_{n/2} s\}$, we get an $n/2$ coloring of the vertices. Since $G' - E(G)$ is the Cayley graph of a nilpotent group, it is 1-factorable (see [15, Corollary 2.4.1]). Therefore, we only need $\Delta(G') - \Delta(G)$ extra colors to color the edges of $G' - E(G)$, thereby giving a total coloring of G' with $x + (\Delta(G') - \Delta(G))$ colors. Thus, if G is a type-I graph, then G' also would be type-I. When G is type-II, it may so happen that G' be type-I or type-II, nevertheless TCC holds for G'. □

For a positive integer $n > 1$, the *unitary Cayley graph* $X_n = \mathrm{Cay}(Z_n, U_n)$ is defined by the additive group of the ring Z_n of integer modulo n and the multiplicative group U_n of its units. If we represent the elements of Z_n by the integers $0, 1, \ldots, n-1$, then it is well known that

$$U_n = \{a \in Z_n : \gcd(a, n) = 1\}.$$

So X_n has the vertex set $V(X_n) = Z_n = \{0, 1, 2, \ldots, n-1\}$ and edge set

$$E(X_n) = \{(a, b) : a, b \in Z_n, \gcd(a-b, n) = 1\}.$$

Boggess et al. [3] studied the structure of unitary Cayley graphs where they have discussed the chromatic number, vertex and edge connectivities, planarity and crossing number. Klotz and Sander [12] have determined the clique number, the independence number and the diameters of unitary Cayley graphs. They have given a necessary and sufficient condition for the perfectness of X_n.

The graph X_n is regular of degree $|U_n| = \varphi(n)$, where $\varphi(n)$ denotes the Euler function. Let the prime factorization of n be $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ where $p_1 < p_2 < \cdots < p_t$. If $n = p$ is a prime number, then $X_n = K_p$ is the complete graph on p vertices. If $n = p^\alpha$ is a prime power then X_n is a complete p-partite graph. In the following theorem, we prove that TCC holds for unitary Cayley graphs.

Theorem 2.11. *A unitary Cayley graph X_n is $(\Delta(X_n)+2)$-total colorable.*

Proof. We know that a unitary Cayley graph can be obtained from a balanced r-partite graph (A complete multipartite graph is balanced if the partite sets all have the same cardinality) by deleting some edges. Suppose $n = p$ is a prime number, then X_n is the complete graph on p vertices. Also, if $n = p^{\alpha}$, a prime power, then X_n is a complete p-partite graph and TCC holds for these two graphs [17].

On the other hand, if $n = 2k$, $k \in N$, then the unitary Cayley graph X_n is a bipartite graph which is $(\Delta(X_n)+2)$-total colorable.

Suppose $n \not\equiv 0 \pmod 2$. Let $p_1 \mid n$ be the smallest odd prime. Since p_1 is the smallest prime, $kp_1, kp_1+1, \ldots, (k+1)p_1-1$, where $k = 0, 1, 2, \ldots, n/p_1 - 1$ induces n/p_1 vertex disjoint cliques each of order p_1. Since p_1 is odd, we can color all the vertices and edges of these n/p_1 cliques using p_1 colors [17]. Now remove the edges of these cliques. The remaining graph is a $(\varphi(n) - p_1 + 1)$-regular graph where the vertices are already colored. We color the edges of this resultant graph with $\varphi(n) - p_1 + 2$ colors. Thus, we only use $\varphi(n) + 2 = \Delta(X_n) + 2$ colors for the total coloring of X_n. This concludes the proof. □

3. Mock threshold graphs and odd graphs

A graph is weakly chordal if neither the graph nor the complement of the graph has an induced cycle on five or more vertices. A simple graph G on $[n] = \{1, 2, \ldots, n\}$ is threshold, if G can be built sequentially from the empty graph by adding vertices one at a time, where each new vertex is either isolated (nonadjacent to all the previous vertices) or dominant (connected to all the previous vertices). A graph G is said to be mock threshold if there is a vertex ordering $v_1, \ldots, v_n$ such that for every i $(1 \leq i \leq n)$ the degree of v_i $d(v_i)$ in the subgraph of G induced by the vertices $\{v_1, \ldots, v_i\}$, denoted by $G[v_1, \ldots, v_i]$, is $0, 1, i-2$, or $i-1$. Mock threshold graphs are a simple generalization of threshold graphs that, are perfect graphs. Mock threshold graphs are perfect and indeed weakly chordal but not necessarily chordal [1]. Similarly, the complement of a mock threshold graph is also mock threshold.

In the following, we prove that the TCC holds for mock threshold graphs.

Note. A total coloring of K_n can be constructed as follows: (This total coloring is due to Hinz and Parisse [10].)

When n is even, we first construct an edge coloring of K_n and extend it. We denote $[n]_0 = \{0, 1, 2, \ldots, n-1\}$. For $k \in [n]_0$, let τ_k be the transposition of k and $n-1$ on

$[n]_0$. For even n, $c_n(i,j) = (\tau_i(j) + \tau_j(i) + 2) \pmod{n+1}$, for $i, j \in [n]_0$, $i \neq j$, defines an $(n+1)$-edge coloring of K_n. In this coloring assignment, line $k \in [n]_0$ will have the missing colors k and $(k+1) \pmod{n}$. We color $c_n(i) = i$ for all $i \in [n]_0$.

When n is odd, we use the same coloring of K_{n-1}. In the coloring assignment of K_{n-1}, still the color $(k+1) \pmod{n}$ is missing in line $k \in [n]_0$. We use these colors to color the edges incident with the n^{th} vertex and color n to the n^{th} vertex.

Theorem 3.1. *Total coloring conjecture holds for mock threshold graphs.*

Proof. Consider the mock threshold graph G with vertex ordering $v_1, v_2, \ldots, v_i, \ldots, v_n$. We prove this theorem using induction on the order of the induced subgraph $G[v_1, v_2, \ldots, v_k]$. For $k \leq 4$, the maximum degree of all induced subgraphs are less than or equal to 3. We know that a graph with maximum degree less than or equal to 3 satisfies TCC [13].

Let us assume that $G[v_1, v_2, \ldots, v_k]$, $k \geq 4$ satisfies TCC.

Claim. The graph $G[v_1, v_2, \ldots, v_k, v_{k+1}]$ satisfies TCC.

The degree of the vertex v_{k+1} in $G[v_1, v_2, \ldots, v_{k+1}]$ can be $0, 1, k-1$ or k. We have the following cases.

Case 1*: $d(v_{k+1}) = 0$.* In this case the vertex v_{k+1} is an isolated vertex. From the induction assumption, it follows that $G[v_1, v_2, \ldots, v_k, v_{k+1}]$ satisfies TCC.

Case 2*: $d(v_{k+1}) = 1$.* In this case, the vertex v_{k+1} is adjacent to a vertex, say v_i, in $G[v_1, v_2, \ldots, v_k]$. Since $G[v_1, v_2, \ldots, v_k]$ is total colorable graph with at most $\Delta(G[v_1, v_2, \ldots, v_k]) + 2$ colors, at each vertex there will be at least one missing color. We assign this missing color to the edge (v_i, v_{k+1}), and for the vertex v_{k+1}, we assign a color of a vertex which is not adjacent to v_{k+1} and not the color of v_i. Therefore, $G[v_1, v_2, \ldots, v_{k+1}]$ satisfies TCC.

Case 3*: $d(v_{k+1}) = k-1$.* Assume that the vertex v_{k+1} is not adjacent to v_i and also assume that $\Delta(G[v_1, v_2, \ldots, v_{k+1}]) = k-1$. We consider the following two cases:

Subcase 1*: k is even.* Construct a complete graph induced by the vertices $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+1}$. Color this even complete graph using colors in the set $\{0, 1, \ldots, k\}$ as given previously [10]. In this coloring, there is one missing color at each of the vertices and they are distinct. Now, color the edges (v_i, v_j), $i \neq j$, $j = 1, 2, \ldots, k+1$, with the missing colors. Assign the color $k-1$ to the vertex v_i. To get a total coloring of $G[v_1, v_2, \ldots, v_k, v_{k+1}]$, we remove the added edges and there is no change in the maximum degree.

Subcase 2*: k is odd.* In this case $k+1$ is even, say $2p$. It is known that a graph of order $2p$ with maximum degree $2p-2$ satisfies TCC (see [7, 9]).

Case 4*: $d(v_{k+1}) = k$.* The maximum degree of $G[v_1, v_2, \ldots, v_k, v_{k+1}]$ is k. Construct a complete graph on the vertex set $\{v_1, v_2, \ldots, v_k, v_{k+1}\}$. We know that the complete graph satisfies TCC. After removing the added edges we get a total coloring of $G[v_1, v_2, \ldots, v_k, v_{k+1}]$. It follows that mock threshold graphs satisfy TCC. □

The Kneser graph $K(n,k)$ is the graph whose vertices correspond to the collection of k-element subsets of a set of n elements, and two vertices are adjacent if and only if the two corresponding sets are disjoint. An odd graph O_n is the Kneser graph $K(2n-1, n-1)$.

Theorem 3.2. *Odd graphs satisfy TCC.*

Proof. Consider a $(2n-1)$-element set X. Let O_n be an odd graph defined on X. Let x be any element of X. Then, among the vertices of O_n, exactly $\binom{2n-2}{n-2}$ vertices correspond to sets that contain x. Because all these sets contain x, they are not disjoint, and form an independent set of O_n. That is, O_n has $2n-1$ different independent sets of size $\binom{2n-2}{n-2}$. Further, every maximum independent set must have this form, so, O_n has exactly $2n-1$ maximum independent sets.

If I is a maximum independent set, formed by the sets that contain x, then the complement of I is the set of vertices that do not contain x. This complementary set induces a matching in G. Each vertex of the independent set is adjacent to n vertices of the matching, and each vertex of the matching is adjacent to $n-1$ vertices of the independent set [8].

Based on the decomposition, we give a total coloring of O_n in the following way:

Assign n colors to the edges between the vertices in the maximum independent set I and the vertices in the matching. Color the edges in matching and the vertices in I with a new color. Color one set of vertices in the matching with another new color and the second set of vertices with the missing colors at these vertices. This will give a total coloring of O_n using at most $n+2 = \Delta(O_n)+2$ colors. The result follows. □

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Shantharam Prajnanaswaroopa, Jayabalan Geetha, Kanagasabapathi Somasundaram
Department of Mathematics, Amrita School of Engineering-Coimbatore, Amrita Vishwa Vidyapeetham, India
E-mail addresses: `s_prajnanaswaroopa@cb.amrita.edu`, `j_geetha@cb.amrita.edu`, `s_sundaram@cb.amrita.edu`

Hung-Lin Fu
Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu 30010, Taiwan
E-mail address: `hlfu@math.nctu.edu.tw`

Narayanan Narayanan
Department of Mathematics, Indian Institute of Technology Madras, Chennai, India
E-mail address: `naru@iitm.ac.in`