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Error Analysis of an Alternating Direction Implicit Difference Method for 2D Subdiffusion Equation with Initial Singularity

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Abstract. The alternating direction implicit (ADI) scheme is used to numerically solve the 2D subdiffusion equation with initial singularity. The time derivative is defined by the commonly used Caputo fractional derivative, and discretised by the L1 scheme on nonuniform mesh. The finite difference method (FDM) is applied to spatial discretization. The local error analyses of fully discrete scheme under the L^2-norm and H^1-norm are strictly established. By selecting the milder grading parameter $r > 2 - \alpha$, the time convergence rate can reach $O(M^{-\min\{2-\alpha,2\alpha\}})$ in positive time. In order to verify the correctness of the theoretical analysis, some numerical results are presented.

1. Introduction

In this work, we study the following 2D subdiffusion equation:

$$\begin{cases} D_t^\alpha w(x,y,t) = \Delta w(x,y,t) + f(x,y,t), & (x,y) \in \Theta,\ 0 < t \le T, \\ w(x,y,0) = w_0(x,y), & (x,y) \in \Theta, \\ w(x,y,t) = 0, & (x,y) \in \partial\Theta,\ 0 < t \le T, \end{cases} \tag{1.1}$$

where $\alpha \in (0,1)$, $\Theta = (0,l) \times (0,l)$, f and w_0 are continuous functions. In (1.1), $D_t^\alpha w$ is the α-order Caputo fractional derivative of t, which is defined as

$$D_t^\alpha w(x,y,t) = \frac{1}{\Gamma(1-\alpha)} \int_{\sigma=0}^{t} (t-\sigma)^{-\alpha} \frac{\partial w(x,y,\sigma)}{\partial \sigma}\, d\sigma.$$

The time fractional subdiffusion equation has been extensively studied by scholars recently, see the latest survey article [9] and references therein. An ADI scheme was proposed for (1.1) by Sun and Zhang [14], but their analyses are based on the constraint

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that the solution is smooth enough in time direction. For time-fractional subdiffusion problem, as is known that, the solution is usually accompanied by some initial weak singularities [8]. For smooth solution, the truncation error of L1 scheme is uniformly bounded by $O(\tau^{2-\alpha})$ on the mesh $t_m = m\tau$, where τ is the step size. However for typical solution exhibiting a weak initial singularity, the truncation error of L1 scheme at $t = t_m$ depends on m (see, e.g., Lemma 3.6 in this paper), which complicates the analysis and one cannot use the usual error analysis designed for smooth solution. On the other hand, graded temporal mesh is employed to compensate for this initial singularity as one can at most get first order accuracy on uniform mesh [5]. For the 2D sub-diffusion equation with Neumann boundary condition, a compact ADI scheme was presented by Cheng et al. [3] under realistic assumption on the solution. The sharp error estimates for the compact ADI-L1 scheme of 2D time fractional integro-differential equations were studied by Wang et al. [13]. In [12], the α-robust H^1-norm convergence analysis of ADI scheme for problem (1.1) with initial singularity was presented. However, both of the above works are for the temporal global error estimates of the proposed schemes, and the schemes often have better convergence rates on positive time [5]. Wang and Chen [11] proved the time local convergence order of $O(M^{-\min\{1,2\alpha\}})$ for an ADI scheme on uniform mesh for problem (1.1), it is no better than 1 for all the values of α. Inspired by the work [7], where local convergence of L1 scheme on graded mesh is proved, in this paper our purpose is to gain local convergence of ADI-L1 scheme for problem (1.1) on graded mesh. In more detail, we apply the L1 scheme on graded mesh to compensate the initial weak singularity, and a fully discrete ADI finite difference scheme is constructed for (1.1). The local convergences of the fully discrete ADI-L1 scheme both in L^2-norm and H^1-norm are rigorously proved.

The rest of this article is as follows. To solve the problem (1.1), we construct a fully discrete ADI-L1 scheme in Section 2. In Section 3, we obtain the local L^2-norm convergence of the ADI-L1 scheme. Section 4 is dedicated to the convergence analysis of ADI-L1 scheme in H^1-norm. We give some numerical results illustrate the correctness of error analysis in Section 5. Section 6 is the conclusion.

Notation. C will be used to represent a general positive constant throughout the paper, and it is independent of the sizes of the mesh.

2. An ADI-L1 scheme

Before formally introducing the discrete scheme, we introduce some symbols to make the article more concise and clear. There are three positive integers N_1, N_2 and M. Let $x_i = ih_1$ and $y_j = jh_2$ for $0 \le i \le N_1$ and $0 \le j \le N_2$, where $h_1 = l/N_1$ and $h_2 = l/N_2$ are the step sizes of spatial mesh. Set $\Theta_h = \{(x_i, y_j) \mid 1 \le i \le N_1 - 1, 1 \le j \le N_2 - 1\}$ and $\partial\Theta_h = \{(x_i, y_j) \mid i = 0, N_1 \text{ or } j = 0, N_2\}$. Let $t_i = T(i/M)^r$, $0 \le i \le M$, where $r \ge 1$

is a parameter which can control the grid. Mark $\tau_m = t_m - t_{m-1}$ for $1 \le m \le M$, and set $\tau = t_1$ for simplicity of the notation, as we will frequently use t_1. Therefore, the grid is $\{(x_i, y_j, t_m) \mid 0 \le i \le N_1, 0 \le j \le N_2, 0 \le m \le M\}$. We use $\varpi_{i,j}^m$ to represent the value of the approximate solution at point (x_i, y_j, t_m). For $1 \le m \le M$, denote

$$\begin{aligned}
\delta_x \varpi_{i-\frac{1}{2},j}^m &= \frac{1}{h_1}\left(\varpi_{i,j}^m - \varpi_{i-1,j}^m\right) && \text{for } 1 \le i \le N_1,\ 0 \le j \le N_2,\\
\delta_x^2 \varpi_{i,j}^m &= \frac{1}{h_1}\left(\delta_x \varpi_{i+\frac{1}{2},j}^m - \delta_x \varpi_{i-\frac{1}{2},j}^m\right) && \text{for } 1 \le i \le N_1 - 1,\ 0 \le j \le N_2,\\
\delta_y \delta_x \varpi_{i-\frac{1}{2},j-\frac{1}{2}}^m &= \frac{1}{h_2}\left(\delta_x \varpi_{i-\frac{1}{2},j}^m - \delta_x \varpi_{i-\frac{1}{2},j-1}^m\right) && \text{for } 1 \le i \le N_1,\ 1 \le j \le N_2.
\end{aligned}$$

According to the same definition method, we can define the form of $\delta_y \varpi_{i,j-\frac{1}{2}}^m$, $\delta_y^2 \varpi_{i,j}^m$, and $\delta_x \delta_y \varpi_{i-\frac{1}{2},j-\frac{1}{2}}^m$.

We approximate Caputo derivative by the commonly used L1 scheme:

$$\begin{aligned}
D_t^\alpha w(x_i, y_j, t_m) &\approx \partial_\tau^\alpha w(x_i, y_j, t_m)\\
&:= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{m-1} \frac{w_{i,j}^{k+1} - w_{i,j}^k}{\tau_{k+1}} \int_{\sigma=t_k}^{t_{k+1}} (t_m - \sigma)^{-\alpha}\, d\sigma\\
&= d_{m,1} w_{i,j}^m - d_{m,m} w_{i,j}^0 - \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) w_{i,j}^{m-k},
\end{aligned}$$

where

$$d_{m,p} = \frac{(t_m - t_{m-p})^{1-\alpha} - (t_m - t_{m-p+1})^{1-\alpha}}{\tau_{m-p+1}\Gamma(2-\alpha)} \quad \text{for } 1 \le p \le m.$$

We use the discrete operator $\Delta_h := \delta_x^2 + \delta_y^2$ to approximate the Laplace operator Δ. The direct approximation of problem (1.1) is as follows:

$$\text{(2.1)} \qquad \begin{cases}
\partial_\tau^\alpha \varpi_{i,j}^m - \Delta_h \varpi_{i,j}^m = f(x_i, y_j, t_m) & \text{for } (x_i, y_j) \in \Theta_h,\ 1 \le m \le M,\\
\varpi_{i,j}^0 = w_0(x_i, y_j) & \text{for } (x_i, y_j) \in \Theta_h,\\
\varpi_{i,j}^m = 0 & \text{for } (x_i, y_j) \in \partial\Theta_h,\ 1 \le m \le M.
\end{cases}$$

For simplicity, let $\varrho_m := d_{m,1}^{-1} = \Gamma(2-\alpha)\tau_m^\alpha$, and it is obvious that its order is $O(M^{-\alpha})$. By adding $\varrho_m^2 \delta_x^2 \delta_y^2 \partial_\tau^\alpha \varpi_{i,j}^m$ to the left of the first relation in (2.1), we can obtain the ADI-L1 scheme:

$$\text{(2.2a)} \quad (1 + \varrho_m^2 \delta_x^2 \delta_y^2) \partial_\tau^\alpha \varpi_{i,j}^m - \Delta_h \varpi_{i,j}^m = f(x_i, y_j, t_m) \quad \text{for } (x_i, y_j) \in \Theta_h,\ 1 \le m \le M,$$

$$\text{(2.2b)} \quad \varpi_{i,j}^0 = w_0(x_i, y_j) \quad \text{for } (x_i, y_j) \in \Theta_h,$$

$$\text{(2.2c)} \quad \varpi_{i,j}^m = 0 \quad \text{for } (x_i, y_j) \in \partial\Theta_h,\ 1 \le m \le M.$$

The (2.2a) can be reformulated in the following expression:

$$\begin{aligned}&(1-\varrho_m\delta_x^2)(1-\varrho_m\delta_y^2)\varpi_{i,j}^m\\&=\varrho_m(1+\varrho_m^2\delta_x^2\delta_y^2)\left(\sum_{k=1}^{m-1}(d_{m,k}-d_{m,k+1})\varpi_{i,j}^k+d_{m,m}\varpi_{i,j}^0\right)+\varrho_m f_{i,j}^m.\end{aligned}$$

Let $W_{i,j}=(1-\varrho_m\delta_y^2)\varpi_{i,j}^m$ for $m=1,\dots,M$. To get the solution of the scheme (2.2), we can transform the spatial two-dimensional problem into two one-dimensional problems by setting the intermediate variable, and then get the numerical solution of the problem. First fix $j\in\{1,2,\dots,N_2-1\}$, for $1\le i\le N_1-1$ we solve

$$\begin{cases}(1-\varrho_m\delta_x^2)W_{i,j}\\\quad=\varrho_m(1+\varrho_m^2\delta_x^2\delta_y^2)\left(\sum_{k=1}^{m-1}(d_{m,k}-d_{m,k+1})\varpi_{i,j}^k+d_{m,m}\varpi_{i,j}^0\right)+\varrho_m f_{i,j}^m,\\W_{0,j}=(1-\varrho_m\delta_y^2)\varpi_{0,j}^m,\quad W_{N_1,j}=(1-\varrho_m\delta_y^2)\varpi_{N_1,j}^m.\end{cases}\tag{2.3}$$

Then fix $i\in\{1,2,\dots,N_1-1\}$, we solve

$$\begin{cases}(1-\varrho_m\delta_y^2)\varpi_{i,j}^m=W_{i,j}\quad\text{for }1\le j\le N_2-1,\\\varpi_{i,0}^m=0,\quad\varpi_{i,N_2}^m=0.\end{cases}\tag{2.4}$$

The two coefficient matrices of (2.3) and (2.4) are diagonally dominant matrices, so scheme (2.2) is uniquely solvable.

3. Local error convergence analysis in the sense of L^2-norm

For all $m\ge 1$ and a constant $\gamma\in R$, we set

$$\Upsilon_\gamma^m:=\begin{cases}\tau t_m^{\alpha-1}&\text{if }\gamma>0,\\\tau t_m^{\alpha-1}[1+\ln(t_m/\tau)]&\text{if }\gamma=0,\\\tau t_m^{\alpha-1}(\tau/t_m)^\gamma&\text{if }\gamma<0.\end{cases}$$

As [10], the positive multipliers are defined as follows: for $m=1,2,\dots,M$ and $k=1,2,\dots,m-1$,

$$\theta_{m,m}:=1,\quad\theta_{m,k}:=\sum_{p=1}^{m-k}\frac{1}{d_{m-p,1}}(d_{m,p}-d_{m,p+1})\theta_{m-p,k}.$$

Lemma 3.1. [2, Corollary 5.4] *For positive multipliers defined above, we have*

$$\varrho_m\sum_{k=1}^m\theta_{m,k}\le\frac{t_m^\alpha}{\Gamma(1+\alpha)}$$

for $m=1,\dots,M$.

Lemma 3.2. [1, Lemma 2.5] *For a grid function* $\{\Psi^m\}_{m=0}^M$*, let* $\Psi^0 = 0$ *and*

$$\Psi^m = \Gamma(2-\alpha)\tau_m^\alpha \sum_{k=1}^m \theta_{m,k}(\tau/t_k)^{\gamma+1} \quad \text{for } m = 1, 2, \ldots, M.$$

Then one has

$$\Psi^m \le C\Upsilon_\gamma^m \quad \text{for } m = 1, \ldots, M.$$

Lemma 3.3. [6, Lemma 4.2] *For a nonnegative grid function* $\{g^m\}_{m=0}^M$ *satisfying* $g_0 \ge 0$*, if* $\{\phi^m\}_{m=1}^\infty$*,* $\{\varphi^m\}_{m=1}^\infty$ *satisfy the following conditions:*

$$(\partial_\tau^\alpha g^m) g^m \le \phi^m g^m + (\varphi^m)^2 \quad \text{for } m = 1, 2, \ldots, M,$$

where $\{\phi^m\}_{m=1}^\infty$*,* $\{\varphi^m\}_{m=1}^\infty$ *are nonnegative, then we have*

$$g^m \le g^0 + \varrho_m \sum_{p=1}^m \theta_{m,p}(\phi^p + \varphi^p) + \max_{1\le p\le m}\{\varphi^p\} \quad \text{for } m = 1, 2, \ldots, M.$$

Set $\overline{\Theta}_h = \{(x_i, y_j) \mid 0 \le i \le N_1, 0 \le j \le N_2\}$ and $\partial\Theta_h = \overline{\Theta}_h \cap \partial\Theta$. Let $\Pi_h = \{\vartheta_{i,j} \mid \vartheta_{i,j} = 0 \text{ if } (x_i, y_j) \in \partial\Theta_h \text{ and } (x_i, y_j) \in \overline{\Theta}_h\}$. For any $w, v \in \Pi_h$, we define

$$(w, v) = h_1 h_2 \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1} w_{i,j} v_{i,j},$$

$$(w, v)_x = h_1 h_2 \sum_{i=1}^{N_1}\sum_{j=1}^{N_2-1} (\delta_x w_{i-\frac12,j})\delta_x v_{i-\frac12,j},$$

$$(w, v)_y = h_1 h_2 \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2} (\delta_y w_{i,j-\frac12})\delta_y v_{i,j-\frac12},$$

$$(w, v)_{xy} = h_1 h_2 \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} (\delta_x\delta_y w_{i-\frac12,j-\frac12})(\delta_x\delta_y v_{i-\frac12,j-\frac12}),$$

and the corresponding norm $\|w\| = (w, w)^{1/2}$, $\|w\|_x = (w, w)_x^{1/2}$, $\|w\|_y = (w, w)_y^{1/2}$, and $\|w\|_{xy} = (w, w)_{xy}^{1/2}$. One can easily verify that

$$(-\delta_x^2 w, v) := h_1 h_2 \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1} (-\delta_x^2 w_{i,j}) v_{i,j} = (w, v)_x, \tag{3.1}$$

$$(-\delta_y^2 w, v) := h_1 h_2 \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1} (-\delta_y^2 w_{i,j}) v_{i,j} = (w, v)_y, \tag{3.2}$$

$$(\delta_x^2\delta_y^2 w, v) := h_1 h_2 \sum_{i=1}^{N_1-1}\sum_{j=1}^{N_2-1} (\delta_x^2\delta_y^2 w_{i,j}) v_{i,j} = (w, v)_{xy}. \tag{3.3}$$

For two mesh functions $w, v \in \Pi_h$, let $(w, v)_{\varrho_m} := (w, v) + \varrho_m^2 (w, v)_{x,y}$, and $\|w\|_{\varrho_m} = \sqrt{(w, w)_{\varrho_m}}$.

Lemma 3.4. *The inner product of the above definition $(w,v)_{\varrho_m}$ has the following properties*

$$(w,v)_{\varrho_m} \le \|w\|_{\varrho_m}\|v\|_{\varrho_m}.$$

Proof. It is obvious that $(w,v)_{\varrho_m}$ satisfies the properties of inner product, and we can use Cauchy–Schwartz inequality to prove this lemma. □

Theorem 3.5. *For $m = 1,2,\dots,M$, the solution of scheme* (2.2) *satisfies*

$$\|\varpi^m\|_{\varrho_m} \le \|\varpi^0\|_{\varrho_m} + \varrho_m \sum_{p=1}^{m} \theta_{m,p}\|f^p\|.$$

Proof. Making L^2-norm inner product with ϖ^m on (2.2a), we can get

$$\begin{aligned}&\left((1+\varrho_m^2\delta_x^2\delta_y^2)\left(d_{m,1}\varpi^m - \sum_{k=1}^{m-1}(d_{m,k}-d_{m,k+1})\varpi^{m-k} - d_{m,m}\varpi^0\right), \varpi^m\right)\\ &= (\delta_x^2\varpi^m,\varpi^m) + (\delta_y^2\varpi^m,\varpi^m) + (f^m,\varpi^m).\end{aligned}$$

Using (3.1)–(3.3), Lemma 3.4, and applying Cauchy–Schwartz inequality, one has

$$\begin{aligned}&d_{m,1}\|\varpi^m\|_{\varrho_m}^2\\ &\le \sum_{k=1}^{m-1}(d_{m,k}-d_{m,k+1})\|\varpi^{m-k}\|_{\varrho_m}\|\varpi^m\|_{\varrho_m} + d_{m,m}\|\varpi^0\|_{\varrho_m}\|\varpi^m\|_{\varrho_m} + \|f^m\|\|\varpi^m\|\\ &\le \sum_{k=1}^{m-1}(d_{m,k}-d_{m,k+1})\|\varpi^{m-k}\|_{\varrho_m}\|\varpi^m\|_{\varrho_m} + d_{m,m}\|\varpi^0\|_{\varrho_m}\|\varpi^m\|_{\varrho_m} + \|f^m\|\|\varpi^m\|_{\varrho_m},\end{aligned}$$

which is equivalent to

$$(\partial_\tau^\alpha\|\varpi^m\|_{\varrho_m})\|\varpi^m\|_{\varrho_m} \le \|f^m\|\|\varpi^m\|_{\varrho_m}.$$

Then by Lemma 3.3, we obtain

$$\|\varpi^m\|_{\varrho_m} \le \|\varpi^0\|_{\varrho_m} + \varrho_m \sum_{p=1}^{m} \theta_{m,p}\|f^p\|. \qquad \square$$

Lemma 3.6. [7, Lemma 3.4] *Suppose $|\partial_t^p w(t)| \le C(1+t^{\alpha-p})$ for $p=0,1,2$, then*

$$\left|\partial_\tau^\alpha w(t_m) - D_t^\alpha w(t_m)\right| \le C(\tau/t_m)^{\min\{\alpha+1,(2-\alpha)/r\}} \quad \text{for } m=1,2,\dots,M.$$

Next we prove the convergence of ADI-L1 scheme, let $\epsilon_{i,j}^m := w(x_i,y_j,t_m) - \varpi_{ij}^m$. If $(x_i,y_j)\in\partial\Theta_h$, $\epsilon_{i,j}^m = 0$, for $1 < m \le M$. Denote $\epsilon_t w_{ij}^m := (1+\varrho_m^2\delta_x^2\delta_y^2)\partial_\tau^\alpha w_{ij}^m - D_t^\alpha w(x_i,y_j,t_m)$, $\epsilon_s w_{ij}^m := \Delta w(x_i,y_j,t_m) - \Delta_h w_{ij}^m$. Then we get the error equation: for $1\le m\le M$,

$$(I+\varrho_m^2\delta_x^2\delta_y^2)\partial_\tau^\alpha\epsilon_{ij}^m - \Delta_h\epsilon_{ij}^m = \epsilon_t w_{ij}^m + \epsilon_s w_{ij}^m. \tag{3.4}$$

Theorem 3.7. *Suppose for $p = 0, 1, 2$, the condition $|\partial_t^p w(t)| \le C(1 + t^{\alpha-p})$ holds, and for each $t, w \in C^4(\overline{\Theta})$. Then for $m = 1, 2, \ldots, M$,*

$$\|\epsilon^m\|_{\varrho_m} \le C\big(h_1^2 + h_2^2 + \mathcal{E}^m + M^{-2\alpha}\big),$$

where $\mathcal{E}^m$ is defined by

$$\mathcal{E}^m := \begin{cases} M^{-r} t_m^{\alpha-1} & \text{if } 1 \le r < 2-\alpha, \\ M^{\alpha-2} t_m^{\alpha-1}[1 + \ln(t_m/\tau)] & \text{if } r = 2-\alpha, \\ M^{\alpha-2} t_m^{\alpha-\frac{2-\alpha}{r}} & \text{if } r \ge 2-\alpha. \end{cases} \tag{3.5}$$

Proof. Denote $\gamma := \min\{\alpha, (2-\alpha)/r - 1\}$. Noting $\epsilon_{i,j}^m = 0$, applying Theorem 3.5 to (3.4) gives

$$\begin{aligned}\|\epsilon^m\|_{\varrho_m} &\le \varrho_m \sum_{p=1}^{m} \theta_{m,p} \|\epsilon_t w^p + \epsilon_s w^p\| \le C\left(\varrho_m \sum_{p=1}^{m} \theta_{m,p}\big((\tau/t_p)^{\gamma+1} + M^{-2\alpha} + h_1^2 + h_2^2\big)\right) \\ &\le C\big(\Upsilon_\gamma^m + M^{-2\alpha} + h_1^2 + h_2^2\big),\end{aligned}$$

where we have used Lemmas 3.1 and 3.2 to obtain the last inequality. It remains to prove that $\Upsilon_\gamma^m \le C\mathcal{E}^m$. This will be three cases:

1. If $1 \le r < 2-\alpha$, i.e., $\gamma > 0$. As $\tau \le CM^{-r}$, one has $\Upsilon_\gamma^m \le CM^{-r}t_m^{\alpha-1}$.
2. If $r = 2-\alpha$, i.e., $\gamma = 0$. By $\tau \le CM^{\alpha-2} = CM^{-r}$, one gets $\Upsilon_\gamma^m \le CM^{\alpha-2}t_m^{\alpha-1}[1 + \ln(t_m/\tau)]$.
3. If $r > 2-\alpha$, i.e., $\gamma < 0$. Thus $\Upsilon_\gamma^m \le C\tau t_m^{\alpha-1}(\tau/t_m)^{\frac{2-\alpha}{r}-1} = C\tau^{\frac{2-\alpha}{r}} t_m^{\alpha-\frac{2-\alpha}{r}} \le CM^{\alpha-2}t_m^{\alpha-\frac{2-\alpha}{r}}$.

The proof is completed. □

Remark 3.8 (Local convergence). For positive time t_m which is away from 0, if $1 \le r \le (2-\alpha)$, $\mathcal{E}^m \simeq M^{-r}$, while if $r > 2-\alpha$, $\mathcal{E}^m \simeq M^{-(2-\alpha)}$, thus the temporal local convergence rate in Theorem 3.7 is $O(M^{-\min\{1,2\alpha\}})$ if $r = 1$, and $O(M^{-\min\{2-\alpha,2\alpha\}})$ if $r > \min\{2-\alpha, 2\alpha\}$.

4. Local error convergence analysis in the sense of H^1-norm

Same as the Section 3, our main work in this section is to analyze the convergence of the local H^1 error of the discrete scheme. For any grid function $w \in \Pi_h$, we define

$$\begin{aligned}\|\nabla_h w^m\| &= \big(\|\delta_x w^m\|^2 + \|\delta_y w^m\|^2\big)^{1/2}, \\ \|w^m\|_{H^1} &= \big(\|w^m\|^2 + \|\nabla_h w^m\|^2\big)^{1/2}, \\ \|w^m\|_{1,\varrho} &= \big(\|\nabla_h w^m\|^2 + \varrho_m^2\big(\|\delta_x\delta_y^2 w^m\|^2 + \|\delta_y\delta_x^2 w^m\|^2\big)\big)^{1/2}.\end{aligned}$$

Lemma 4.1. [12, Lemma 3.2] *For any $w, v \in \Pi_h$, we have*

$$\big(-(w^m + \varrho_m^2 \delta_x^2 \delta_y^2 w^m), \Delta_h v^m \big) \leq \|w^m\|_{1,\varrho} \|v^m\|_{1,\varrho},$$

when $w = v$, the equal sign in the above inequality can be taken.

Theorem 4.2. *For $m = 1, 2, \dots, M$, the solution of ADI-L1 scheme (2.2) satisfies*

$$\big(\partial_\tau^\alpha \|\varpi^m\|_{1,\varrho}\big) \|\varpi^m\|_{1,\varrho} \leq \frac{1}{4} \|f^m\|^2.$$

Proof. Making discrete inner product with $-\Delta_h \varpi^m$ on (2.2a), we have

$$\left((1 + \varrho_m^2 \delta_x^2 \delta_y^2) \left(d_{m,1} \varpi^m - \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \varpi^{m-k} - d_{m,m} \varpi^0 \right), -\Delta_h \varpi^m \right) + \|\Delta_h \varpi^m\|^2$$
$$= (f^m, -\Delta_h \varpi^m).$$

By Lemma 4.1, one gets

$$\begin{aligned} & d_{m,1} \|\varpi^m\|_{1,\varrho}^2 + \|\Delta_h \varpi^m\|^2 \\ \leq & \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \|\varpi^{m-k}\|_{1,\varrho} \|\varpi^m\|_{1,\varrho} + d_{m,m} \|\varpi^0\|_{1,\varrho} \|\varpi^m\|_{1,\varrho} + (f^m, -\Delta_h \varpi^m). \end{aligned}$$

Applying Young's inequality and Cauchy–Schwartz inequality, we can get

$$(f^m, -\Delta_h \varpi^m) \leq \|f^m\| \|\Delta_h \varpi^m\| \leq \frac{1}{4} \|f^m\|^2 + \|\Delta_h \varpi^m\|^2.$$

Then we have

$$d_{m,1} \|\varpi^m\|_{1,\varrho}^2 \leq \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \|\varpi^{m-k}\|_{1,\varrho} \|\varpi^m\|_{1,\varrho} + d_{m,m} \|\varpi^0\|_{1,\varrho} \|\varpi^m\|_{1,\varrho} + \frac{1}{4} \|f^m\|^2,$$

that is

$$\big(\partial_\tau^\alpha \|\varpi^m\|_{1,\varrho}\big) \|\varpi^m\|_{1,\varrho} \leq \frac{1}{4} \|f^m\|^2.$$

Here, the proof of the theorem is completed. □

Theorem 4.3. *Assume that $|\partial_t^p w(t)| \leq C(1 + t^{\alpha - p})$ for $p = 0, 1, 2$, and $w \in C^4(\overline{\Theta})$ for each t. For $m = 1, 2, \dots, M$, one has*

$$\|\epsilon^m\|_{1,\varrho} \leq C\big(h_1^2 + h_2^2 + \mathcal{E}^m + M^{-2\alpha}\big),$$

where $\mathcal{E}^m$ is defined in (3.5).

Proof. On both sides of (3.4), taking discrete L^2 inner product with $-\Delta_h \epsilon^m$, we obtain

$$
\begin{aligned}
& d_{m,1}\|\epsilon^m\|_{1,\varrho}^2 + \|\Delta_h \epsilon^m\|^2 \\
\le & \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \|\epsilon^{m-k}\|_{1,\varrho} \|\epsilon^m\|_{1,\varrho} + (\epsilon_t w^m + \epsilon_s w^m, -\Delta_h \epsilon^m) \\
= & \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \|\epsilon^{m-k}\|_{1,\varrho} \|\epsilon^m\|_{1,\varrho} + (\nabla_h \epsilon_t w^m, \nabla_h \epsilon^m) - (\epsilon_s w^m, \Delta_h \epsilon^m) \\
\le & \sum_{k=1}^{m-1} (d_{m,k} - d_{m,k+1}) \|\epsilon^{m-k}\|_{1,\varrho} \|\epsilon^m\|_{1,\varrho} + \|\nabla_h \epsilon_t w^m\| \|\epsilon^m\|_{1,\varrho} + \frac{1}{4}\|\epsilon_s w^m\|^2 + \|\Delta_h \epsilon^m\|^2,
\end{aligned}
$$

which can be rewritten as

$$
\left(\partial_\tau^\alpha \|\epsilon^m\|_{1,\varrho}\right) \|\epsilon^m\|_{1,\varrho} \le \|\nabla_h \epsilon_t w^m\| \|\epsilon^m\|_{1,\varrho} + \frac{1}{4}\|\epsilon_s w^m\|^2 .
$$

Denote $\gamma := \min\left\{\alpha, \frac{2-\alpha}{r} - 1\right\}$. By Lemma 3.3, we get

$$
\begin{aligned}
\|\epsilon^m\|_{1,\varrho} &\le \varrho_m \sum_{p=1}^{m} \theta_{m,p} \left(\|\nabla_h \epsilon_t w^p\| + \frac{1}{2}\|\epsilon_s w^p\| \right) + \frac{1}{2} \max_{1\le p\le m} \{\|\epsilon_s w^p\|\} \\
&\le C\varrho_m \sum_{p=1}^{m} \theta_{m,p} \left((\tau/t_p)^{\gamma+1} + M^{-2\alpha} + h_1^2 + h_2^2 \right) + C(h_1^2 + h_2^2) \\
&\le C\left(\Upsilon_\gamma^m + M^{-2\alpha} + h_1^2 + h_2^2\right),
\end{aligned}
$$

where we have used Lemmas 3.1 and 3.2 to obtain the last inequality. The proof is completed by noting that $\Upsilon_\gamma^m \le C\mathcal{E}^m$ which was proved in the proof of Theorem 3.7. □

5. Numerical experiments

Example 5.1. In (1.1), take $l = \pi$, $T = 1$, $f = \Gamma(1+\alpha)\sin x \sin y + 2t^\alpha \sin x \sin y$ and $w_0(x, y) = 0$. Moreover, $w(x, y, t) = t^\alpha \sin x \sin y$ is the exact solution.

Denote $\|w(t^M) - \varpi^M\|$ as the L^2 local error when $t = t_M$, and $\|w(t^M) - \varpi^M\|_{H^1}$ as the local H^1-norm error when $t = t_M$. In numerical experiments, we take $M = 2N_1 = 2N_2$ such that the error in time direction is dominant. Tables 5.1–5.6 present the local errors and convergence orders in the sense of L^2-norm and H^1-norm under different grading parameters r. From these data, we can see if $r = 1$, the local convergence order is $O(M^{-\min\{1,2\alpha\}})$, and we can attain $O(M^{-\min\{2-\alpha,2\alpha\}})$ by selecting milder grading parameters r, which is consistent with Remark 3.8.

Table 5.1: Example 5.1 local L^2-norm error when $r = 1$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.0619e-3		5.0977e-3		2.2769e-2	
16	9.6441e-4	0.1389	2.9001e-3	0.8138	1.4135e-2	0.6878
32	5.6231e-4	0.7783	1.4520e-3	0.9981	8.3622e-3	0.7573
64	2.9188e-4	0.9460	6.8971e-4	1.0739	4.8540e-3	0.7847
128	1.4553e-4	1.0041	3.1997e-4	1.1080	2.7959e-3	0.7958

Table 5.2: Example 5.1 local L^2-norm error when $r = 2 - \alpha$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.1606e-3		6.7663e-3		3.0825e-2	
16	8.9805e-4	0.3701	3.5498e-3	0.9306	1.8986e-2	0.6992
32	4.7998e-4	0.9038	1.6883e-3	1.0722	1.1230e-2	0.7575
64	2.2957e-4	1.0641	7.6773e-4	1.1369	6.5337e-3	0.7814
128	1.0510e-4	1.1272	3.4125e-4	1.1698	3.7741e-3	0.7918

Table 5.3: Example 5.1 local L^2-norm error when $r = 2$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	3.3358e-3		1.0589e-2		3.6399e-2	
16	1.5970e-3	1.0627	5.2466e-3	1.0132	2.2339e-2	0.7043
32	6.8957e-4	1.2116	2.4300e-3	1.1104	1.3206e-2	0.7584
64	2.8563e-4	1.2715	1.0900e-3	1.1566	7.6842e-3	0.7812
128	1.1646e-4	1.2943	4.8108e-4	1.1800	4.4401e-3	0.7913

Table 5.4: Example 5.1 local H^1-norm error when $r = 1$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.0970e-3		5.2665e-3		2.3524e-2	
16	9.7307e-4	0.1730	2.9261e-3	0.8479	1.4262e-2	0.7219
32	5.6362e-4	0.7878	1.4553e-3	1.0076	8.3817e-3	0.7669
64	2.9205e-4	0.9485	6.9012e-4	1.0764	4.8569e-3	0.7872
128	1.4555e-4	1.0047	3.2002e-4	1.1087	2.7964e-3	0.7965

Table 5.5: Example 5.1 local H^1-norm error when $r = 2 - \alpha$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.1991e-3		6.9904e-3	3.1846e-2		
16	9.0612e-4	0.4042	3.5817e-3	0.9647	1.9156e-2	0.7333
32	4.8110e-4	0.9134	1.6922e-3	1.0818	1.1257e-2	0.7670
64	2.2970e-4	1.0666	7.6818e-4	1.1394	6.5375e-3	0.7840
128	1.0511e-4	1.1279	3.4130e-4	1.1704	3.7746e-3	0.7924

Table 5.6: Example 5.1 local H^1-norm error when $r = 2$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	3.4462e-3		1.0940e-2		3.7604e-2	
16	1.6113e-3	1.0968	5.2937e-3	1.0473	2.2540e-2	0.7384
32	6.9117e-4	1.2211	2.4356e-3	1.1200	1.3237e-2	0.7679
64	2.8580e-4	1.2740	1.0907e-3	1.1591	7.6888e-3	0.7837
128	1.1648e-4	1.2950	4.8115e-4	1.1807	4.4408e-3	0.7919

Example 5.2. In problem (1.1), take $T = 1$, $l = \pi$, $w_0 = \sin x \sin y$, and $f = t^2$.

Different from Example 5.1, the analytical solution of Example 5.2 is unknown, we employ the two-mesh method in [4] to compute the error. Let ϖ_h^m be the solution of our scheme (2.2), where $0 \le i \le N_1$, $0 \le j \le N_2$, and $0 \le m \le M$. Next, we obtain a second mesh by

$$t_m = T\left(\frac{m}{2M}\right)^r \text{ for } 0 \le M \le 2M, \quad N_1' = 2N_1, \quad N_2' = 2N_2.$$

On the second mesh we get the numerical solution $\overline{\varpi}_h^m$, where $0 \le i \le N_1'$, $0 \le j \le N_2'$, $0 \le m \le 2M$. We then define

$$E_h^M = \left\|\varpi_h^M - \overline{\varpi}_h^{2M}\right\|.$$

Here $\|\cdot\|$ represents the two error norms studied in this paper. The convergence order is computed by

$$\log_2 \frac{E_h^M}{E_h^{2M}}.$$

We take $M = 2N_1 = 2N_2$ such that the error in time direction is dominant. Tables 5.7–5.12 shows the L^2 and H^1 local differences and convergence orders of the Example 5.2 using different grading parameters r, which conform with Remark 3.8.

Table 5.7: Example 5.2 local L^2-norm error when $r = 1$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.5397e-2		8.7923e-3		1.3278e-2	
16	8.7493e-3	0.8154	5.1096e-3	7.8304	6.6887e-3	0.9898
32	4.6814e-3	0.9022	2.4213e-3	1.0774	3.5064e-3	0.9317
64	2.3767e-3	0.9780	1.1891e-3	1.0260	2.0342e-3	0.7856
128	1.1623e-3	1.0319	5.7834e-4	1.0398	1.1987e-3	0.7629

Table 5.8: Example 5.2 local L^2-norm error when $r = 2 - \alpha$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	1.5841e-2		9.4798e-3		1.4145e-2	
16	8.9291e-3	0.8271	5.7728e-3	0.7156	7.9401e-3	0.8331
32	4.6646e-3	0.9367	2.5915e-3	1.1555	4.3763e-3	0.8594
64	2.3033e-3	1.0181	1.1465e-3	1.1766	2.5909e-3	0.7563
128	1.0919e-3	1.0769	4.9941e-4	1.1989	1.5628e-3	0.7294

Table 5.9: Example 5.2 local L^2-norm error when $r = 2$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	2.2732e-2		9.8769e-3		1.3659e-2	
16	1.2585e-2	0.8530	6.8302e-3	0.5321	8.3271e-3	0.7140
32	6.3801e-3	0.9801	3.2965e-3	1.0510	4.8445e-3	0.7815
64	3.0432e-3	1.0680	1.4999e-3	1.1361	2.9455e-3	0.7178
128	1.3995e-3	1.1207	6.6132e-4	1.1815	1.8048e-3	0.7067

Table 5.10: Example 5.2 local H^1-norm error when $r = 1$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	2.5300e-2		1.7821e-2		2.0590e-2	
16	1.0562e-2	1.2602	7.3169e-3	1.2843	9.1840e-3	1.1647
32	4.9178e-3	1.1028	2.7781e-3	1.3971	4.0817e-3	1.1700
64	2.4035e-3	1.0329	1.2361e-3	1.1683	2.1498e-3	0.9250
128	1.1654e-3	1.0443	5.8413e-4	1.0814	1.2214e-3	0.8156

Table 5.11: Example 5.2 local H^1-norm error when $r = 2 - \alpha$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	2.6082e-2		1.8776e-2		2.1805e-2	
16	1.0830e-2	1.2680	8.4225e-3	1.1566	1.0660e-2	1.0323
32	4.9156e-3	1.1396	3.0502e-3	1.4653	5.0400e-3	1.0807
64	2.3307e-3	1.0766	1.2167e-3	1.3259	2.7370e-3	0.8808
128	1.0949e-3	1.0900	5.0966e-4	1.2554	1.5922e-3	0.7815

Table 5.12: Example 5.2 local H^1-norm error when $r = 2$.

$M \setminus \alpha$	0.8		0.6		0.4	
	Error	Rate	Error	Rate	Error	Rate
8	3.5123e-2		1.8959e-2		2.1042e-2	
16	1.5076e-2	1.2202	9.7744e-3	0.9558	1.1224e-2	0.9067
32	6.7290e-3	1.1638	3.8612e-3	1.3400	5.5665e-3	1.0117
64	3.0825e-3	1.1263	1.5927e-3	1.2776	3.1064e-3	0.8415
128	1.4037e-3	1.1349	6.7588e-4	1.2366	1.8378e-3	0.7572

6. Conclusion

In this work, we have constructed a fully discrete ADI-L1 scheme for 2D subdiffusion equation with initial singularity. Error analysis of the proposed fully discrete scheme in the sense of L^2 and H^1-norms are strictly proved. The temporal convergence rate can attain $O(M^{-\min\{2-\alpha,2\alpha\}})$ on positive time by selecting milder grading parameter $r > 2-\alpha$. Finally, the correctness of the theoretical analysis is supported by numerical examples.

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