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The lifespan of small solutions to a system of cubic nonlinear Schrödinger equations in one space dimension

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Abstract. We consider the initial value problem for a two-component system of cubic nonlinear Schrödinger equations in one space dimension. We provide a detailed lower bound estimate for the lifespan of the solution to the system, which can be computed explicitly from the initial data, the masses and the nonlinear term.

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§1. Introduction

We consider the following initial value problem:

$$\begin{cases} \mathcal{L}_{m_j} u_j = F_j(u), & t > 0,\ x \in \mathbb{R}, \\ u_j(0,x) = \varepsilon \varphi_j(x), & x \in \mathbb{R} \end{cases} \tag{1.1}$$

for $j = 1, \ldots, N$, where $\mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2$, $i = \sqrt{-1}$, $m_j \in \mathbb{R}\backslash\{0\}$ and $u = (u_j(t,x))_{1\le j\le N}$ is a $\mathbb{C}^N$-valued unknown function. The nonlinear term $F = (F_j)_{1\le j\le N}$ is assumed to be a cubic homogeneous polynomial in $(u, \overline{u})$. Also we assume that the system (1.1) satisfies the so-called *gauge invariance*:

$$F_j(e^{im_1\theta} z_1, \ldots, e^{im_N\theta} z_N) = e^{im_j\theta} F_j(z_1, \ldots, z_N)$$

for $j = 1, \ldots, N$ and any $\theta \in \mathbb{R}$, $z = (z_j)_{1\le j\le N} \in \mathbb{C}^N$. $\varepsilon > 0$ is a small parameter which is responsible for the size of the initial data, and $\varphi = (\varphi_j(x))_{1\le j\le N}$ is a $\mathbb{C}^N$-valued known function which belongs to $(H^1 \cap H^{0,1}(\mathbb{R}))^N$. Here and later on as well, H^s denotes the standard L^2-based Sobolev space of order s, and the weighted Sobolev space $H^{s,\sigma}$ is defined by $\{\phi \in L^2 \,|\, \langle\cdot\rangle^\sigma \phi \in H^s\}$, equipped with the norm $\|\phi\|_{H^{s,\sigma}} = \|\langle\cdot\rangle^\sigma\phi\|_{H^s}$, where $\langle x\rangle = \sqrt{1+x^2}$. We are interested in large-time behavior of the small amplitude solution for (1.1).

Let us recall the backgrounds briefly. It is well-known that cubic nonlinearity is critical when we consider large-time behavior of solutions to nonlinear Schrödinger equation in one space dimension. Because the best possible decay in L^2_x of general cubic nonlinear terms is $O(t^{-1})$, standard perturbative approach is valid only for $t \lesssim \exp(o(\varepsilon^{-2}))$ in general, and our problem is to make clear how the nonlinearity affects the behavior of the solutions for $t \gtrsim \exp(O(\varepsilon^{-2}))$. We begin with the single case ($N = 1$):

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda|u|^2u, \quad t > 0,\ x \in \mathbb{R} \tag{1.2}$$

with $\lambda \in \mathbb{R}$. According to Hayashi-Naumkin [5], the solution to (1.2) with small initial data exists globally in time and the global solution behaves like

$$u(t,x) = \frac{1}{\sqrt{it}}\tilde{\alpha}(x/t)\exp\left(i\frac{x^2}{2t} - i\lambda|\tilde{\alpha}(x/t)|^2\log t\right) + o(t^{-1/2})$$

as $t \to +\infty$ uniformly in $x \in \mathbb{R}$, where $\tilde{\alpha}(\xi)$ is a suitable $\mathbb{C}$-valued function of $\xi \in \mathbb{R}$ satisfying $|\tilde{\alpha}(\xi)| \lesssim \varepsilon$. An important consequence of this asymptotic expression is that the solution decays like $O(t^{-1/2})$ in $L^\infty(\mathbb{R}_x)$, while it does not behave like the free solution unless $\lambda = 0$. In other words, the additional logarithmic factor in the phase reflects the long-range character of the cubic nonlinear Schrödinger equations in one space dimension. If $\lambda \in \mathbb{C}$ in (1.2), another kind of long-range effect can be observed. Shimomura [20] showed that the small data solution to (1.2) exists globally in time and decays like $O(t^{-1/2}(\log t)^{-1/2})$ in $L^\infty(\mathbb{R}_x)$ as $t \to \infty$ if $\operatorname{Im}\lambda < 0$. This gain of additional logarithmic time decay should be interpreted as another kind of long-range effect. If $\operatorname{Im}\lambda > 0$, Sunagawa [21] and Sagawa-Sunagawa [17] have derived the following more precise estimate for the lifespan T_ε of the solution to (1.2) with initial data $u(0,x) = \varepsilon\phi(x)$:

$$\liminf_{\varepsilon\to+0}(\varepsilon^2\log T_\varepsilon) \geq \frac{1}{2\operatorname{Im}\lambda \sup_{\xi\in\mathbb{R}}|\hat{\phi}(\xi)|^2}, \tag{1.3}$$

where $1/0$ is understood as $+\infty$, and $\hat{\phi}$ denotes the Fourier transform of ϕ, i.e.,

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iy\xi}\phi(y)\,dy, \quad \xi \in \mathbb{R}.$$

This estimate tells us the dependence of T_ε on $\operatorname{Im}\lambda$. Roughly speaking, the estimate (1.3) is derived from the ordinary differential equation

$$\begin{cases} i\partial_t f(t,\xi) = \frac{\lambda}{t}|f(t,\xi)|^2 f(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ f(1,\xi) = \varepsilon\hat{\phi}(\xi), & \xi \in \mathbb{R}. \end{cases}$$

This equation can be solved explicitly as follows:

$$|f(t,\xi)|^2 = \frac{\varepsilon^2|\hat{\phi}(\xi)|^2}{1 - 2\operatorname{Im}\lambda|\hat{\phi}(\xi)|^2\varepsilon^2\log t}$$

as long as the denominator is strictly positive. Hence the solution $f(t,\xi)$ blows up at

$$\varepsilon^2 \log t = \frac{1}{2\operatorname{Im}\lambda \sup_{\xi\in\mathbb{R}} |\hat{\phi}(\xi)|^2}.$$

This observation implies that small data solution $u(t,x)$ of (1.2) with $\operatorname{Im}\lambda > 0$ may blow up in finite time. An example of blowing-up solution to (1.2) with arbitrarily small $\varepsilon > 0$ has been given by Kita [11] under a particular choice of ϕ when $\operatorname{Im}\lambda > 0$. However, it seems difficult to specify the lifespan for the blowing-up solution given in [11], and the optimality of (1.3) is left to be unknown. Next let us turn our attentions to the system case ($N \geq 2$). An interesting feature in the system case is that the behavior of solutions are affected by the combinations of the masses as well as the structure of the nonlinearity (see e.g., [2], [3], [4], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16], [19], [22], etc.). In [13], several structural conditions on F have been introduced under which small data global existence holds, and time-decay properties of the global solutions have been investigated. As a result, we come up with the following question: *what happens if the structural conditions on F given in* [13] *are violated*? However it seems difficult to treat the general N-component system (1.1). As the first step we consider the following two-component system:

$$\text{(1.4)} \quad \begin{cases} \mathcal{L}_{m_1} u = \lambda |v|^2 u, & t > 0,\ x \in \mathbb{R}, \\ \mathcal{L}_{m_2} v = \mu |u|^2 v, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \varepsilon\varphi(x),\ v(0,x) = \varepsilon\psi(x), & x \in \mathbb{R} \end{cases}$$

with m_1, $m_2 \in \mathbb{R}\backslash\{0\}$, λ, $\mu \in \mathbb{C}$ and φ, $\psi \in H^1 \cap H^{0,1}(\mathbb{R})$. The approach of Li-Sunagawa [13] implies small data global existence and time decay of the global solution for (1.4) under the either of the following three conditions:

- $\operatorname{Im}\lambda < 0$,
- $\operatorname{Im}\mu < 0$,
- $\operatorname{Im}\lambda = \operatorname{Im}\mu = 0$.

According to [13], large-time behavior of the solution for (1.4) deeply relates to the following system of ordinary differential equations:

$$\text{(1.5)} \begin{cases} i\partial_t f(t,\xi) = \frac{\lambda}{t}|g(t,\xi)|^2 f(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ i\partial_t g(t,\xi) = \frac{\mu}{t}|f(t,\xi)|^2 g(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ f(1,\xi) = \varepsilon\mathcal{F}_{m_1}\varphi(\xi),\ g(1,\xi) = \varepsilon\mathcal{F}_{m_2}\psi(\xi), & \xi \in \mathbb{R}, \end{cases}$$

where $\mathcal{F}_m$ denotes the scaled Fourier transform which will be defined in the next section. We note that global existence and boundedness of the solution to the reduced system (1.5) holds in this case. We check it for $\operatorname{Im}\lambda < 0$ (the same is true for the other cases). Multiplying the equations of system (1.5) by $\overline{f}$ and $\overline{g}$ respectively, and taking the imaginary part of the result, we obtain

$$\begin{cases} \partial_t(|f(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|f(t,\xi)|^2|g(t,\xi)|^2, & t>1,\ \xi\in\mathbb{R}, \\ \partial_t(|g(t,\xi)|^2) = \frac{2\operatorname{Im}\mu}{t}|f(t,\xi)|^2|g(t,\xi)|^2, & t>1,\ \xi\in\mathbb{R}, \\ |f(1,\xi)|^2 = \varepsilon^2|\mathcal{F}_{m_1}\varphi(\xi)|^2,\ |g(1,\xi)|^2 = \varepsilon^2|\mathcal{F}_{m_2}\psi(\xi)|^2, & \xi\in\mathbb{R}. \end{cases}$$

Therefore we see

$$\partial_t(|\operatorname{Im}\mu||f|^2 - \operatorname{Im}\lambda|g|^2) = \frac{2\operatorname{Im}\lambda(|\operatorname{Im}\mu| - \operatorname{Im}\mu)}{t}|f|^2|g|^2 \le 0$$

for $\operatorname{Im}\mu \neq 0$, and

$$\partial_t(|f|^2 + |g|^2) = \frac{2\operatorname{Im}\lambda}{t}|f|^2|g|^2 \le 0$$

for $\operatorname{Im}\mu = 0$. Hence we obtain

$$|f(t,\xi)|^2 + |g(t,\xi)|^2 \le C\varepsilon^2(|\mathcal{F}_{m_1}\varphi(\xi)|^2 + |\mathcal{F}_{m_2}\psi(\xi)|^2)$$

for $t \ge 1$ and some constant $C > 0$. This observation yields global existence and time decay of solutions to (1.4) (see [13] for details). However the remaining cases are left unsolved so far, that is,

- $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$,
- $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu = 0$,
- $\operatorname{Im}\lambda = 0$ and $\operatorname{Im}\mu > 0$.

The aim of this paper is to clarify large-time behavior of the solution to (1.4) with $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$. Since the solution of the reduced system (1.5) blows up at finite time in this case (see Section 3 for details), it could be natural to expect that the lifespan of the solution to the original system (1.4) is characterized by the blow-up time of the solution to the reduced system (1.5). We will justify the half of this expectation.

To state the main result, let us define $\tau_0 \in (0, +\infty]$ by

$$\tau_0 := \frac{1}{2}\inf_{\xi\in\mathbb{R}}\left\{\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}\right\}. \tag{1.6}$$

We remark that if $\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2 = \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2$ at $\xi^* \in \mathbb{R}$, then we define

$$\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi^*)|^2} = \frac{1}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi^*)|^2}.$$

We also remark that the right-hand side of (1.6) is always non-negative. Because of $|\mathcal{F}_{m_1}\varphi(\xi)| < +\infty$, $|\mathcal{F}_{m_2}\psi(\xi)| < +\infty$ and mean value theorem, we have

$$\frac{\log(\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2) - \log(\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2)}{\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2 - \operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}$$
$$\geq \min\left\{\inf_{\xi\in\mathbb{R}}\left(\frac{1}{2\operatorname{Im}\mu|\mathcal{F}_{m_1}\varphi(\xi)|^2}\right), \inf_{\xi\in\mathbb{R}}\left(\frac{1}{2\operatorname{Im}\lambda|\mathcal{F}_{m_2}\psi(\xi)|^2}\right)\right\} > 0.$$

The main result of this paper is as follows:

Theorem 1.1. *Assume that* φ, $\psi \in H^1 \cap H^{0,1}(\mathbb{R})$, *and that* λ, $\mu \in \mathbb{C}$ *with* $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\mu > 0$. *Let* T_ε *be the supremum of* $T > 0$ *such that* (1.4) *admits a unique solution in* $\left(C([0,T); H^1 \cap H^{0,1}(\mathbb{R}))\right)^2$. *Then we have*

$$\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon \geq \tau_0, \tag{1.7}$$

where $\tau_0 \in (0, +\infty]$ *is given by* (1.6).

Remark 1.2. *From the main result, we clarify that the lower bound estimate for the lifespan of small solutions to* (1.4) *holds when* $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\mu > 0$. *Moreover the estimate* (1.7) *is different from single case one* (1.3) *in general. It is caused by the initial data and the structure of the nonlinearities on the system* (1.4) *(see Section 3 for details). Therefore Theorem 1.1 tells us another kind of large-time behavior of solutions which does not correspond to the single case and heavily depends on the initial data and the structure of the nonlinearities on the system. This is new knowledge on the system case. However the author does not know whether* (1.7) *is optimal or not.*

Remark 1.3. *As for the remaining cases, that is,*

- $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\mu = 0$,
- $\operatorname{Im}\lambda = 0$ *and* $\operatorname{Im}\mu > 0$,

solutions of the reduced system (1.5) *grow up at* $t \to +\infty$. *Therefore it is natural to expect that solutions of the original system* (1.4) *also grow up at* $t \to +\infty$. *However the author does not know whether this expectation is true or not. Even the small data global existence is not trivial at all, and what we can show by the present approach is only* $\liminf_{\varepsilon\to+0} \varepsilon^2 \log T_\varepsilon = +\infty$.

We close this section with the contents of this paper. In the next section, we state preliminaries. Section 3 is devoted to a lemma on some system of ordinary differential equations. In this section, we derive the Riccati-type differential equation from the reduced system (1.5). This is the new ingredient of the proof. After that, we will get an a priori estimate in Section 4, and the main theorem will be proved in Section 5. In what follows, we denote several positive constants by C, which may vary from one line to another.

§2. Preliminaries

In this section, we summarize basic facts related to the Schrödinger operator $\mathcal{L}_m = i\partial_t + \frac{1}{2m}\partial_x^2$. We set $\mathcal{J}_m(t) = x + \frac{it}{m}\partial_x$. It is well-known that this operator has good compatibility with $\mathcal{L}_m$ as follows:

$$[\mathcal{L}_m, \mathcal{J}_m(t)] = 0,\ [\partial_x, \mathcal{J}_m(t)] = 1,$$

where $[\cdot,\cdot]$ stands for the commutator of two linear operators. Next we set the free Schrödinger evolution operator

$$(\mathcal{U}_m(t)\phi)(x) := e^{i\frac{t}{2m}\partial_x^2}\phi(x) = \sqrt{\frac{|m|}{2\pi t}}e^{-i\frac{\pi}{4}\operatorname{sgn}(m)}\int_{\mathbb{R}} e^{im\frac{(x-y)^2}{2t}}\phi(y)dy$$

for $m \in \mathbb{R}\backslash\{0\}$ and $t > 0$. We also introduce the scaled Fourier transform $\mathcal{F}_m$ by

$$(\mathcal{F}_m\phi)(\xi) := |m|^{1/2}e^{-i\frac{\pi}{4}\operatorname{sgn}(m)}\hat{\phi}(m\xi) = \sqrt{\frac{|m|}{2\pi}}e^{-i\frac{\pi}{4}\operatorname{sgn}(m)}\int_{\mathbb{R}} e^{-imy\xi}\phi(y)dy$$

as well as auxiliary operators

$$(\mathcal{M}_m(t)\phi)(x) := e^{im\frac{x^2}{2t}}\phi(x),\ (\mathcal{D}(t)\phi)(x) := \frac{1}{\sqrt{t}}\phi\left(\frac{x}{t}\right),$$

$$\mathcal{W}_m(t)\phi := \mathcal{F}_m\mathcal{M}_m(t)\mathcal{F}_m^{-1}\phi$$

so that $\mathcal{U}_m(t)$ can be decomposed into

$$\mathcal{U}_m(t) = \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{F}_m\mathcal{M}_m(t) = \mathcal{M}_m(t)\mathcal{D}(t)\mathcal{W}_m(t)\mathcal{F}_m.$$

In what follows, we will occasionally omit "(t)" from $\mathcal{J}_m(t)$, $\mathcal{U}_m(t)$, $\mathcal{M}_m(t)$ and $\mathcal{W}_m(t)$, if it causes no confusion.

Lemma 2.1. *Let m, μ_1, μ_2, μ_3 be non-zero real constants satisfying $m = \mu_1 + \mu_2 + \mu_3$. For smooth $\mathbb{C}$-valued functions f_1, f_2 and f_3, we have*

$$\mathcal{J}_m(f_1 f_2 \overline{f_3}) = \frac{\mu_1}{m}(\mathcal{J}_{\mu_1} f_1) f_2 \overline{f_3} + \frac{\mu_2}{m} f_1 (\mathcal{J}_{\mu_2} f_2)\overline{f_3} + \frac{\mu_3}{m} f_1 f_2 (\overline{\mathcal{J}_{-\mu_3} f_3}).$$

Lemma 2.2. *Let m be a non-zero real constant. We have*

$$\|\phi - \mathcal{M}_m\mathcal{D}\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} \le Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

and

$$\|\phi\|_{L^\infty} \le t^{-1/2}\|\mathcal{F}_m\mathcal{U}_m^{-1}\phi\|_{L^\infty} + Ct^{-3/4}(\|\phi\|_{L^2} + \|\mathcal{J}_m\phi\|_{L^2})$$

for $t \ge 1$.

Lemma 2.3. *Let m be a non-zero real constant. For smooth $\mathbb{C}$-valued functions f_1, f_2 and f_3, we have*

$$\|\mathcal{F}_m\mathcal{U}_m^{-1}(f_1 f_2 f_3)\|_{L^\infty} \le C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^\infty}.$$

We skip the proof of these lemmas (see e.g., §3 of [13] and its references).

§3. A technical lemma

In this section, we introduce a lemma on some system of ordinary differential equations which will be used effectively in the next section. Throughout this section, we always assume that λ, $\mu \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\mu > 0$. Let φ_0, $\psi_0 : \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}} |\varphi_0(\xi)| < \infty, \ \sup_{\xi\in\mathbb{R}} |\psi_0(\xi)| < \infty.$$

We define $\tau_1 \in (0, \infty]$ by

$$\tau_1 := \frac{1}{2} \inf_{\xi\in\mathbb{R}} \left\{ \frac{\log(\operatorname{Im}\mu|\varphi_0(\xi)|^2) - \log(\operatorname{Im}\lambda|\psi_0(\xi)|^2)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 - \operatorname{Im}\lambda|\psi_0(\xi)|^2} \right\}.$$

We remark that if $\operatorname{Im}\mu|\varphi_0(\xi^*)|^2 = \operatorname{Im}\lambda|\psi_0(\xi^*)|^2$ at $\xi^* \in \mathbb{R}$, then we define

$$\frac{\log(\operatorname{Im}\mu|\varphi_0(\xi^*)|^2) - \log(\operatorname{Im}\lambda|\psi_0(\xi^*)|^2)}{\operatorname{Im}\mu|\varphi_0(\xi^*)|^2 - \operatorname{Im}\lambda|\psi_0(\xi^*)|^2} = \frac{1}{\operatorname{Im}\mu|\varphi_0(\xi^*)|^2}.$$

Let $(\alpha_0(t,\xi),\ \beta_0(t,\xi))$ be a solution to

$$\begin{cases} i\partial_t\alpha_0(t,\xi) = \frac{\lambda}{t}|\beta_0(t,\xi)|^2\alpha_0(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ i\partial_t\beta_0(t,\xi) = \frac{\mu}{t}|\alpha_0(t,\xi)|^2\beta_0(t,\xi), & t > 1,\ \xi \in \mathbb{R}, \\ \alpha_0(1,\xi) = \varepsilon\varphi_0(\xi),\ \beta_0(1,\xi) = \varepsilon\psi_0(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a parameter. If $\varphi_0(\xi^*) = 0$ or $\psi_0(\xi^*) = 0$ at $\xi^* \in \mathbb{R}$, then we can immediately solve the equation (3.1) to find that $|\alpha_0(t,\xi^*)|^2 + |\beta_0(t,\xi^*)|^2 \le C\varepsilon^2$. In what follows, we consider (3.1) at $\xi \in \mathbb{R}$ with $\varphi_0(\xi) \neq 0$ and $\psi_0(\xi) \neq 0$. At first we consider the case $\operatorname{Im}\mu|\varphi_0(\xi)|^2 > \operatorname{Im}\lambda|\psi_0(\xi)|^2$. Multiplying the equations of system (3.1) by $\overline{\alpha_0}$ and $\overline{\beta_0}$ respectively, and taking the imaginary part of the result, we have

$$\begin{cases} \partial_t(|\alpha_0(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|\alpha_0(t,\xi)|^2|\beta_0(t,\xi)|^2, & t > 1,\ \xi \in \mathbb{R}, \\ \partial_t(|\beta_0(t,\xi)|^2) = \frac{2\operatorname{Im}\mu}{t}|\alpha_0(t,\xi)|^2|\beta_0(t,\xi)|^2, & t > 1,\ \xi \in \mathbb{R}, \\ \alpha_0(1,\xi) = \varepsilon\varphi_0(\xi),\ \beta_0(1,\xi) = \varepsilon\psi_0(\xi), & \xi \in \mathbb{R}. \end{cases}$$

Therefore we see

$$\partial_t(\operatorname{Im}\mu|\alpha_0(t,\xi)|^2 - \operatorname{Im}\lambda|\beta_0(t,\xi)|^2) = 0,$$

so that

$$\begin{aligned} \operatorname{Im}\mu|\alpha_0(t,\xi)|^2 - \operatorname{Im}\lambda|\beta_0(t,\xi)|^2 &= \varepsilon^2(\operatorname{Im}\mu|\varphi_0(\xi)|^2 - \operatorname{Im}\lambda|\psi_0(\xi)|^2) \\ &=: \varepsilon^2 G(\xi), \end{aligned} \tag{3.2}$$

to obtain the Riccati-type differential equation

$$\partial_t(|\beta_0(t,\xi)|^2) = \frac{2}{t}|\beta_0(t,\xi)|^2\{\operatorname{Im}\lambda|\beta_0(t,\xi)|^2 + \varepsilon^2 G(\xi)\}.$$

Solving this Riccati-type equation, and applying the result to (3.2), we have

$$|\alpha_0(t,\xi)|^2 = \varepsilon^2\left(\frac{\operatorname{Im}\lambda}{\operatorname{Im}\mu}|\psi_0(\xi)|^2\frac{G(\xi)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2} + \frac{G(\xi)}{\operatorname{Im}\mu}\right),$$
$$|\beta_0(t,\xi)|^2 = \varepsilon^2|\psi_0(\xi)|^2\frac{G(\xi)}{\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2}$$

as long as the denominators are strictly positive. Similarly if $\operatorname{Im}\mu|\varphi_0(\xi)|^2 < \operatorname{Im}\lambda|\psi_0(\xi)|^2$, we can see that

$$|\alpha_0(t,\xi)|^2 = \varepsilon^2|\varphi_0(\xi)|^2\frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda|\psi_0(\xi)|^2 t^{-2\varepsilon^2\tilde{G}(\xi)} - \operatorname{Im}\mu|\varphi_0(\xi)|^2},$$
$$|\beta_0(t,\xi)|^2 = \varepsilon^2\left(\frac{\operatorname{Im}\mu}{\operatorname{Im}\lambda}|\varphi_0(\xi)|^2\frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda|\psi_0(\xi)|^2 t^{-2\varepsilon^2\tilde{G}(\xi)} - \operatorname{Im}\mu|\varphi_0(\xi)|^2} + \frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda}\right),$$

where $\tilde{G}(\xi) := \operatorname{Im}\lambda|\psi_0(\xi)|^2 - \operatorname{Im}\mu|\varphi_0(\xi)|^2$. At last, we consider the remaining case $\operatorname{Im}\mu|\varphi_0(\xi)|^2 = \operatorname{Im}\lambda|\psi_0(\xi)|^2$. From (3.2), we can see that $\operatorname{Im}\mu|\alpha_0(t,\xi)|^2 = \operatorname{Im}\lambda|\beta_0(t,\xi)|^2$ to obtain

$$\partial_t(|\beta_0(t,\xi)|^2) = \frac{2\operatorname{Im}\lambda}{t}|\beta_0(t,\xi)|^4.$$

Solving this equation, we have

$$|\alpha_0(t,\xi)|^2 = \frac{\varepsilon^2|\varphi_0(\xi)|^2}{1-2\varepsilon^2|\psi_0(\xi)|^2\operatorname{Im}\lambda\log t},\quad |\beta_0(t,\xi)|^2 = \frac{\varepsilon^2|\psi_0(\xi)|^2}{1-2\varepsilon^2|\psi_0(\xi)|^2\operatorname{Im}\lambda\log t}.$$

Note that the solution $(\alpha_0(t,\xi),\, \beta_0(t,\xi))$ blows up at the time $t = e^{\tau_1/\varepsilon^2}$, which comes from the minimum time that the denominators $\operatorname{Im}\mu|\varphi_0(\xi)|^2 t^{-2\varepsilon^2 G(\xi)} - \operatorname{Im}\lambda|\psi_0(\xi)|^2 = 0$ at some $\xi \in \mathbb{R}$. This is the reason why τ_0 appears in the lower bound estimate (1.7). Therefore we see that

$$\sup_{(t,\xi)\in[1,e^{\sigma/\varepsilon^2}]\times\mathbb{R}}(|\alpha_0(t,\xi)|^2 + |\beta_0(t,\xi)|^2) \le C_1^2\varepsilon^2 \tag{3.3}$$

for $\sigma \in (0,\tau_1)$, where

$$C_1 = \sqrt{\max\{A,B,D\}}$$

and

$$A = \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}>1}}\left[(|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2)\frac{\frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2} - 1}{\left(\frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}\right)^{1-\frac{\sigma}{\tau_1}} - 1} + \frac{G(\xi)}{\operatorname{Im}\mu}\right],$$

$$B = \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}<1}} \left[(|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2) \frac{\frac{\operatorname{Im}\lambda|\psi_0(\xi)|^2}{\operatorname{Im}\mu|\varphi_0(\xi)|^2} - 1}{\left(\frac{\operatorname{Im}\lambda|\psi_0(\xi)|^2}{\operatorname{Im}\mu|\varphi_0(\xi)|^2}\right)^{1-\frac{\sigma}{\tau_1}} - 1} + \frac{\tilde{G}(\xi)}{\operatorname{Im}\lambda} \right],$$

$$D = \frac{1}{1-\frac{\sigma}{\tau_1}} \sup_{\substack{\xi\in\mathbb{R},\\ \frac{\operatorname{Im}\mu|\varphi_0(\xi)|^2}{\operatorname{Im}\lambda|\psi_0(\xi)|^2}=1}} (|\varphi_0(\xi)|^2 + |\psi_0(\xi)|^2).$$

Next we consider a perturbation of (3.1). Let $T > 1$ and let $\varphi_1,\ \psi_1 : \mathbb{R} \to \mathbb{C}$, $\rho,\ \nu : [1, T) \times \mathbb{R} \to \mathbb{C}$ be continuous functions satisfying

$$\sup_{\xi\in\mathbb{R}}(|\varphi_1(\xi)|+|\psi_1(\xi)|) \le C_2\varepsilon^{1+\delta}, \qquad \sup_{(t,\xi)\in[1,T)\times\mathbb{R}} t^{1+\omega}(|\rho(t,\xi)|+|\nu(t,\xi)|) \le C_3\varepsilon^{1+\delta}$$

with some positive constants C_2, C_3, δ and ω. Let $(\alpha_1(t,\xi),\ \beta_1(t,\xi))$ be the solution to

$$\begin{cases} i\partial_t\alpha_1(t,\xi) = \frac{\lambda}{t}|\beta_1(t,\xi)|^2\alpha_1(t,\xi) + \rho(t,\xi), & t>1,\ \xi\in\mathbb{R}, \\ i\partial_t\beta_1(t,\xi) = \frac{\mu}{t}|\alpha_1(t,\xi)|^2\beta_1(t,\xi) + \nu(t,\xi), & t>1,\ \xi\in\mathbb{R}, \\ \alpha_1(1,\xi) = \varepsilon\varphi_0(\xi) + \varphi_1(\xi),\ \beta_1(1,\xi) = \varepsilon\psi_0(\xi) + \psi_1(\xi), & \xi\in\mathbb{R}. \end{cases}$$

The following lemma asserts that an estimate similar to (3.3) remains valid if (3.1) is perturbed by ρ, ν and φ_1, ψ_1:

Lemma 3.1. *Let $\sigma \in (0, \tau_1)$. We set $T_* = \min\{T, e^{\sigma/\varepsilon^2}\}$. For $\varepsilon \in (0, M^{-1/\delta}]$, we have*

$$\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}} (|\alpha_1(t,\xi)| + |\beta_1(t,\xi)|) \le \sqrt{2}C_1\varepsilon + M\varepsilon^{1+\delta},$$

where

$$M = \left(2C_2 + \frac{C_3}{\omega}\right) e^{\frac{\operatorname{Im}\lambda+\operatorname{Im}\mu}{2}(1+3C_1+4C_1^2)\sigma}.$$

Proof. We put $w(t,\xi) = \alpha_1(t,\xi) - \alpha_0(t,\xi)$, $z(t,\xi) = \beta_1(t,\xi) - \beta_0(t,\xi)$ and

$$T_{**} = \sup\Big\{\tilde{T} \in [1, T_*) \ \Big|\ \sup_{(t,\xi)\in[1,\tilde{T})\times\mathbb{R}} (|w(t,\xi)| + |z(t,\xi)|) \le M\varepsilon^{1+\delta}\Big\}.$$

Note that $T_{**} > 1$, because of the estimate

$$|w(1,\xi)| + |z(1,\xi)| = |\varphi_1(\xi)| + |\psi_1(\xi)| \le C_2\varepsilon^{1+\delta} \le \frac{M}{2}\varepsilon^{1+\delta}$$

and the continuity of w and z. Since w satisfies

$$i\partial_t w = \frac{\lambda}{t}\Big(|z+\beta_0|^2(w+\alpha_0) - |\beta_0|^2\alpha_0\Big) + \rho,$$

we see that

$$
\begin{aligned}
&\partial_t|w|^2 \\
&= 2\operatorname{Im}\Big(\overline{w}\cdot i\partial_t w\Big) \\
&\le \frac{2\operatorname{Im}\lambda}{t}\left\{\Big(M^2\varepsilon^{2+2\delta}+3C_1M\varepsilon^{2+\delta}+2C_1^2\varepsilon^2\Big)|w||z|+C_1^2\varepsilon^2|w|^2\right\}+|w||\rho|
\end{aligned}
$$

for $t\in[1,T_{**})$. Similarly we see that

$$
\partial_t|z|^2 \le \frac{2\operatorname{Im}\mu}{t}\left\{\Big(M^2\varepsilon^{2+2\delta}+3C_1M\varepsilon^{2+\delta}+2C_1^2\varepsilon^2\Big)|w||z|+C_1^2\varepsilon^2|z|^2\right\}+|z||\nu|
$$

for $t\in[1,T_{**})$. Therefore we have

$$
\partial_t(|w|^2+|z|^2)\le \frac{2}{t}\tilde{C}\varepsilon^2(|w|^2+|z|^2)+\frac{C_3\varepsilon^{1+\delta}}{t^{1+\omega}}(|w|^2+|z|^2)^{1/2}
$$

where $\tilde{C}=\frac{\operatorname{Im}\lambda+\operatorname{Im}\mu}{2}(1+3C_1+4C_1^2)$. By the Gronwall-type argument, we obtain

$$
\begin{aligned}
(|w|^2+|z|^2)^{1/2} &\le \left((|\varphi_1(\xi)|^2+|\psi_1(\xi)|^2)^{1/2}+\int_1^t\frac{C_3\varepsilon^{1+\delta}}{2s^{1+\omega+\tilde{C}\varepsilon^2}}ds\right)e^{\tilde{C}\varepsilon^2\log t} \\
&\le \left(C_2\varepsilon^{1+\delta}+\frac{C_3\varepsilon^{1+\delta}}{2(\omega+\tilde{C}\varepsilon^2)}\right)e^{\tilde{C}\sigma} \\
&\le \frac{M}{2}\varepsilon^{1+\delta}
\end{aligned}
$$

for $(t,\xi)\in[1,T_{**})\times\mathbb{R}$. Therefore

$$
|w(t,\xi)|+|z(t,\xi)|\le\sqrt{2}(|w|^2+|z|^2)^{1/2}\le\frac{M}{\sqrt{2}}\varepsilon^{1+\delta}.
$$

This contradicts the definition of T_{**} if $T_{**}<T_*$. Therefore we conclude $T_{**}=T_*$. In other words, we have

$$
\sup_{(t,\xi)\in[1,T_*)\times\mathbb{R}}|w(t,\xi)|+|z(t,\xi)|\le M\varepsilon^{1+\delta},
$$

whence

$$
|\alpha_1(t,\xi)|+|\beta_1(t,\xi)|\le|\alpha_0(t,\xi)|+|\beta_0(t,\xi)|+|w(t,\xi)|+|z(t,\xi)|\le\sqrt{2}C_1\varepsilon+M\varepsilon^{1+\delta}
$$

for $(t,\xi)\in[1,T_*)\times\mathbb{R}$. This completes the proof. □

§4. A priori estimate

This section is devoted to getting an a priori estimate for the solution to (1.1). Throughout this section, we fix $\sigma \in (0, \tau_0)$ and $T \in (0, e^{\sigma/\varepsilon^2}]$, where τ_0 is defined by (1.6). Let u, $v \in C([0,T); H^1 \cap H^{0,1}(\mathbb{R}))$ be a pair of solutions to (1.1) for $t \in [0,T)$. We set

$$\alpha(t,\xi) := \mathcal{F}_{m_1}\big[\mathcal{U}_{m_1}(t)^{-1}u(t,\cdot)\big](\xi),\ \ \beta(t,\xi) := \mathcal{F}_{m_2}\big[\mathcal{U}_{m_2}(t)^{-1}v(t,\cdot)\big](\xi).$$

We also define

$$\begin{aligned} E(T) := &\sup_{(t,\xi)\in[0,T)\times\mathbb{R}} (|\alpha(t,\xi)| + |\beta(t,\xi)|) \\ &+ \sup_{0\le t<T} \big[(1+t)^{-\gamma}(\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|\mathcal{J}_{m_1}u(t)\|_{L^2} + \|\mathcal{J}_{m_2}v(t)\|_{L^2})\big] \end{aligned}$$

with $\gamma \in (0, 1/12)$. The goal of this section is to prove the following:

Lemma 4.1. *Let σ, T and γ be as above. Then there exist positive constants ε_0 and K, not depending on T, such that*

$$E(T) \le \varepsilon^{2/3} \tag{4.1}$$

implies the stronger estimate

$$E(T) \le K\varepsilon,$$

provided that $\varepsilon \in (0, \varepsilon_0]$.

We divide the proof of this lemma into two subsections. We remark that many parts of the proof below are similar to that of Section 3 in [13] (see also [18]), although we need modifications to fit for our purpose.

4.1. L^2-estimates

In the first part, we consider the bound for $\|u(t)\|_{H^1}$, $\|v(t)\|_{H^1}$, $\|\mathcal{J}_{m_1}u(t)\|_{L^2}$ and $\|\mathcal{J}_{m_2}v(t)\|_{L^2}$. We first remark that Lemma 2.2 and the assumption (4.1) lead to

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \le \frac{C\varepsilon^{2/3}}{t^{1/2}}$$

for $t \ge 1$. Indeed the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \le C(\|u(t)\|_{H^1} + \|v(t)\|_{H^1}) \le C\varepsilon^{2/3}$$

for $t \le 1$. Hence we have

$$\|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} \le \frac{C\varepsilon^{2/3}}{\sqrt{1+t}}$$

for $t \in [0, T)$. Now we see from the standard energy method that

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(t)\|_{H^1} + \|v(t)\|_{H^1}\right) \\
&\le |\lambda|(2\|u(t)\|_{L^\infty}\|v(t)\|_{L^\infty}\|v(t)\|_{H^1} + \|v(t)\|_{L^\infty}^2\|u(t)\|_{H^1}) \\
&\quad + |\mu|(2\|u(t)\|_{L^\infty}\|v(t)\|_{L^\infty}\|u(t)\|_{H^1} + \|u(t)\|_{L^\infty}^2\|v(t)\|_{H^1}) \\
&\le \frac{C\varepsilon^2}{(1+t)^{1-\gamma}},
\end{aligned}$$

whence

$$\begin{aligned}
\|u(t)\|_{H^1} + \|v(t)\|_{H^1} &\le \varepsilon(\|\varphi\|_{H^1} + \|\psi\|_{H^1}) + \int_0^t \frac{C\varepsilon^2}{(1+s)^{1-\gamma}}ds \\
&\le C\varepsilon(1+t)^\gamma.
\end{aligned} \tag{4.2}$$

Next we deduce from Lemma 2.1 that

$$\mathcal{J}_{m_1}(|v|^2u) = \frac{m_2}{m_1}(\mathcal{J}_{m_2}v)\overline{v}u - \frac{m_2}{m_1}(\overline{\mathcal{J}_{m_2}v})vu + |v|^2(\mathcal{J}_{m_1}u).$$

We also remember the commutation relation $[\mathcal{L}_{m_1}, \mathcal{J}_{m_1}] = 0$. From them it follows that

$$\mathcal{L}_{m_1}\mathcal{J}_{m_1}u = \lambda\left(\frac{m_2}{m_1}(\mathcal{J}_{m_2}v)\overline{v}u - \frac{m_2}{m_1}(\overline{\mathcal{J}_{m_2}v})vu + |v|^2(\mathcal{J}_{m_1}u)\right).$$

Therefore the standard energy method leads to

$$\|\mathcal{J}_{m_1}u\|_{L^2} \le \varepsilon\|x\varphi\|_{L^2} + \int_0^t \frac{C\varepsilon^2}{(1+s)^{1-\gamma}}ds \le C\varepsilon(1+t)^\gamma. \tag{4.3}$$

In the same way, we have

$$\|\mathcal{J}_{m_2}v\|_{L^2} \le C\varepsilon(1+t)^\gamma. \tag{4.4}$$

Substituting (4.2), (4.3) and (4.4), we arrive at the desired estimate

$$\|u(t)\|_{H^1} + \|v(t)\|_{H^1} + \|\mathcal{J}_{m_1}u(t)\|_{L^2} + \|\mathcal{J}_{m_2}v(t)\|_{L^2} \le C\varepsilon(1+t)^\gamma$$

for $t \in [0, T)$.

4.2. Estimates for α and β

In this part, we will show $|\alpha(t,\xi)| + |\beta(t,\xi)| \le C\varepsilon$ for $(t,\xi) \in [0,T) \times \mathbb{R}$ under the assumption (4.1). When $0 \le t \le 1$, the desired estimate follows immediately from the Sobolev embedding and (4.1). Hence we have only to consider the case of $T > 1$ and $t \in [1,T)$.

$$\begin{aligned} i\partial_t\alpha(t,\xi) &= \mathcal{F}_{m_1}\mathcal{U}_{m_1}^{-1}[\mathcal{L}_{m_1}u] = \mathcal{F}_{m_1}\mathcal{U}_{m_1}^{-1}[\lambda|v|^2u] \\ &= \frac{\lambda}{t}|\beta(t,\xi)|^2\alpha(t,\xi) + \rho_1(t,\xi), \end{aligned}$$

where

$$\rho_1(t,\xi) = \frac{\lambda}{t}\mathcal{W}_{m_1}^{-1}\left[|\mathcal{W}_{m_2}\beta|^2\mathcal{W}_{m_1}\alpha\right] - \frac{\lambda}{t}|\beta|^2\alpha.$$

In the same way, we have

$$\begin{aligned} i\partial_t\beta(t,\xi) &= \mathcal{F}_{m_2}\mathcal{U}_{m_2}^{-1}[\mathcal{L}_{m_2}v] = \mathcal{F}_{m_2}\mathcal{U}_{m_2}^{-1}[\mu|u|^2v] \\ &= \frac{\mu}{t}|\alpha(t,\xi)|^2\beta(t,\xi) + \rho_2(t,\xi), \end{aligned}$$

where

$$\rho_2(t,\xi) = \frac{\mu}{t}\mathcal{W}_{m_2}^{-1}\left[|\mathcal{W}_{m_1}\alpha|^2\mathcal{W}_{m_2}\beta\right] - \frac{\mu}{t}|\alpha|^2\beta.$$

Note that Lemma 2.3 and $\|(\mathcal{W}_m - 1)f\|_{L^\infty} + \|(\mathcal{W}_m^{-1} - 1)f\|_{L^\infty} \le Ct^{-1/4}\|f\|_{H^1}$ lead to

$$|\rho_1(t,\xi)| + |\rho_2(t,\xi)| \le \frac{C\varepsilon^2}{t^{1+\omega}}$$

with $\omega = 1/4 - 3\gamma > 0$. Moreover we have

$$\begin{aligned} &|\alpha(1,\xi) - \varepsilon\mathcal{F}_{m_1}\varphi(\xi)| \\ &\le C\|u(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\|_{L^2}^{1/2}\|\mathcal{J}_{m_1}(1)\{u(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\}\|_{L^2}^{1/2} \\ &= C\|u(1,\cdot) - \mathcal{U}_{m_1}(1)\varepsilon\varphi\|_{L^2}^{1/2}\|\mathcal{J}_{m_1}(1)u(1,\cdot) - \mathcal{U}_{m_1}(1)x\varepsilon\varphi\|_{L^2}^{1/2} \\ &\le C\left(\int_0^1 \|\lambda|v(s)|^2u(s)\|_{L^2}ds\right)^{1/2}\varepsilon^{1/2} \\ &\le C\varepsilon^2, \end{aligned}$$

where we apply the Gagliardo-Nirenberg inequality $\|\phi\|_{L^\infty} \le C\|\phi\|_{L^2}^{1/2}\|\partial_x\phi\|_{L^2}^{1/2}$ and the relation $\mathcal{J}_m(t) = \mathcal{U}_m(t)x\mathcal{U}_m(t)^{-1}$. In the same way, we have

$$|\beta(1,\xi) - \varepsilon\mathcal{F}_{m_2}\psi(\xi)| \le C\varepsilon^2.$$

Therefore we can apply Lemma 3.1 with $\varphi_0(\xi) = \mathcal{F}_{m_1}\varphi(\xi)$, $\psi_0(\xi) = \mathcal{F}_{m_2}\psi(\xi)$, $\delta = 1$, $\omega = 1/4 - 3\gamma > 0$ and $\tau_1 = \tau_0$ to obtain

$$|\alpha(t,\xi)| + |\beta(t,\xi)| \le C\varepsilon.$$

§5. Proof of the main theorem

Now we prove Theorem 1.1. At first, existence of local solutions to (1.4) is proved in a standard way applying the contraction mapping principle (see [1]). Let T_ε be the lifespan defined in the statement of Theorem 1.1. Next we set

$$T^* = \sup\{T \in [0, T_\varepsilon) \mid E(T) \leq \varepsilon^{2/3}\}.$$

Note that $T^* > 0$ if ε is suitably small, because of the estimate $E(0) \leq C\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$ and the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$. Now, we take $\sigma \in (0, \tau_0)$ and assume $T^* \leq e^{\sigma/\varepsilon^2}$. Then Lemma 4.1 with $T = T^*$ yields

$$E(T^*) \leq K\varepsilon \leq \frac{1}{2}\varepsilon^{2/3}$$

if $\varepsilon \leq \min\{\varepsilon_0, (2K)^{-3}\}$. By the continuity of $[0, T_\varepsilon) \ni T \mapsto E(T)$, we can choose $\Delta > 0$ such that

$$E(T^* + \Delta) \leq \varepsilon^{2/3}.$$

This contradicts the definition of T^*. Therefore we must have $T^* \geq e^{\sigma/\varepsilon^2}$ if ε is suitably small. As a consequence, we obtain

$$\liminf_{\varepsilon \to +0} \varepsilon^2 \log T_\varepsilon \geq \sigma.$$

Since $\sigma \in (0, \tau_0)$ is arbitrary, we arrive at the desired conclusion. □

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References

[1] T. Cazenave, Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics **10**, American Mathematical Society, Providence, RI, 2003.

[2] M. Colin and T. Colin, On a quasilinear Zakharov system describing laser-plasma interactions, Differential Integral Equations, **17** (2004), 297–330.

[3] N. Hayashi, C. Li and P. I. Naumkin, On a system of nonlinear Schrödinger equations in 2D, Differential Integral Equations, **24** (2011), 417–434.

[4] N. Hayashi, C. Li and T. Ozawa, Small data scattering for a system of nonlinear Schrödinger equations, Differ. Equ. Appl., **3** (2011), 415–426.

[5] N. Hayashi and P. I. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, Amer. J. Math., **120** (1998), 369–389.

[6] N. Hayashi, T. Ozawa and K. Tanaka, On a system of nonlinear Schrödinger equations with quadratic interaction, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), 661–690.

[7] H. Hirayama, Well-posedness and scattering for a system of quadratic derivative nonlinear Schrödinger equations with low regularity initial data, Commun. Pure Appl. Anal., **13** (2014), 1563–1591.

[8] M. Ikeda, S. Katayama and H. Sunagawa, Null structure in a system of quadratic derivative nonlinear Schrödinger equations, Annales Henri Poincaré **16** (2015), 535–567.

[9] S. Katayama, C. Li and H. Sunagawa, A remark on decay rates of solutions for a system of quadratic nonlinear Schrödinger equations in 2D, Differential Integral Equations, **27** (2014), 301–312.

[10] D. Kim, A note on decay rates of solutions to a system of cubic nonlinear Schrödinger equations in one space dimension, Asymptotic Analysis, **98** (2016), 79–90.

[11] N. Kita, A work in preparation.

[12] C. Li, Decay of solutions for a system of nonlinear Schrödinger equations in 2D, Discrete Conti. Dyn. Syst., **32** (2012), 4265–4285.

[13] C. Li and H. Sunagawa, On Schrödinger systems with cubic dissipative nonlinearities of derivative type, Nonlinearity, **29** (2016), 1537–1563.

[14] C. Li and H. Sunagawa, Remarks on derivative nonlinear Schrödinger systems with multiple masses, to appear in the proceedings of the conference "Asymptotic Analysis for Nonlinear Dispersive and Wave Equations" held at Osaka in September 2014 (arXiv:1603.04966).

[15] T. Ogawa and K. Uriya, Final state problem for a quadratic nonlinear Schrödinger system in two space dimensions with mass resonance, J. Differential equations **258** (2015), 483–503.

[16] T. Ozawa and H. Sunagawa, Small data blow-up for a system of nonlinear Schrödinger equations, J. Math. Anal. Appl. **399** (2013), 147–155.

[17] Y. Sagawa and H. Sunagawa, The lifespan of small solutions to cubic derivative nonlinear Schrödinger equations in one space dimension, Discrete Contin. Dyn. Syst., **36** (2016), 5743–5761.

[18] Y. Sagawa, H. Sunagawa and S. Yasuda, A sharp lower bound for the lifespan of small solutions to the Schrödinger equation with a subcritical power nonlinearity, Differential Integral Equations, **31** (2018), 685–700.

[19] D. Sakoda and H. Sunagawa, Small data global existence for a class of quadratic derivative nonlinear Schrödinger systems in two space dimensions, preprint, arXiv:1804.05540v2.

[20] A. Shimomura, Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities, Comm. Partial Differential Equations, **31** (2006), 1407–1423.

[21] H. Sunagawa, Lower bounds of the lifespan of small data solutions to the nonlinear Schrödinger equations, Osaka J. Math., **43** (2006), 771–789.

[22] K. Uriya, Final state problem for a system of nonlinear Schrödinger equations with three wave interaction, J. Evol. Equ., **16** (2016), 173–191.

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