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Characterization of the Lorentzian para-Sasakian manifolds admitting a quarter-symmetric non-metric connection

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Abstract. We set the goal to study the properties of LP-Sasakian manifolds equipped with a quarter-symmetric non-metric connection. It is proved that the LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection is partially Ricci semisymmetric with respect to the quarter-symmetric non-metric connection if and only if it is an η-Einstein manifold. We also study the properties of semisymmetric, Ricci recurrent LP-Sasakian manifolds and η-parallel Ricci tensor with respect to the quarter-symmetric non-metric connection. In the end, the non-trivial example of a 4-dimensional LP-Sasakian manifold with a quarter-symmetric non-metric connection is given.

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§1. Introduction

Motivated by the Sasakian structures, Matsumoto [11], in 1989, introduced the notion of Lorentzian para-Sasakian structures (briefly, LP-Sasakian structures). Mihai et al. [13] presented the same notion and found many fruitful results. Since then, many geometers studied the properties of LP-Sasakian manifolds and obtained several geometrical and physical results. We refer [1], [6], [12], [14], [16], [20], [24] and the references there in.

The pioneer work of Cartan [3] opened the door to study the symmetric spaces. A semi-Riemannian manifold M is said to be semisymmetric if the non-vanishing curvature tensor R with respect to the Levi-Civita connection ∇ satisfies $R\cdot R = 0$. Szabó [22] gave the complete intrinsic classification of the semisymmetric manifolds, which generalize the notion of the locally symmetric manifolds ($\nabla R = 0$). Every semisymmetric manifold is Ricci semisymmetric ($R \cdot S = 0$) although the converse is not true in general. Here S denotes the Ricci tensor with respect to ∇.

The notion of quarter-symmetric metric connection $\tilde{\nabla}$ on a Riemannian manifold M was given by Golab [9] in 1975. After that many researchers defined and studied the properties of the quarter-symmetric connection on different structures. We cite [4], [5], [7], [15], [19], [21] and their references. A Linear connection $\tilde{\nabla}$ on a Riemannian manifold M is said to be a quarter-symmetric connection if the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is defined by $\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$ and satisfies

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{1.1}$$

for all vector fields X and Y on M. The quarter-symmetric connection $\tilde{\nabla}$ defined on M is said to be metric if $\tilde{\nabla} g = 0$, otherwise it is non-metric. The present paper deals with the study of LP-Sasakian manifolds equipped with a quarter-symmetric non-metric connection.

Motivated by the above studies, we will plan our work as: In Section 2, we brief the basic known results of the LP-Sasakian manifolds, some classes of the symmetric spaces, and the Weyl conformal and projective curvature tensors. Section 3 deals with the study of quarter-symmetric non-metric connection on the LP-Sasakian manifolds. The properties of partially Ricci semisymmetric and semisymmetric LP-Sasakian manifolds with respect to the quarter-symmetric non-metric connection are studied in Section 4 and Section 5, respectively. We prove the existence of the Ricci recurrent LP-Sasakian manifold and the properties of η-parallel Ricci tensor on an LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection in Section 6. To validate the existence of quarter-symmetric non-metric connection on an LP-Sasakian manifold, a non-trivial example of the 4-dimensional LP-Sasakian manifold is given in Section 7.

§2. Lorentzian para-Sasakian manifolds

Let M be an n-dimensional differentiable manifold of differentiability class C^{r+1}. If M admits a $(1,1)$-type vector valued linear function ϕ, a 1-form η, and the associated vector field ξ, which satisfies

$$\phi^2 = I + \eta \otimes \xi \;\; \textit{and} \;\; \eta(\xi) = -1, \tag{2.1}$$

then M is called a Lorentzian almost para-contact manifold [11] and the structure (ϕ, ξ, η, g) is known as a Lorentzian almost para-contact structure on M. In view of (2.1), we immediate get

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \;\; \textit{and} \;\; \textit{rank}\ \phi = n-1. \tag{2.2}$$

If the Lorentzian metric g of M holds

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields X and Y on M, then (M, g) is known as a Lorentzian almost para-contact metric manifold. If, in addition, M satisfies

$$\nabla_X \xi = \phi X \iff (\nabla_X \eta)(Y) = g(\phi X, Y), \tag{2.4}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.5}$$

for all vector fields X and Y on M, then it reduces to a Lorentzian para-Sasakian manifold (briefly, LP-Sasakian manifold) [11]. We list the following known results of the LP-Sasakian manifolds (see [6])

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.9}$$

An n-dimensional semi-Riemannian manifold (M, g) is said to be partially Ricci semisymmetric if $R(\xi, X) \cdot S = 0$ holds for all vector field X on M.

The idea of η-parallel Ricci tensor on a Sasakian manifold was given by Kon [10]. An LP-Sasakian manifold M possesses an η-parallel Ricci tensor if the non-vanishing Ricci tensor S of M satisfies $(\nabla_Z S)(\phi X, \phi Y) = 0$ for all vector fields X, Y and Z.

An n-dimensional LP-Sasakian manifold M endowed with the non-zero Ricci tensor S is said to be a Ricci recurrent if $(\nabla_X S)(Y, Z) = A(X)S(Y, Z)$ holds for all X, Y and Z on M [17]. Here A is a non-zero 1-form.

If the non-vanishing Ricci tensor S of an n-dimensional LP-Sasakian manifold M satisfies $S = ag + b\eta \otimes \eta$ for the smooth functions a and b, then M is said to be an η-Einstein manifold. It is obvious that the smooth functions a and b on M are connected by $a - b = n - 1$. In particular, if $b = 0$ and a is a non-zero constant then M is known as an Einstein manifold.

A conformal curvature tensor C on an n-dimensional semi-Riemannian manifold M is defined by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z &- \frac{1}{n-2}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &-g(X, Z)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \tag{2.10}$$

for all vector fields X, Y and Z on M [8]. Here Q is the Ricci operator corresponding to the Ricci tensor S and r is the scalar curvature of M, defined as $r = \{\epsilon_i S(e_i, e_i)\}_{i=1}^{n}$, where $\{e_i,\ i = 1, 2, ...n\}$ is an orthonormal frame of reference on M and $\epsilon_i = g(e_i, e_i)$.

Apart from the conformal curvature tensor C, we may recall another important curvature tensor, called the projective curvature tensor P of M, and is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}$$

for all vector fields X, Y and Z on M [23]. A semi-Riemannian manifold is said to be projectively flat or ξ-projectively flat if and only if $P = 0$ or $P(X,Y)\xi = 0$, respectively.

§3. Quarter-symmetric non-metric connection

Sular et al. [21], in 2008, defined and studied the properties of quarter-symmetric metric connection on a Kenmotsu manifold. The properties of the same connection on the different structures have been studied by many geometers, for instance see [2], [18] and their references. Let $\tilde{\nabla}$ be a linear connection on an LP-Sasakian manifold M. If $\tilde{\nabla}$ is connected with the Levi-Civita connection ∇ of M by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y \tag{3.1}$$

for all vector fields X and Y, then the linear connection $\tilde{\nabla}$ on M is said to be a quarter-symmetric non-metric connection and it satisfies the equation (1.1) and

$$(\tilde{\nabla}_X g)(Y,Z) = 2\eta(X)g(\phi Y, Z) \tag{3.2}$$

for all vector fields X, Y and Z on M. The relation between the curvature tensors $\tilde{R}$ and R with respect to the connections $\tilde{\nabla}$ and ∇, respectively, is given by

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z &+ \{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi \\ &-\{\eta(Y)X - \eta(X)Y\}\eta(Z)\end{aligned} \tag{3.3}$$

for all vector fields X, Y and Z on M [2]. Contracting (3.3) along the vector field X, we find

$$\tilde{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\,\eta(Y)\eta(Z), \tag{3.4}$$

which is equivalent to

$$\tilde{Q}Y = QY - Y - n\,\eta(Y)\xi \tag{3.5}$$

implies

$$\tilde{r} = r.$$

Here $\tilde{S}$, $\tilde{Q}$ and $\tilde{r}$ denote the Ricci tensor, Ricci operator and scalar curvature with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$. From the last equation, it is obvious that the scalar curvature with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ coincide on M. Setting $Z = \xi$ in (3.3) and then the equations (2.1), (2.3) and (2.6) follows

$$\tilde{R}(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\}. \tag{3.6}$$

This reflects that the LP-Sasakian manifold equipped with $\tilde{\nabla}$ is regular. Also we have

$$\tilde{R}(\xi,X)Y = -2\eta(Y)\{X + \eta(X)\xi\}, \tag{3.7}$$

$$\tilde{S}(X,\xi) = 2(n-1)\eta(X). \tag{3.8}$$

The projective curvature tensor $\tilde{P}$ with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ on an LP-Sasakian manifold M is defined by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{3.9}$$

for arbitrary vector fields X, Y and Z [23]. From (3.6)-(3.9), we can find that

$$\tilde{P}(X,Y)\xi = 0.$$

Thus we can state:

Lemma 3.1. *An n-dimensional LP-Sasakian manifold M endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is ξ-projective flat with respect to $\tilde{\nabla}$.*

§4. Partially Ricci semisymmetric LP-Sasakian manifolds with respect to a quarter-symmetric non-metric connection

The objective of this section is to study the properties of partially Ricci semisymmetric LP-Sasakian manifolds equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$. Before going to prove our results, we give the following definition.

Definition 4.1. *An n-dimensional LP-Sasakian manifold M equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is said to be partially Ricci semisymmetric with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ if $\tilde{R}(\xi, X) \cdot \tilde{S} = 0$ holds for all vector field X on M.*

Theorem 4.2. *An n-dimensional LP-Sasakian manifold M, $n > 2$, equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is partially Ricci semisymmetric with respect to $\tilde{\nabla}$ if and only if M is an η-Einstein manifold with respect to the Levi-Civita connection ∇.*

Proof. We have

$$(\tilde{R}(X,Y) \cdot \tilde{S})(Z,U) = -\tilde{S}(\tilde{R}(X,Y)Z,U) - \tilde{S}(Z, \tilde{R}(X,Y)U).$$

Setting $X = \xi$ in the above equation and then using (3.4), (3.7) and (3.8) we obtain

$$\begin{aligned}(\tilde{R}(\xi,Y) \cdot \tilde{S})(Z,U) &= 2\eta(Z)\{S(Y,U) - g(Y,U) + (n-2)\eta(Y)\eta(U)\} \\ &\quad +2\eta(U)\{S(Y,Z) - g(Y,Z) + (n-2)\eta(Y)\eta(Z)\}.\end{aligned} \tag{4.1}$$

If possible, we suppose that the LP-Sasakian manifold M is partially Ricci semisymmetric with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$, then the equation (4.1) turns into the form

$$\begin{aligned}&\eta(Z)\{S(Y,U) - g(Y,U) + (n-2)\eta(Y)\eta(U)\} \\ &\qquad +\eta(U)\{S(Y,Z) - g(Y,Z) + (n-2)\eta(Y)\eta(Z)\} = 0.\end{aligned}$$

Replacing U with ξ in the above equation and then using (2.1), (2.3) and (2.8), we get

$$S(Y,Z) = g(Y,Z) - (n-2)\eta(Y)\eta(Z), \tag{4.2}$$

which shows that the manifold M under consideration is an η-Einstein manifold with respect to the Levi-Civita connection ∇. Conversely, if possible, we consider that the LP-Sasakian manifold M equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$ satisfies the equation (4.2). Thus the equations (4.1) and (4.2) reflect that $\tilde{R}(\xi, Y) \cdot \tilde{S} = 0$. Hence the statement of the Theorem 4.2 is proved. □

In consequence of (4.2), equation (3.4) assumes the form

$$\tilde{S}(Y,Z) = -2(n-1)\eta(Y)\eta(Z). \tag{4.3}$$

Thus we can state the following corollary.

Corollary 4.3. *A partially Ricci semisymmetric LP-Sasakian manifold M, $n > 2$, endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ satisfies $\tilde{S} = -2(n-1)\eta \otimes \eta$.*

In the next theorem, we establish the relation between the Weyl conformal curvature and projective curvature tensors for the connections ∇ and $\tilde{\nabla}$, respectively.

Theorem 4.4. *Let M be an n-dimensional LP-Sasakian manifold equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$. Then M is partially Ricci semisymmetric with respect to the connection $\tilde{\nabla}$ if and only if the Weyl conformal curvature tensor with respect to the Levi-Civita connection ∇ is equal to the projective curvature tensor with respect to $\tilde{\nabla}$ and $r = 2(n-1)$.*

Proof. We suppose that the LP-Sasakian manifold M of dimension n, $n > 2$, equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is partially Ricci semisymmetric with respect to $\tilde{\nabla}$. Then the Theorem 4.2 tells us that the equations (4.2) and (4.3) are satisfied. From the equation (4.2), it is obvious that

$$QY = Y - (n-2)\eta(Y)\xi \quad \text{and} \quad r = 2(n-1). \tag{4.4}$$

Using the equations (4.2) and (4.4) in the equation (2.10), we have

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - [&\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &- g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{4.5}$$

The equation (3.3) together with the equation (4.5) give

$$C(X,Y)Z = \tilde{R}(X,Y)Z + 2\{\eta(Y)X - \eta(X)Y\}\eta(Z). \tag{4.6}$$

In light of the equations (3.9) and (4.2)-(4.6), we get

$$\tilde{P} = C \quad \text{and} \quad \tilde{r} = r = 2(n-1). \tag{4.7}$$

To prove the converse part, we assume that the LP-Sasakian manifold M endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ satisfies the equation (4.7). From (2.10), (3.9) and (4.7), we have

$$\begin{aligned} R(X,Y)Z - \frac{1}{n-2}\{&S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\} + \frac{2}{n-2}\{g(Y,Z)X - g(X,Z)Y\} \\ &= \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\}. \end{aligned}$$

Substituting $Z = \xi$ in the above equation and then the equations (2.3), (2.6), (2.8), (3.8) along with the Lemma 3.1 follow that

$$\eta(Y)QX - \eta(X)QY = \eta(Y)X - \eta(X)Y.$$

Replacing X with ξ in the above equation, we obtain

$$QY = Y - (n-2)\eta(Y)\xi,$$

which is equivalent to the equation (4.2) and hence the Theorem 4.2 shows that the manifold M under the consideration is partially Ricci semisymmetric with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$. Thus the proof is completed. □

§5. Semisymmetric LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection

In this section, we will study the properties of semisymmetric LP-Sasakian manifold with respect to the quarter-symmetric non-metric connection. We prove the following theorems.

Theorem 5.1. *Every n-dimensional semisymmetric LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection $\tilde{\nabla}$ is conformally flat with respect to the Levi-Civita connection.*

Proof. Let M be an n-dimensional LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$. It is obvious that

$$\begin{aligned}(\tilde{R}(X,Y)\cdot\tilde{R})(Z,U)V = {} & \tilde{R}(X,Y)\,\tilde{R}(Z,U)V - \tilde{R}(\tilde{R}(X,Y)Z,U)V \\ & -\tilde{R}(Z,\tilde{R}(X,Y)U)V - \tilde{R}(Z,U)\tilde{R}(X,Y)V.\end{aligned}$$

If possible, we assume that M is semisymmetric with respect to $\tilde{\nabla}$, that is, $\tilde{R}\cdot\tilde{R} = 0$. Using this fact along with $X = \xi$ in the above equation, we get

$$\begin{aligned}&\tilde{R}(\xi,Y)\,\tilde{R}(Z,U)V - \tilde{R}(\tilde{R}(\xi,Y)Z,U)V \\ &\qquad -\tilde{R}(Z,\tilde{R}(\xi,Y)U)V - \tilde{R}(Z,U)\tilde{R}(\xi,Y)V = 0.\end{aligned}$$

In view of (2.1), (2.2) and (3.7), the above equation takes the form

$$\begin{aligned}&\eta(\tilde{R}(Z,U)V)\{Y+\eta(Y)\xi\} = \eta(Z)\{\tilde{R}(Y,U)V + \eta(Y)\tilde{R}(\xi,U)V\} \\ &\quad +\eta(U)\{\tilde{R}(Z,Y)V + \eta(Y)\tilde{R}(Z,\xi)V\} + \eta(V)\{\tilde{R}(Z,U)Y + \eta(Y)\tilde{R}(Z,U)\xi\}.\end{aligned}$$

Setting $V = \xi$ in the above equation and then using the equation (2.1), we get

$$\begin{aligned}&\eta(\tilde{R}(Z,U)\xi)\{Y+\eta(Y)\xi\} = \eta(Z)\{\tilde{R}(Y,U)\xi + \eta(Y)\tilde{R}(\xi,U)\xi\} \\ &+\eta(U)\{\tilde{R}(Z,Y)\xi + \eta(Y)\tilde{R}(Z,\xi)\xi\} - \{\tilde{R}(Z,U)Y + \eta(Y)\tilde{R}(Z,U)\xi\}.\end{aligned} \tag{5.1}$$

From the equations (2.1), (2.3), (3.6) and (3.7), we can find that

$$\eta(\tilde{R}(X,Y)\xi) = 0, \quad \tilde{R}(X,\xi)\xi = -2\{X + \eta(X)\xi\} = -\tilde{R}(\xi, X)\xi. \tag{5.2}$$

With the help of the equations (2.2), (3.6), (3.7) and (5.2), the equation (5.1) takes the form

$$\tilde{R}(Z,U)Y = 2\eta(Y)\{\eta(Z)U - \eta(U)Z\}, \tag{5.3}$$

which is equivalent to

$$R(Z,U)Y = \{\eta(U)g(Y,Z) - \eta(Z)g(U,Y)\}\xi + \eta(Y)\{\eta(Z)U - \eta(U)Z\}, \tag{5.4}$$

where equation (3.3) is used. Contracting the equation (5.4) along the vector field Z, we find

$$S(U,Y) = g(U,Y) - (n-2)\eta(Y)\eta(U). \tag{5.5}$$

This shows that the manifold under consideration is an η-Einstein manifold. From (5.5) we have

$$QU = U - (n-2)\eta(U)\xi \Longrightarrow r = 2(n-1). \tag{5.6}$$

Thus the scalar curvature of the semisymmetric LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection is constant. In consequence of (2.10) and (5.4)-(5.6), we conclude that $C = 0$. That is the semisymmetric LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is conformally flat. Hence we have the statement of Theorem 5.1. □

It is well known that a conformally flat LP-Sasakian manifold is locally isometric to a sphere $S^n(1)$. Thus we have the following corollary.

Corollary 5.2. *An n-dimensional semisymmetric LP-Sasakian manifold M endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is locally isometric to the sphere $S^n(1)$.*

Theorem 5.3. *Suppose an n-dimensional semisymmetric LP-Sasakian manifold M admits a quarter-symmetric non-metric connection $\tilde{\nabla}$. Then M is projectively flat with respect to $\tilde{\nabla}$.*

Proof. Let an n-dimensional LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$ is semisymmetric with respect to $\tilde{\nabla}$. Thus the equation (5.3) holds on M. The contraction of (5.3) along the vector field Z gives

$$\tilde{S}(U,Y) = -2(n-1)\eta(U)\eta(Y) \Longleftrightarrow \tilde{Q}U = -2(n-1)\eta(U)\xi. \tag{5.7}$$

Equations (3.9), (5.3) and (5.7) prove the statement of the Theorem 5.3. □

§6. Some properties of the Ricci tensor with respect to a quarter-symmetric non-metric connection

This section deals with the study of η-parallel and recurrent Ricci tensors of an n-dimensional LP-Sasakian manifold M equipped with a quarter-symmetric non-metric connection $\tilde{\nabla}$.

In light of the equations (2.2)-(2.5), (3.1) and (3.5), we obtain

$$(\tilde{\nabla}_X \tilde{Q})(Y) = (\nabla_X Q)(Y) - n(\nabla_X \eta)(Y)\xi - n\eta(Y)\phi X - 2\eta(X)\phi Y. \tag{6.1}$$

From (3.1) and (3.2), we have

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = g((\tilde{\nabla}_X \tilde{Q})(Y), Z) + 2\eta(X)\tilde{S}(Y, \phi Z). \tag{6.2}$$

The inner product of (6.1) with Z and then use of (6.2) give

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Y, Z) &= (\nabla_X S)(Y, Z) - n(\nabla_X \eta)(Y)\eta(Z) - n\eta(Y)g(\phi X, Z) \\ &\quad + 2\eta(X)\{\tilde{S}(Y, \phi Z) - g(Y, \phi Z)\}.\end{aligned} \tag{6.3}$$

Changing Y and Z with ϕY and ϕZ in the equation (6.3), we get

$$(\tilde{\nabla}_X \tilde{S})(\phi Y, \phi Z) = (\nabla_X S)(\phi Y, \phi Z) + 2\eta(X)\{\tilde{S}(Z, \phi Y) - g(Z, \phi Y)\}. \tag{6.4}$$

We suppose that the η-parallel Ricci tensor with respect to the quarter-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection of an LP-Sasakian manifold coincide, then the equations (2.9), (3.4) and (6.4) give

$$S(Y, Z) = 2g(Y, Z) - (n-3)\eta(Y)\eta(Z). \tag{6.5}$$

This shows that the manifold M under assumption is an η-Einstein manifold, provided $n > 3$. The converse part is obvious from (2.9), (3.4), (6.4) and (6.5). Thus we are in position to state the following theorem.

Theorem 6.1. *If an LP-Sasakian manifold M of dimension $n(> 3)$ admits a quarter-symmetric non-metric connection $\tilde{\nabla}$. Then the η-parallel Ricci tensors of M with respect to the quarter-symmetric non-metric connection and the Levi-Civita connection are equal if and only if M is an η-Einstein manifold.*

Now, we discuss the existence of Ricci recurrent LP-Sasakian manifold endowed with a quarter-symmetric non-metric connection $\tilde{\nabla}$. Let an LP-Sasakian manifold M equipped with $\tilde{\nabla}$ is Ricci recurrent with respect to $\tilde{\nabla}$, that is, M possesses a Ricci tensor $\tilde{S}$ satisfies

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = A(X)\tilde{S}(Y, Z). \tag{6.6}$$

Setting $Z = \xi$ in (6.6), we have

$$(\tilde{\nabla}_X \tilde{S})(Y, \xi) = A(X)\tilde{S}(Y, \xi).$$

With the help of (2.1), (2.2), (3.8) and (6.3), the above equation assumes the form

$$(\nabla_X S)(Y, \xi) + n(\nabla_X \eta)(Y) = 2(n-1)A(X)\eta(Y). \tag{6.7}$$

It is well known that

$$\begin{aligned}(\nabla_X S)(Y, \xi) &= \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi) \\ &= (n-1)(\nabla_X \eta)(Y) - S(Y, \phi X).\end{aligned} \tag{6.8}$$

From (6.7) and (6.8), we have

$$S(Y, \phi X) = (2n-1)g(\phi X, Y) - 2(n-1)A(X)\eta(Y). \tag{6.9}$$

Replacing Y with ϕY in (6.9), we find

$$S(X, Y) = (2n-1)g(X, Y) + n\eta(X)\eta(Y).$$

This reflects that the manifold under consideration is an η-Einstein manifold. Again changing X by ϕX in (6.9), we conclude that

$$A(\phi X) = 0 \Longrightarrow AX = -A(\xi)\eta(X). \tag{6.10}$$

Again, setting $X = \xi$ in (6.9), we get $A(\xi) = 0$ and therefore the equation (6.10) reveals that $A(X) = 0$. Thus we can state the following theorem.

Theorem 6.2. *There does not exit a Ricci recurrent LP-Sasakian manifold with respect to a quarter-symmetric non-metric connection.*

§7. Example

In this section, we give an example of LP-Sasakian manifold M of dimension 4 and prove the existence of a quarter-symmetric non-metric connection on M.

Let M be a differentiable manifold of dimension 4, defined by

$$M = \{(x_1, x_2, x_3, x_4) : x_i \in \mathcal{R},\ x_4 \neq 0,\ i = 1, 2, 3, 4\},$$

where $\mathcal{R}$ denotes the set of real numbers. Let

$$e_1 = \frac{x_1}{x_4}\frac{\partial}{\partial x_1}, \qquad e_2 = \frac{x_2}{x_4}\frac{\partial}{\partial x_2}, \quad e_3 = \frac{x_3}{x_4}\frac{\partial}{\partial x_3}, \qquad e_4 = x_4\frac{\partial}{\partial x_4}$$

be a set of linearly independent vector fields of M. From the above equations, we can easily find that the non-vanishing components of the Lie bracket are

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3.$$

Let $\xi = e_4$ and g is the Lorentzian metric of M defined by

$$g(e_i, e_j) = \begin{cases} -1, & \text{if} \quad i = j = 4 \\ 1, & \text{if} \quad i = j = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases},$$

where $i, j = 1, 2, 3, 4$. The associated 1-form η corresponding to the metric g is given by $g(X, e_4) = \eta(X)$ and the linear function ϕ of M is defined by $\phi e_i = e_i$ for $i = 1, 2, 3$ and $\phi e_4 = 0$. With help of the above equations, we can observe that the equations (2.1)-(2.3) hold for all $e_i, i = 1, 2, 3, 4$. Above relations together with the Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

give

$$\begin{array}{llll} \nabla_{e_1} e_1 = e_4, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = e_4, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = e_4, & \nabla_{e_3} e_4 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = 0, \end{array}$$

where ∇ denotes the Levi-Civita connection corresponding to the metric g. This shows that $\nabla_X e_4 = \phi X$ for all X of M. Thus (M, g) is a 4-dimensional LP-Sasakian manifold. In consequence of the above equations and (3.1), we get

$$\begin{array}{llll} \tilde{\nabla}_{e_1} e_1 = e_4, & \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_1} e_3 = 0, & \tilde{\nabla}_{e_1} e_4 = e_1, \\ \tilde{\nabla}_{e_2} e_1 = 0, & \tilde{\nabla}_{e_2} e_2 = e_4, & \tilde{\nabla}_{e_2} e_3 = 0, & \tilde{\nabla}_{e_2} e_4 = e_2, \\ \tilde{\nabla}_{e_3} e_1 = 0, & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_3 = e_4, & \tilde{\nabla}_{e_3} e_4 = e_3, \\ \tilde{\nabla}_{e_4} e_1 = e_1, & \tilde{\nabla}_{e_4} e_2 = e_2, & \tilde{\nabla}_{e_4} e_3 = e_3, & \tilde{\nabla}_{e_4} e_4 = 0. \end{array}$$

Equation (1.1) together with the above equations give the non-zero components of torsion tensor with respect to the connection $\tilde{\nabla}$ as:

$$\tilde{T}(e_1, e_4) = -e_1, \quad \tilde{T}(e_2, e_4) = -e_2, \quad \tilde{T}(e_3, e_4) = -e_3.$$

Let X and Y are arbitrary vector fields of M, then it can be expressed as $X = X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4$, $Y = Y^1 e_1 + Y^2 e_2 + Y^3 e_3 + Y^4 e_4$, where X^i and Y^i are scalars for $i = 1, 2, 3, 4$. We have

$$\tilde{T}(X, Y) = (X^1 Y^4 - X^4 Y^1) e_1 + (X^2 Y^4 - X^4 Y^2) e_2 + (X^3 Y^4 - X^4 Y^3) e_3$$

and

$$\eta(Y)\phi X - \eta(X)\phi Y = (X^1Y^4 - X^4Y^1)e_1 \\ +(X^2Y^4 - X^4Y^2)e_2 + (X^3Y^4 - X^4Y^3)e_3.$$

Hence the linear connection $\tilde{\nabla}$ defined by (3.1) on M is a quarter-symmetric connection. It is obvious that $(\tilde{\nabla}_{e_4}g)(e_1, e_1) = -2 \neq 0$. Therefore the quarter-symmetric connection $\tilde{\nabla}$ is non-metric.

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