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Maximal regularity for the heat equation with various boundary conditions in an infinite layer

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Abstract. This paper treats resolvent L_q estimate and maximal L_p-L_q regularity for the heat equation with various boundary conditions in an infinite layer. We need to consider two boundary conditions on upper boundary and lower boundary. We are able to choose any pair of Dirichlet, Neumann and Robin boundary conditions. We construct the solutions of Fourier multiplier operators and we use a theorem for an integral operator, which derives L_q-boundedness and L_p-L_q boundedness. The key is that the holomorphic symbols can be properly estimated from above.

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§1. Introduction and Main theorems

This paper is concerned with the linear heat equations with various boundary conditions in an infinite layer. Since the layer has two boundaries, we need to consider various types of the conditions. We give the solutions for any pair of boundary conditions of Dirichlet, Neumann and Robin. Let $0 < \delta < \infty$, $\Omega := \mathbb{R}^{n-1} \times (0, \delta)$, $\Gamma_\delta := \mathbb{R}^{n-1} \times \{\delta\}$, $\Gamma_0 := \mathbb{R}^{n-1} \times \{0\}$. We use $\alpha_i, \beta_i \geq 0$ with $(\alpha_i, \beta_i) \neq (0, 0)$ $(i = 1, 2)$ to distinguish the boundary conditions. The equations are as follows;

$$
\begin{cases}
\partial_t u - \Delta u = f & \text{in } \Omega,\ t > 0, \\
\alpha_1 u + \beta_1 \partial_\nu u = g & \text{on } \Gamma_0,\ t > 0, \\
\alpha_2 u + \beta_2 \partial_\nu u = h & \text{on } \Gamma_\delta,\ t > 0, \\
u|_{t=0} = u_0 & \text{in } \Omega.
\end{cases} \tag{1.1}
$$

The function u is unknown, while f, g, h, u_0 are given functions. We use the notation $\partial_\nu := \frac{\partial}{\partial \nu} = \nu \cdot \nabla$ with outward unit normal vector ν. The case

$(\beta_1, \beta_2) = (0,0)$ implies Dirichlet boundary condition, and $(\alpha_1, \alpha_2) = (0,0)$ implies Neumann boundary condition. However, we emphasize that we are able to select various cases since $\alpha_i, \beta_i \geq 0$ with $(\alpha_i, \beta_i) \neq (0,0)$ $(i = 1,2)$.

To solve the heat equation (1.1), we divide the problem into the resolvent problem and the problem with trivial initial data, i.e., we regard $(f, g, h, u_0) = (0, 0, 0, u_0) + (f, g, h, 0)$. The first one is treated by the semigroup approach. To do so we prove the resolvent estimate. The second one is treated by the maximal regularity approach, where the time interval is $\mathbb{R}$. Mainly we tackle the following resolvent problem;

$$\begin{cases} \lambda u - \Delta u = f & \text{in } \Omega, \\ \alpha_1 u + \beta_1 \partial_\nu u = g & \text{on } \Gamma_0, \\ \alpha_2 u + \beta_2 \partial_\nu u = h & \text{on } \Gamma_\delta. \end{cases} \tag{1.2}$$

Since the problem (1.2) is Laplace-transformed equation of (1.1), two solutions are almost same. Resolvent estimate with the case $g = h = 0$ implies the generation of analytic semigroup. Moreover we can see maximal regularity estimate for the problem (1.1) with $t \in \mathbb{R}$.

To state resolvent estimate, we introduce some notations. Given a domain D, Lebesgue and Sobolev spaces are denoted by $L_q(D)$ and $W_q^m(D)$ with norms $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$. The same manner is applied in the X-valued spaces $L_p(\mathbb{R}, X)$ and $W_p^m(\mathbb{R}, X)$. For a scalar function f, we use the following symbols;

$$\nabla f = (\partial_1 f, \ldots, \partial_n f), \quad \nabla^2 f = (\partial_i \partial_j f \mid i, j = 1, \ldots, n).$$

Even though $\mathbf{g} = (g_1, \ldots, g_{\tilde{n}}) \in X^{\tilde{n}}$ for some $\tilde{n}$, we use the notations $\mathbf{g} \in X$ and $\|\mathbf{g}\|_X$ as $\sum_{j=1}^{\tilde{n}} \|g_j\|_X$ for simplicity.

Let $\Sigma_{\varepsilon,\gamma} := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \varepsilon, |\lambda| \geq \gamma\}$ and $\Sigma_\varepsilon := \Sigma_{\varepsilon,0}$. Throughout this paper, we use generic constants c and C which may different from line to line.

The following is the generalized resolvent estimate.

Theorem 1.1 (Resolvent estimate)**.** *Let* $1 < q < \infty, \gamma_0 > 0, 0 < \varepsilon < \pi/2$. *For any* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$,

$$f \in L_q(\Omega), \quad g \in \begin{cases} W_q^2(\Omega) \text{ if } \beta_1 = 0, \\ W_q^1(\Omega) \text{ if } \beta_1 > 0, \end{cases} \qquad h \in \begin{cases} W_q^2(\Omega) \text{ if } \beta_2 = 0, \\ W_q^1(\Omega) \text{ if } \beta_2 > 0, \end{cases}$$

the problem (1.2) *has a unique solution* $u \in W_q^2(\Omega)$ *with the resolvent estimate*

$$\|(\lambda u, \lambda^{1/2} \nabla u, \nabla^2 u)\|_{L_q(\Omega)} \leq C_{n,q,\delta,\varepsilon,\gamma_0} (\|f\|_{L_q(\Omega)} + \mathrm{I}_{\beta_1}(g) + \mathrm{II}_{\beta_2}(h))$$

with

$$
\mathrm{I}_{\beta_1}(g) := \begin{cases} \|(\lambda g, \lambda^{1/2}\nabla g, \nabla^2 g)\|_{L_q(\Omega)} & \text{if } \beta_1 = 0, \\ \|(\lambda^{1/2} g, \nabla g)\|_{L_q(\Omega)} & \text{if } \beta_1 > 0, \end{cases}
$$

$$
\mathrm{II}_{\beta_2}(h) := \begin{cases} \|(\lambda h, \lambda^{1/2}\nabla h, \nabla^2 h)\|_{L_q(\Omega)} & \text{if } \beta_2 = 0, \\ \|(\lambda^{1/2} h, \nabla h)\|_{L_q(\Omega)} & \text{if } \beta_2 > 0. \end{cases}
$$

Remark 1.2. Since we give functions g and h as boundary data, only $g(x', 0)$ and $h(x', \delta)$ are important. We have to take $\gamma_0 > 0$. However, this is the reason why we use cut-off technique as we can see in the proof. We can take $\gamma_0 = 0$ by assuming $g(x', \delta) = 0$ and $h(x', 0) = 0$. This assumption is not essential. So, we can use this boundary conditions $g(x', \delta) = 0$ and $h(x', 0) = 0$. This remark is also applied to the following maximal regularity theorem.

By $g = h = 0$, we have the generation of analytic semigroup.

Corollary 1.3. *Let $1 < q < \infty$ and $\alpha_i, \beta_i \geq 0$ with $(\alpha_i, \beta_i) \neq (0, 0)$ $(i = 1, 2)$. Then the operator A on $L_q(\Omega)$ defined by*

$$
\begin{aligned}
D(A) &:= \{u \in W_q^2(\Omega) \mid \alpha_1 u + \beta_1 \partial_\nu u = 0 \text{ on } \Gamma_0, \alpha_2 u + \beta_2 \partial_\nu u = 0 \text{ on } \Gamma_\delta\}, \\
Au &:= \Delta u
\end{aligned}
$$

generates a bounded analytic semigroup e^{tA}.

The other main theorem is maximal L_p-L_q regularity estimate. Although we usually consider time interval $\mathbb{R}_+$ for initial value problems, we regard functions on $\mathbb{R}$ to use Fourier transform. To do so and to consider Laplace transforms as Fourier transforms, we introduce some function spaces;

$$
\begin{aligned}
L_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbb{R}, X),\ f(t) = 0 \text{ for } t < 0\}, \\
W^m_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f \in L_{p,0,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t} \partial_t^j f(t) \in L_p(\mathbb{R}, X),\ j = 1, \dots, m\},
\end{aligned}
$$

for some $\gamma_0 \geq 0$. Let $\mathcal{L}$ and $\mathcal{L}_\lambda^{-1}$ denote two-sided Laplace transform and its inverse, defined as

$$
\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) dt, \quad \mathcal{L}_\lambda^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\lambda) d\tau,
$$

where $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s \geq 0$ and X-valued function f, we use the following Bessel potential spaces to treat fractional orders;

$$
\begin{aligned}
H^s_{p,0,\gamma_0}(\mathbb{R}, X) := \{f : \mathbb{R} \to X \mid \Lambda^s_\gamma f := \mathcal{L}_\lambda^{-1}[|\lambda|^s \mathcal{L}[f](\lambda)](t) \in L_{p,0,\gamma}(\mathbb{R}, X) \\
\text{for any } \gamma \geq \gamma_0\}.
\end{aligned}
$$

We consider the heat equation on $t \in \mathbb{R}$.

$$\begin{cases} \partial_t U - \Delta U = F & \text{in } \Omega,\ t \in \mathbb{R}, \\ \alpha_1 U + \beta_1 \partial_\nu U = G & \text{on } \Gamma_0,\ t \in \mathbb{R}, \\ \alpha_2 U + \beta_2 \partial_\nu U = H & \text{on } \Gamma_\delta,\ t \in \mathbb{R}. \end{cases} \tag{1.3}$$

Theorem 1.4 (Maximal regularity)**.** *Let $1 < p, q < \infty, \gamma_0 > 0$. For any*

$$\begin{aligned} &F \in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)), \\ &G \in \begin{cases} W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\Omega)) & \text{if } \beta_1 = 0, \\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\Omega)) & \text{if } \beta_1 > 0, \end{cases} \\ &H \in \begin{cases} W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\Omega)) & \text{if } \beta_2 = 0, \\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\Omega)) & \text{if } \beta_2 > 0 \end{cases} \end{aligned}$$

the equation (1.3) *with initial value $U|_{t=0} = 0$ has a unique solution*

$$U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\Omega))$$

with maximal regularity

$$\begin{aligned} &\|e^{-\gamma t}(\partial_t U, \gamma U, \Lambda_\gamma^{1/2} \nabla U, \nabla^2 U)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ \le & C_{n,\delta,p,q,\gamma_0} (\|e^{-\gamma t} F\|_{L_p(\mathbb{R}, L_q(\Omega))} + \hat{\mathrm{I}}_{\beta_1}(G) + \hat{\mathrm{II}}_{\beta_2}(H)) \end{aligned}$$

for $\gamma \ge \gamma_0$, where

$$\begin{aligned} \hat{\mathrm{I}}_{\beta_1}(G) &:= \begin{cases} \|e^{-\gamma t}(\partial_t G, \Lambda_\gamma^{1/2} \nabla G, \nabla^2 G)\|_{L_p(\mathbb{R}, L_q(\Omega))} & \text{if } \beta_1 = 0, \\ \|e^{-\gamma t}(\Lambda_\gamma^{1/2} G, \nabla G)\|_{L_p(\mathbb{R}, L_q(\Omega))} & \text{if } \beta_1 > 0, \end{cases} \\ \hat{\mathrm{II}}_{\beta_2}(H) &:= \begin{cases} \|e^{-\gamma t}(\partial_t H, \Lambda_\gamma^{1/2} \nabla H, \nabla^2 H)\|_{L_p(\mathbb{R}, L_q(\Omega))} & \text{if } \beta_2 = 0, \\ \|e^{-\gamma t}(\Lambda_\gamma^{1/2} H, \nabla H)\|_{L_p(\mathbb{R}, L_q(\Omega))} & \text{if } \beta_2 > 0. \end{cases} \end{aligned}$$

Let us review relevant works for resolvent estimate and maximal regularity. There are some classical books [8, 9]. Based on operator-valued Fourier multiplier theorem in [11], maximal L_p-L_q regularity for parabolic equation has been proved in [1, 7]. The point to be noted is in the assumption. Namely, we have to check $\mathcal{R}$-boundedness of Fourier multiplier symbols. To do so, there are some methods given by [6] and [10]. We use the good points of both in this paper. Although the notion of $\mathcal{R}$-boundedness is difficult, $\mathcal{R}$-boundedness and usual boundedness for an operator is equivalent on the Hilbert space. We can use Fourier multiplier theorem twice since we regard Laplace transform as Fourier transform. Moreover we know that if the Fourier multiplier symbol is bounded and holomorphic, then it becomes a sufficient condition to use

Fourier multiplier theorem on $\mathbb{R}^n$. In our previous papers [3, 5] which considered on the half space and the layer domain, we constructed theorems to prove L_q-boundedness and L_p-L_q boundedness. Two papers proved resolvent estimate and maximal regularity estimate for the Stokes equations with various boundary conditions on the half space, and with Dirichlet-Neumann boundary condition on the layer domain. This paper also treats resolvent estimate and maximal regularity for the heat equations with various boundary conditions on the layer domain.

This paper consists of five sections and one appendix. In the next section, we reduce the problems (1.1)-(1.3) to the homogeneous source terms except for boundary data, i.e., we show that $f = 0$, $F = 0$ are enough to prove main theorems. To do so, we have to consider the heat equations on the whole space. The estimates on $\mathbb{R}^n$ are proved in the appendix. The proof is straightforward since the symbols are bounded and holomorphic. In Section 3, We give the solution formula for the problems (1.1)-(1.3). Thanks to the preparation in Section 2, the solution operator is given from boundary data to the solution. In Section 4, we prove the main theorems. After we review a theorem, we divide the solution into two terms which come from two boundary conditions. We also divide the symbols while paying attention to the regularity of the boundary. Since we need to estimate the symbols, we prepare some lemma. Then we are able to get resolvent estimate and maximal regularity estimate at the same time. In the last section, we prove the solution is unique. The method is to use dual problem and fundamental lemma of calculus of variations.

§2. Reduction to the problem only with boundary data

In this section we show that it is enough to consider the case $f = 0$ in (1.2) and $F = 0$ in (1.3) by subtracting solutions of inhomogeneous data.

Let $f \in L_q(\Omega)$ with $1 < q < \infty$. We extend the function f by

$$Ef := \begin{cases} f(x) & x \in \Omega, \\ 0 & x \notin \Omega. \end{cases}$$

Then we have $Ef \in L_q(\mathbb{R}^n)$. We solve the equation $(\lambda - \Delta)v = Ef$ in $\mathbb{R}^n$. We are able to get $v \in W_q^2(\mathbb{R}^n)$ and

$$\begin{aligned} \|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\Omega)} &\leq \|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\mathbb{R}^n)} \\ &\leq C\|Ef\|_{L_q(\mathbb{R}^n)} \\ &= C\|f\|_{L_q(\Omega)}. \end{aligned}$$

The proof of the second inequality is given in Appendix A.

First we consider the case $\beta_1 = \beta_2 = 0$. Keeping in mind $w = u - v$, let w be the solution of

$$\begin{aligned}
(\lambda - \Delta)w = (\lambda - \Delta)u - (\lambda - \Delta)v = f - Ef = 0 \quad \text{in } \Omega,\\
\alpha_1 w|_{\Gamma_0} = \alpha_1(u - v)|_{\Gamma_0} = g - \alpha_1 v|_{\Gamma_0} =: \tilde{g},\\
\alpha_2 w|_{\Gamma_\delta} = \alpha_2(u - v)|_{\Gamma_\delta} = h - \alpha_2 v|_{\Gamma_\delta} =: \tilde{h}.
\end{aligned}$$

We see

$$\begin{aligned}
\mathrm{I}_0(\tilde{g}) + \mathrm{II}_0(\tilde{h}) &\leq \mathrm{I}_0(g) + \mathrm{II}_0(h) + \alpha_1 \mathrm{I}_0(v) + \alpha_2 \mathrm{II}_0(v)\\
&\leq C(\|f\|_{L_q(\Omega)} + \mathrm{I}_0(g) + \mathrm{II}_0(h)).
\end{aligned}$$

This means that, if we have theorem 1.1 with $f = 0$,

$$\begin{aligned}
&\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\Omega)}\\
&\leq \|(\lambda w, \lambda^{1/2}\nabla w, \nabla^2 w)\|_{L_q(\Omega)} + \|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\Omega)}\\
&\leq C(\|f\|_{L_q(\Omega)} + \mathrm{I}_0(\tilde{g}) + \mathrm{II}_0(\tilde{h}))\\
&\leq C(\|f\|_{L_q(\Omega)} + \mathrm{I}_0(g) + \mathrm{II}_0(h)),
\end{aligned}$$

which implies that $f = 0$ is enough to prove theorem 1.1, where the solution u is $u = w + v$.

Next, we consider the general case. Let v be the solution of

$$\begin{aligned}
(\lambda - \Delta)v = f \qquad &\text{in } \Omega,\\
v = 0 \qquad &\text{on } \Gamma_0,\\
v = 0 \qquad &\text{on } \Gamma_\delta.
\end{aligned}$$

From above discussion, we have $\|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}$. Let w be the solution of

$$\begin{aligned}
(\lambda - \Delta)w = (\lambda - \Delta)u - (\lambda - \Delta)v = f - Ef = 0 \qquad &\text{in } \Omega,\\
\alpha_1 w - \beta_1\partial_n w = \alpha_1(u - v) - \beta_1\partial_n(u - v) = g + \beta_1\partial_n v =: \tilde{g} \qquad &\text{on } \Gamma_0,\\
\alpha_2 w + \beta_2\partial_n w = \alpha_2(u - v) + \beta_2\partial_n(u - v) = h - \beta_2\partial_n v =: \tilde{h} \qquad &\text{on } \Gamma_\delta.
\end{aligned}$$

Then we show $u = w + v$ is the original solution. This is the reason why

$$\begin{aligned}
&\mathrm{I}_{\beta_1}(\tilde{g}) \leq \mathrm{I}_{\beta_1}(g) + \beta_1 \mathrm{I}_{\beta_1}(v)\\
&\begin{cases}
= \mathrm{I}_{\beta_1}(g) & \text{if } \beta_1 = 0,\\
\leq \mathrm{I}_{\beta_1}(g) + \beta_1\|(\lambda^{1/2}\partial_n v, \nabla(\partial_n v))\|_{L_q(\Omega)} \leq C(\|f\|_{L_q(\Omega)} + \mathrm{I}_{\beta_1}(g)) & \text{if } \beta_1 > 0.
\end{cases}
\end{aligned}$$

The term of h is treated similarly.

Therefore $f = 0$ is enough to prove theorem 1.1. After replacing the norms, $F = 0$ is enough to prove theorem 1.4.

§3. Solution formula from boundary data

We give the solution of the resolvent problem (1.2) with $f = 0$ and the non-stationary problem (1.3) with $F = 0$. We apply partial Fourier transform with respect to tangential direction $x' \in \mathbb{R}^{n-1}$. We use the notations

$$\begin{aligned}\hat{v}(\xi', x_n) :=& \mathcal{F}_{x'} v(\xi', x_n) := \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} v(x', x_n) dx', \\ & \mathcal{F}^{-1}_{\xi'} w(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} w(\xi', x_n) d\xi'\end{aligned}$$

for functions $v, w : \Omega \to \mathbb{C}$.

Set

$$A := \sqrt{\sum_{j=1}^{n-1} \xi_j^2}, \qquad B := \sqrt{\lambda + A^2}$$

with positive real parts. Here we consider $\xi' = (\xi_1, \dots, \xi_{n-1})^T$ as complex values;

$$\xi_j \in \tilde{\Sigma}_\eta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \eta\} \cup \{z \in \mathbb{C} \setminus \{0\} \mid \pi - \eta < |\arg z|\}$$

for $\eta \in (0, \pi/4)$.

We need to solve the following second order ordinary differential equations;

$$\begin{cases} (B^2 - \partial_n^2)\hat{u} = 0 & \text{in } 0 < x_n < \delta, \\ \alpha_1 \hat{u} - \beta_1 \partial_n \hat{u} = \hat{g} & \text{on } x_n = 0, \\ \alpha_2 \hat{u} + \beta_2 \partial_n \hat{u} = \hat{h} & \text{on } x_n = \delta. \end{cases}$$

From the first equation, we get the general solution

$$\hat{u}(\xi', x_n) = C_1 e^{Bx_n} + C_2 e^{-Bx_n},$$

where the coefficients $C_{1,2}$ are determined by the boundary conditions;

$$\begin{pmatrix} \alpha_1 - \beta_1 B & \alpha_1 + \beta_1 B \\ (\alpha_2 + \beta_2 B)e^{B\delta} & (\alpha_2 - \beta_2 B)e^{-B\delta} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \hat{g} \\ \hat{h} \end{pmatrix}.$$

Let

$$\begin{aligned} D :=& (\alpha_1 - \beta_1 B)(\alpha_2 - \beta_2 B)e^{-B\delta} - (\alpha_1 + \beta_1 B)(\alpha_2 + \beta_2 B)e^{B\delta} \\ =& -2\{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta)\} \end{aligned}$$

be the determinant of the coefficient matrix. We prove $D \neq 0$ in lemma 4.3. Therefore we have

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} (\alpha_2 - \beta_2 B)e^{-B\delta} & -(\alpha_1 + \beta_1 B) \\ -(\alpha_2 + \beta_2 B)e^{B\delta} & \alpha_1 - \beta_1 B \end{pmatrix} \begin{pmatrix} \hat{g} \\ \hat{h} \end{pmatrix}$$

and

$$
\begin{aligned}
&\hat{u}(\xi', x_n)\\
=&\frac{1}{D}\{(\alpha_2 - \beta_2 B)e^{-B\delta}\hat{g} - (\alpha_1 + \beta_1 B)\hat{h}\}e^{Bx_n} +\\
&+ \frac{1}{D}\{-(\alpha_2 + \beta_2 B)e^{B\delta}\hat{g} + (\alpha_1 - \beta_1 B)\hat{h}\}e^{-Bx_n}\\
=&\frac{1}{D}\{(\alpha_2 - \beta_2 B)e^{-B(\delta - x_n)} - (\alpha_2 + \beta_2 B)e^{B(\delta - x_n)}\}\hat{g}\\
&+ \frac{1}{D}\{-(\alpha_1 + \beta_1 B)e^{Bx_n} + (\alpha_1 - \beta_1 B)e^{-Bx_n}\}\hat{h}\\
=&-\frac{1}{D}\{\alpha_2(e^{B(\delta - x_n)} - e^{-B(\delta - x_n)}) + \beta_2 B(e^{B(\delta - x_n)} + e^{-B(\delta - x_n)})\}\hat{g}\\
&- \frac{1}{D}\{\alpha_1(e^{Bx_n} - e^{-Bx_n}) + \beta_1 B(e^{Bx_n} + e^{-Bx_n})\}\hat{h}\\
=&\frac{\alpha_2 \sinh B(\delta - x_n) + \beta_2 B \cosh B(\delta - x_n)}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta}\hat{g}\\
&\qquad + \frac{\alpha_1 \sinh Bx_n + \beta_1 B \cosh Bx_n}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta}\hat{h}.
\end{aligned}
$$

Similarly, the solution of (1.3) with $F = 0$ is

$$
\begin{aligned}
&\widehat{\mathcal{L}U}(\xi', x_n, \lambda)\\
=&\frac{\alpha_2 \sinh B(\delta - x_n) + \beta_2 B \cosh B(\delta - x_n)}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta}\widehat{\mathcal{L}G}\\
&\qquad + \frac{\alpha_1 \sinh Bx_n + \beta_1 B \cosh Bx_n}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta}\widehat{\mathcal{L}H}.
\end{aligned}
$$

§4. Proof of estimates

We prepare a theorem to prove the main theorems. Let us define the operators T and $\tilde{T}_\gamma$ by

$$
\begin{aligned}
T[m]f(x) &:= \int_0^\delta [\mathcal{F}_{\xi'}^{-1} m(\xi', x_n, y_n)\mathcal{F}_{x'} f](x, y_n)dy_n,\\
\tilde{T}_\gamma[m_\lambda]g(x,t) &:= \mathcal{L}_\lambda^{-1}\int_0^\delta [\mathcal{F}_{\xi'}^{-1} m_\lambda(\xi', x_n, y_n)\mathcal{F}_{x'}\mathcal{L}g](x, y_n, \lambda)dy_n,\\
&= [e^{\gamma t}\mathcal{F}_{\tau\to t}^{-1} T[m_\lambda]\mathcal{F}_{t\to\tau}(e^{-\gamma t}g)](x,t),
\end{aligned}
$$

where $\lambda = \gamma + i\tau \in \Sigma_\varepsilon$, symbols m, m_λ are $\mathbb{C}$-valued functions, and $f : \Omega \to \mathbb{C}$ and $g : \Omega \times \mathbb{R} \to \mathbb{C}$.

The following theorem is a basis of $L_q(\Omega)$-boundedness to prove resolvent estimates, and $L_p(\mathbb{R}, L_q(\Omega))$-boundedness to prove maximal regularity estimates, where the domain Ω is layer $\mathbb{R}^{n-1} \times (0, \delta)$, while $\Omega = \mathbb{R}^n_+$ in [3]. We define

$$d_1(x_n) = \delta - x_n, \quad d_2(x_n) = x_n.$$

Let $H^\infty(\tilde{\Sigma}^{n-1}_\eta)$ be the space of bounded and holomorphic functions on $\tilde{\Sigma}^{n-1}_\eta$.

Theorem 4.1 ([5, Theorem 5.1]). (i) *Let m satisfy the following two conditions:*
(a) *There exists $\eta \in (0, \pi/2)$ such that*

$$\{m(\cdot, x_n, y_n); x_n, y_n \in (0, \delta)\} \subset H^\infty(\tilde{\Sigma}^{n-1}_\eta).$$

(b) *There exist $\eta \in (0, \pi/2)$, $\ell, \ell' \in \{1, 2\}$ and $C > 0$ such that*

$$\sup_{\xi' \in \tilde{\Sigma}^{n-1}_\eta} |m(\xi', x_n, y_n)| \leq C(d_\ell(x_n) + d_{\ell'}(y_n))^{-1}$$

for all $x_n, y_n \in (0, \delta)$.
Then $T[m]$ is a bounded linear operator on $L_q(\Omega)$ for every $1 < q < \infty$.
(ii) *Let $\gamma_0 \geq 0$ and let m_λ satisfy the following two conditions:*
(c) *There exist $\eta \in (0, \pi/2 - \varepsilon)$ and $\ell, \ell' \in \{1, 2\}$ such that for each $x_n, y_n \in (0, \delta)$ and $\gamma \geq \gamma_0$,*

$$\tilde{\Sigma}^n_\eta \ni (\tau, \xi') \mapsto m_\lambda(\xi', x_n, y_n) \in \mathbb{C}$$

is bounded and holomorphic.
(d) *There exist $\eta \in (0, \pi/2 - \varepsilon)$, $\ell, \ell' \in \{1, 2\}$ and $C > 0$ such that*

$$\sup\{|m_\lambda(\xi', x_n, y_n)| \mid (\tau, \xi') \in \tilde{\Sigma}^n_\eta\} \leq C(d_\ell(x_n) + d_{\ell'}(y_n))^{-1}$$

for all $\gamma \geq \gamma_0$ and $x_n, y_n \in (0, \delta)$.
Then $\tilde{T}_\gamma[m_\lambda]$ satisfies

$$\|e^{-\gamma t}\tilde{T}_\gamma[m_\lambda]g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\|e^{-\gamma t}g\|_{L_p(\mathbb{R}, L_q(\Omega))}$$

for every $\gamma \geq \gamma_0$ and $1 < p, q < \infty$.

We define the symbols

$$\phi_{\beta_1,\beta_2}(x_n) = \frac{\alpha_2 \sinh Bx_n + \beta_2 B \cosh Bx_n}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta},$$
$$\psi_{\beta_1,\beta_2}(x_n) = \frac{\alpha_1 \sinh Bx_n + \beta_1 B \cosh Bx_n}{(\alpha_1\alpha_2 + \beta_1\beta_2 B^2)\sinh B\delta + (\alpha_1\beta_2 + \alpha_2\beta_1)B\cosh B\delta},$$

so that

$$u(x) = \mathcal{F}^{-1}_{\xi'}\phi_{\beta_1,\beta_2}(d_1(x_n))\mathcal{F}_{x'}g(x', 0) + \mathcal{F}^{-1}_{\xi'}\psi_{\beta_1,\beta_2}(d_2(x_n))\mathcal{F}_{x'}h(x', \delta)$$

$$=: u_g + u_h.$$

Although ϕ_{β_1,β_2} and ψ_{β_1,β_2} are the functions of λ and ξ', we omit their variables for simplicity.

Let $\zeta_0 \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that $0 \le \zeta_0 \le 1$ and

$$\zeta_0(y) = \begin{cases} 1 & y < \frac{1}{3}\delta, \\ 0 & y > \frac{2}{3}\delta \end{cases}$$

and let $\zeta_\delta := 1 - \zeta_0$. Recall the fundamental theorem $f(x', \delta) - f(x', 0) = \int_0^\delta \partial_n f(x', y_n) dy_n$, and note

$$\begin{aligned} f(x', \delta) &= \int_0^\delta \partial_{y_n}(\zeta_\delta(y_n) f(x', y_n)) dy_n, \\ f(x', 0) &= -\int_0^\delta \partial_{y_n}(\zeta_0(y_n) f(x', y_n)) dy_n \end{aligned}$$

and $d_1(\delta) = d_2(0) = 0$.

By using the following identities

$$B^2 = \lambda + \sum_{m=1}^{n-1} \xi_m^2, \qquad 1 = \frac{B^2}{B^2} = \frac{\lambda^{1/2}}{B^2}\lambda^{1/2} - \sum_{m=1}^{n-1} \frac{i\xi_m}{B^2}(i\xi_m),$$

we calculate as follows;

$$\begin{aligned} u(x) = &-\int_0^\delta [\partial_{y_n}(\mathcal{F}_{\xi'}^{-1} \phi_{\beta_1,\beta_2}(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'} \zeta_0 g)] dy_n \\ &+ \int_0^\delta [\partial_{y_n}(\mathcal{F}_{\xi'}^{-1} \psi_{\beta_1,\beta_2}(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'} \zeta_\delta h)] dy_n \\ = &-\int_0^\delta \mathcal{F}_{\xi'}^{-1} [\partial_n(\phi_{\beta_1,\beta_2}(d_1(x_n) - d_2(y_n))) \mathcal{F}_{x'} \zeta_0 g] dy_n \\ &- \int_0^\delta \mathcal{F}_{\xi'}^{-1} [\phi_{\beta_1,\beta_2}(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'} \partial_n(\zeta_0 g)] dy_n \\ &+ \int_0^\delta \mathcal{F}_{\xi'}^{-1} [\partial_n(\psi_{\beta_1,\beta_2}(d_2(x_n) - d_1(y_n))) \mathcal{F}_{x'} \zeta_\delta h] dy_n \\ &+ \int_0^\delta \mathcal{F}_{\xi'}^{-1} [\psi_{\beta_1,\beta_2}(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'} \partial_n(\zeta_\delta h)] dy_n. \end{aligned}$$

Case $\beta_1 = 0$;

$$u_g = \int_0^\delta \mathcal{F}_{\xi'}^{-1} [B^{-2}(\partial_n \phi_{0,\beta_2})(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'}((\lambda - \Delta')\zeta_0 g)] dy_n$$

$$
\begin{aligned}
&- \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} B^{-2} \phi_{0,\beta_2}(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'}(\lambda^{1/2} \partial_n(\zeta_0 g))] dy_n \\
&+ \sum_{m=1}^{n-1} \int_0^\delta \mathcal{F}_{\xi'}^{-1}[i\xi_m B^{-2} \phi_{0,\beta_2}(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'}(\partial_m \partial_n(\zeta_0 g))] dy_n.
\end{aligned}
$$

Case $\beta_1 > 0$;

$$
\begin{aligned}
u_g = &\int_0^\delta \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} B^{-2} (\partial_n \phi_{\beta_1,\beta_2})(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'}(\lambda^{1/2} \zeta_0 g)] dy_n \\
&- \sum_{m=1}^{n-1} \int_0^\delta \mathcal{F}_{\xi'}^{-1}[i\xi_m B^{-2} (\partial_n \phi_{\beta_1,\beta_2})(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'}(\partial_m(\zeta_0 g))] dy_n \\
&- \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\phi_{\beta_1,\beta_2}(d_1(x_n) - d_2(y_n)) \mathcal{F}_{x'} \partial_n(\zeta_0 g)] dy_n.
\end{aligned}
$$

Case $\beta_2 = 0$;

$$
\begin{aligned}
u_h = &\int_0^\delta \mathcal{F}_{\xi'}^{-1}[B^{-2} (\partial_n \psi_{\beta_1,0})(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'}((\lambda - \Delta')\zeta_\delta h)] dy_n \\
&+ \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} B^{-2} \psi_{\beta_1,0}(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'}(\lambda^{1/2} \partial_n(\zeta_\delta h))] dy_n \\
&- \sum_{m=1}^{n-1} \int_0^\delta \mathcal{F}_{\xi'}^{-1}[i\xi_m B^{-2} \psi_{\beta_1,0}(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'}(\partial_m \partial_n(\zeta_\delta h))] dy_n.
\end{aligned}
$$

Case $\beta_2 > 0$;

$$
\begin{aligned}
u_h = &\int_0^\delta \mathcal{F}_{\xi'}^{-1}[\lambda^{1/2} B^{-2} (\partial_n \psi_{\beta_1,\beta_2})(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'}(\lambda^{1/2} \zeta_\delta h)] dy_n \\
&- \sum_{m=1}^{n-1} \int_0^\delta \mathcal{F}_{\xi'}^{-1}[i\xi_m B^{-2} (\partial_n \psi_{\beta_1,\beta_2})(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'}(\partial_m(\zeta_\delta h))] dy_n \\
&+ \int_0^\delta \mathcal{F}_{\xi'}^{-1}[\psi_{\beta_1,\beta_2}(d_2(x_n) - d_1(y_n)) \mathcal{F}_{x'} \partial_n(\zeta_\delta h)] dy_n.
\end{aligned}
$$

Let $S_{u_g}(\lambda, \xi', x_n, y_n)$ and $S_{u_h}(\lambda, \xi', x_n, y_n)$ be any of symbols;

$$
S_{u_g}(\lambda, \xi', x_n, y_n) := \begin{cases}
B^{-2}(\partial_n \phi_{0,\beta_2})(d_1(x_n) - d_2(y_n)), & (\beta_1 = 0), \\
\lambda^{1/2} B^{-2} \phi_{0,\beta_2}(d_1(x_n) - d_2(y_n)), & (\beta_1 = 0), \\
i\xi_m B^{-2} \phi_{0,\beta_2}(d_1(x_n) - d_2(y_n)), & (\beta_1 = 0), \\
\lambda^{1/2} B^{-2} (\partial_n \phi_{\beta_1,\beta_2})(d_1(x_n) - d_2(y_n)), & (\beta_1 > 0), \\
i\xi_m B^{-2} (\partial_n \phi_{\beta_1,\beta_2})(d_1(x_n) - d_2(y_n)), & (\beta_1 > 0), \\
\phi_{\beta_1,\beta_2}(d_1(x_n) - d_2(y_n)), & (\beta_1 > 0),
\end{cases}
$$

$$S_{u_h}(\lambda, \xi', x_n, y_n) := \begin{cases} B^{-2}(\partial_n \psi_{\beta_1,0})(d_2(x_n) - d_1(y_n)), & (\beta_2 = 0), \\ \lambda^{1/2} B^{-2} \psi_{\beta_1,0}(d_2(x_n) - d_1(y_n)), & (\beta_2 = 0), \\ i\xi_m B^{-2} \psi_{\beta_1,0}(d_2(x_n) - d_1(y_n)), & (\beta_2 = 0), \\ \lambda^{1/2} B^{-2} \psi_{\beta_1,0}(d_2(x_n) - d_1(y_n)), & (\beta_2 > 0), \\ i\xi_m B^{-2}(\partial_n \psi_{\beta_1,\beta_2})(d_2(x_n) - d_1(y_n)), & (\beta_2 > 0), \\ \psi_{\beta_1,\beta_2}(d_2(x_n) - d_1(y_n)), & (\beta_2 > 0). \end{cases}$$

Here m runs from 1 to $n-1$.

We shall confirm that all of the symbols are bounded in the sense that

$$\begin{aligned} &\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde{\Sigma}_\eta^{n-1} \\ r,r'=1,\dots,n-1}} \Big\{ (|\lambda| + |\lambda|^{1/2}|\xi_r| + |\xi_r||\xi_{r'}|)|S_{u_g}| \\ &\qquad\qquad + (|\lambda|^{1/2} + |\xi_r|)|\partial_{x_n} S_{u_g}| + |\partial_{x_n}^2 S_{u_g}| \Big\} \\ &< C(d_2(x_n) + d_2(y_n))^{-1}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\sup_{\substack{(\lambda,\xi')\in\Sigma_\varepsilon\times\tilde{\Sigma}_\eta^{n-1} \\ r,r'=1,\dots,n-1}} \Big\{ (|\lambda| + |\lambda|^{1/2}|\xi_r| + |\xi_r||\xi_{r'}|)|S_{u_h}| \\ &\qquad\qquad + (|\lambda|^{1/2} + |\xi_r|)|\partial_{x_n} S_{u_h}| + |\partial_{x_n}^2 S_{u_h}| \Big\} \\ &< C(d_1(x_n) + d_1(y_n))^{-1}, \end{aligned} \tag{4.2}$$

for suitable $\eta \in (0, \pi/4)$ depending on ε.

Let $\tilde{A} := \sqrt{\sum_{j=1}^{n-1} |\xi_j|^2}$.

Lemma 4.2 ([3, Lemma 6.3])**.** *Let $0 < \varepsilon < \pi/2$ and $0 < \eta < \varepsilon/4$. Then there exist positive constants c and C such that for any $(\lambda, \xi', x_n) \in \Sigma_\varepsilon \times \tilde{\Sigma}_\eta^{n-1} \times (0, \delta)$, $\alpha, \beta \geq 0$,*

$$\begin{gathered} c\tilde{A} \leq \operatorname{Re} A \leq |A| \leq \tilde{A}, \\ c(|\lambda|^{1/2} + \tilde{A}) \leq \operatorname{Re} B \leq |B| \leq C(|\lambda|^{1/2} + \tilde{A}), \\ c(\alpha + \beta|B|) \leq |\alpha + \beta B| \leq \alpha + \beta|B|. \end{gathered}$$

Lemma 4.3. *Let $0 < \varepsilon < \pi/2$, $0 < \eta < \varepsilon/4$. Then for any $(\lambda, \xi') \in \Sigma_\varepsilon \times \tilde{\Sigma}_\eta^{n-1}$, we have the estimate*

$$|D| \geq 2|\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B| \sinh((\operatorname{Re} B)\delta).$$

Proof.

$$D = \begin{vmatrix} \alpha_1 - \beta_1 B & \alpha_1 + \beta_1 B \\ (\alpha_2 + \beta_2 B)e^{B\delta} & (\alpha_2 - \beta_2 B)e^{-B\delta} \end{vmatrix}$$

$$= (\alpha_1 - \beta_1 B)(\alpha_2 - \beta_2 B)e^{-B\delta} - (\alpha_1 + \beta_1 B)(\alpha_2 + \beta_2 B)e^{B\delta}.$$

Therefore

$$\begin{aligned}|D| &= |(\alpha_1 + \beta_1 B)(\alpha_2 + \beta_2 B)e^{B\delta} - (\alpha_1 - \beta_1 B)(\alpha_2 - \beta_2 B)e^{-B\delta}| \\ &= |e^{B\delta}||(\alpha_1 + \beta_1 B)(\alpha_2 + \beta_2 B) - (\alpha_1 - \beta_1 B)(\alpha_2 - \beta_2 B)e^{-2B\delta}| \\ &\geq e^{(\operatorname{Re} B)\delta}\{|\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B| - |\alpha_1 - \beta_1 B||\alpha_2 - \beta_2 B|e^{-2\operatorname{Re} B\delta}\}.\end{aligned}$$

Here we use $|\alpha_1 + \beta_1 B| \geq |\alpha_1 - \beta_1 B|, |\alpha_2 + \beta_2 B| \geq |\alpha_2 - \beta_2 B|$.

$$\begin{aligned}|D| &\geq e^{(\operatorname{Re} B)\delta}\{|\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B| - |\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B|e^{-2\operatorname{Re} B\delta}\} \\ &= |\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B|(e^{(\operatorname{Re} B)\delta} - e^{-(\operatorname{Re} B)\delta}) \\ &\geq 2|\alpha_1 + \beta_1 B||\alpha_2 + \beta_2 B|(\sinh(\operatorname{Re} B)\delta).\end{aligned}$$

□

We prepare another estimate.

Lemma 4.4. *Let $0 < \varepsilon < \pi/2$, $0 < \eta < \varepsilon/4$. Then there exists a positive constant C such that for any $(\lambda, \xi', x_n) \in \Sigma_\varepsilon \times \tilde{\Sigma}_\eta^{n-1}$, we have the estimate*

$$\left|\frac{\sinh Bx_n}{\sinh \operatorname{Re} B\delta}\right|, \left|\frac{\cosh Bx_n}{\sinh \operatorname{Re} B\delta}\right| \leq \frac{C}{(|\lambda|^{1/2} + \tilde{A})(\delta - x_n)} \qquad (-\delta < x_n < \delta).$$

Proof.

$$\begin{aligned}&\left|\frac{\sinh Bx_n}{\sinh \operatorname{Re} B\delta}\right|(|\lambda|^{1/2} + \tilde{A})(\delta - x_n) \\ &\leq C \operatorname{Re} B(\delta - x_n)\frac{|e^{Bx_n} - e^{-Bx_n}|}{e^{\operatorname{Re} B\delta} - e^{-\operatorname{Re} B\delta}} \\ &\leq C \operatorname{Re} B(\delta - x_n)\frac{|e^{Bx_n}| + |e^{-Bx_n}|}{e^{\operatorname{Re} B\delta} - e^{-\operatorname{Re} B\delta}} \\ &\leq C \operatorname{Re} B(\delta - x_n)\frac{e^{\operatorname{Re} Bx_n}}{e^{\operatorname{Re} B\delta}}\frac{1 + e^{-2\operatorname{Re} Bx_n}}{1 - e^{-2\operatorname{Re} B\delta}} \\ &\leq C \operatorname{Re} B(\delta - x_n)e^{-\operatorname{Re} B(\delta - x_n)}\frac{1 + e^{-2\operatorname{Re} Bx_n}}{1 - e^{-2\operatorname{Re} B\delta}} \\ &\leq C \qquad (-\delta < x_n < \delta).\end{aligned}$$

The other $\left|\frac{\cosh Bx_n}{\sinh \operatorname{Re} B\delta}\right|$ is the same. □

Using these lemma, we have

$|\phi_{\beta_1,\beta_2}(x_n)|$

$$\begin{aligned}&\leq C(|\alpha_1+\beta_1 B||\alpha_2+\beta_2 B|\sinh \operatorname{Re} B\delta)^{-1}|\alpha_2\sinh Bx_n+\beta_2 B\cosh Bx_n|\\&\leq C((\alpha_1+\beta_1|B|)(|\lambda|^{1/2}+\tilde{A})(\delta-x_n))^{-1},\end{aligned}$$

and similarly,

$$\begin{aligned}|\psi_{\beta_1,\beta_2}(x_n)|&\leq C((\alpha_2+\beta_2|B|)(|\lambda|^{1/2}+\tilde{A})(\delta-x_n))^{-1},\\|\partial_n\phi_{\beta_1,\beta_2}(x_n)|&\leq C((\alpha_1+\beta_1|B|)(\delta-x_n))^{-1},\\|\partial_n\psi_{\beta_1,\beta_2}(x_n)|&\leq C((\alpha_2+\beta_2|B|)(\delta-x_n))^{-1}.\end{aligned}$$

This implies that

$$\begin{aligned}|S_{u_g}|&\leq\frac{C}{(|\lambda|^{1/2}+\tilde{A})^2(\delta-d_1(x_n)+d_2(y_n))}\\&=\frac{C}{(|\lambda|^{1/2}+\tilde{A})^2(d_2(x_n)+d_2(y_n))},\\|\partial_{x_n}S_{u_g}|&\leq\frac{C}{(|\lambda|^{1/2}+\tilde{A})(d_2(x_n)+d_2(y_n))},\\|\partial_{x_n}^2S_{u_g}|&\leq\frac{C}{d_2(x_n)+d_2(y_n)},\\|S_{u_h}|&\leq\frac{C}{(|\lambda|^{1/2}+\tilde{A})^2(\delta-d_2(x_n)+d_1(y_n))}\\&=\frac{C}{(|\lambda|^{1/2}+\tilde{A})^2(d_1(x_n)+d_1(y_n))},\\|\partial_{x_n}S_{u_h}|&\leq\frac{C}{(|\lambda|^{1/2}+\tilde{A})(d_1(x_n)+d_1(y_n))},\\|\partial_{x_n}^2S_{u_h}|&\leq\frac{C}{d_1(x_n)+d_1(y_n)},\end{aligned}$$

and the inequalities (4.1) and (4.2).

We are in the position that we able to use theorem 4.1. We obtain the following inequality

$$\|(\lambda u,\lambda^{1/2}\nabla u,\nabla^2 u)\|_{L_q(\Omega)}\leq C(\mathrm{I}_{\beta_1}(\zeta_0 g)+\mathrm{II}_{\beta_2}(\zeta_\delta h)).$$

To remove the cut-off function, we use $\zeta_0,\zeta_\delta\in W^2_\infty(\mathbb{R})$, $\|(\zeta_0,\zeta_\delta)\|_{W^2_\infty(\mathbb{R})}\leq C$ and $0<\gamma_0<|\lambda|$ as follows; When $\beta_1=0$,

$$\begin{aligned}\|\lambda(\zeta_0 g)\|_{L_q(\Omega)}&\leq\|\zeta_0\|_{L_\infty(0,\delta)}\|\lambda g\|_{L_q(\Omega)}\leq C\|\lambda g\|_{L_q(\Omega)},\\\|\lambda^{1/2}\nabla(\zeta_0 g)\|_{L_q(\Omega)}&\leq\|\nabla\zeta_0\|_{L_\infty(0,\delta)}\|\lambda^{1/2}g\|_{L_q(\Omega)}+\|\zeta_0\|_{L_\infty(\Omega)}\|\lambda^{1/2}\nabla g\|_{L_q(\Omega)},\\&\leq C\|(\lambda g,\lambda^{1/2}\nabla g)\|_{L_q(\Omega)},\end{aligned}$$

$$
\begin{aligned}
\|\nabla^2(\zeta_0 g)\|_{L_q(\Omega)} &\le \|\nabla^2\zeta_0\|_{L_\infty(0,\delta)}\|g\|_{L_q(\Omega)} + 2\|\nabla\zeta_0\|_{L_\infty(0,\delta)}\|\nabla g\|_{L_q(\Omega)} \\
&\quad + \|\zeta_0\|_{L_\infty(0,\delta)}\|\nabla^2 g\|_{L_q(\Omega)} \\
&\le C\|(\lambda g, \lambda^{1/2}\nabla g, \nabla^2 g)\|_{L_q(\Omega)},
\end{aligned}
$$

where we used $1 \le \gamma_0^{-1/2}|\lambda|^{1/2}$. When $\beta_1 > 0$, we show similarly,

$$
\begin{aligned}
\|\lambda^{1/2}(\zeta_0 g)\|_{L_q(\Omega)} &\le \|\zeta_0\|_{L_\infty(0,\delta)}\|\lambda^{1/2} g\|_{L_q(\Omega)} \le C\|\lambda^{1/2} g\|_{L_q(\Omega)}, \\
\|\nabla(\zeta_0 g)\|_{L_q(\Omega)} &\le \|\nabla\zeta_0\|_{L_\infty(0,\delta)}\|g\|_{L_q(\Omega)} + \|\zeta_0\|_{L_\infty(0,\delta)}\|\nabla g\|_{L_q(\Omega)} \\
&\le C\|(\lambda^{1/2} g, \nabla g)\|_{L_q(\Omega)}.
\end{aligned}
$$

The term h is same as the term g. Therefore we are able to prove

$$
\|(\lambda u, \lambda^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\Omega)} \le C(\mathrm{I}_{\beta_1}(g) + \mathrm{II}_{\beta_2}(h)).
$$

This concludes the proof of theorem 1.1 with $f = 0$.

By theorem 4.1 again, we are able to prove theorem 1.4 with $F = 0$. Since $\gamma \le |\lambda|$, we have the estimate for the term γU.

§5. Uniqueness

In this section we prove the heat equation with various boundary conditions has at most one solution.

Theorem 5.1. *Let $1 < q < \infty$ and $u \in W_q^2(\Omega)$ satisfy*

$$
\begin{aligned}
(\lambda - \Delta)u &= 0 && \text{in } 0 < x_n < \delta, \\
\alpha_1 u - \beta_1 \partial_n u &= 0 && \text{on } x_n = 0, \\
\alpha_2 u + \beta_2 \partial_n u &= 0 && \text{on } x_n = \delta,
\end{aligned}
$$

then $u = 0$.

Proof. The method is based on fundamental lemma of calculus of variations. For any $\phi \in C_0^\infty(\Omega)$, take $\varphi \in W_{q'}^2(\Omega)$ such that

$$
\begin{aligned}
(\lambda - \Delta)\varphi &= \phi && \text{in } 0 < x_n < \delta, \\
\alpha_1 \varphi - \beta_1 \partial_n \varphi &= 0 && \text{on } x_n = 0, \\
\alpha_2 \varphi + \beta_2 \partial_n \varphi &= 0 && \text{on } x_n = \delta,
\end{aligned}
$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark that

$$
\int_\Omega u \frac{\partial^2 \varphi}{\partial x_n^2} dx = \int_{\mathbb{R}^{n-1}} \left(\int_0^\delta u \frac{\partial^2 \varphi}{\partial x_n^2} dx_n \right) dx'
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{n-1}} \left(\left[u \frac{\partial \varphi}{\partial x_n} \right]_0^\delta - \int_0^\delta \frac{\partial u}{\partial x_n} \frac{\partial \varphi}{\partial x_n} dx_n \right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\left[u \frac{\partial \varphi}{\partial x_n} \right]_0^\delta - \left[\frac{\partial u}{\partial x_n} \varphi \right]_0^\delta + \int_0^\delta \frac{\partial^2 u}{\partial x_n^2} \varphi dx_n \right) dx'.
\end{aligned}
$$

For the case $\beta_1, \beta_2 > 0$,

$$
\begin{aligned}
&\left[u \frac{\partial \varphi}{\partial x_n} \right]_0^\delta - \left[\frac{\partial u}{\partial x_n} \varphi \right]_0^\delta \\
&= (\frac{\alpha_2}{\beta_2} \varphi u|_{x_n=\delta}) - (\frac{\alpha_1}{\beta_1} \varphi u|_{x_n=0}) - (\frac{\alpha_2}{\beta_2} u \varphi|_{x_n=\delta}) + (\frac{\alpha_1}{\beta_1} u \varphi|_{x_n=0}) \\
&= 0.
\end{aligned}
$$

For the degenerate case, we can lead the same result more simply. Therefore we have

$$
\begin{aligned}
\int_\Omega u \phi dx &= \int_\Omega u(\lambda \varphi - \Delta \varphi) dx \\
&= \int_\Omega (\lambda u - \Delta u) \varphi dx \\
&= 0,
\end{aligned}
$$

which implies $u = 0$. □

§A. Appendix: Proof of estimates in the whole space

Proof. We can solve the equation $(\lambda - \Delta)v = Ef$ in $\mathbb{R}^n$, and then we see

$$
\begin{aligned}
\lambda v &= \mathcal{F}^{-1} \lambda B^{-2} \mathcal{F}(Ef), \\
\lambda^{1/2} \partial_j v &= \mathcal{F}^{-1} \lambda^{1/2} i \xi_j B^{-2} \mathcal{F}(Ef), \\
\partial_j \partial_k v &= -\mathcal{F}^{-1} \xi_j \xi_k B^{-2} \mathcal{F}(Ef),
\end{aligned}
$$

where $B = \sqrt{\lambda + \sum_{j=1}^n \xi_j^2}$. Here we note that the symbols λB^{-2}, $\lambda^{1/2} i \xi_j B^{-2}$, $\xi_j \xi_k B^{-2}$ satisfy holomorphic in $\lambda \in \Sigma_\varepsilon, \xi \in \tilde{\Sigma}_\eta^n$ and boundedness

$$
\begin{aligned}
|\lambda B^{-2}| &\le C|\lambda|(|\lambda|^{1/2} + \tilde{A})^{-2} \le C, \\
|\lambda^{1/2} i \xi_j B^{-2}| &\le C|\lambda|^{1/2} |i\xi_j| (|\lambda|^{1/2} + \tilde{A})^{-2} \le C, \\
|\xi_j \xi_k B^{-2}| &\le C|\xi_j||\xi_k|(|\lambda|^{1/2} + \tilde{A})^{-2} \le C.
\end{aligned}
$$

This H^∞ property implies L_q-boundedness for the Fourier multiplier operator by [6, Proposition 4.2.1] and [10, Proposition 4.3.10]. This concludes that the resolvent L_q estimate

$$\|(\lambda v, \lambda^{1/2}\nabla v, \nabla^2 v)\|_{L_q(\mathbb{R}^n)} \leq C\|Ef\|_{L_q(\mathbb{R}^n)}$$

holds. The maximal L_p-L_q estimate is also same. □

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