SUT Journal of Mathematics
Vol. 53, No. 2 (2017), 127–134

Remark on the roots of generalized Lens equations

Mutsuo Oka

(Received June 9, 2017; Revised November 27, 2017)

Abstract. We consider roots of a generalized Lens polynomial $L(z,\bar{z}) = \bar{z}^m q(z) - p(z)$ and also harmonically splitting Lens type polynomial $L^{hs}(z,\bar{z}) = r(\bar{z})q(z) - p(z)$ with $\deg q(z) = n$, $\deg p(z) \le n$ and $\deg r(\bar{z}) = m$. We have shown that there exists a harmonically splitting polynomial $r(\bar{z})q(z) - p(z)$ which takes $5n + m - 6$ roots, using a bifurcation family of polynomial. In this note, we show that this number of roots can be taken by a generalized Lens polynomial $\bar{z}^m q(z) - p(z)$ after a slight modification of the bifurcation family of a Rhie polynomial.

AMS 2010 *Mathematics Subject Classification.* 14P05, 14N99

Key words and phrases. Lens equation, Generalized Lens equation, Roots with sign

§1. Introduction

Consider a mixed polynomial of one variable $f(z,\bar{z}) = \sum_{\nu,\mu} a_{\nu,\mu} z^\nu \bar{z}^\mu$. We consider the number of roots of $f = 0$. Assume that $z = \alpha$ is an isolated zero of $f = 0$. Put $f(z,\bar{z}) = g(x,y) + ih(x,y)$ with $z = x + iy$ where $g = \Re(f)$ and $h = \Im(f)$. We call α a positive simple root (respectively a negative simple root), if the Jacobian $J(g,h)$ is positive (resp. negative) at $z = \alpha$.

1.1. Number of roots with sign

Let $f(z,\bar{z})$ be a given mixed polynomial of one variable, we consider the filtration by the degree:

$$f(z,\bar{z}) = f_d(z,\bar{z}) + f_{d-1}(z,\bar{z}) + \cdots + f_0(z,\bar{z}).$$

Here $f_\ell(z,\bar{z}) := \sum_{\nu+\mu=\ell} c_{\nu,\mu} z^\nu \bar{z}^\mu$. We consider the case $f_d(z,\bar{z}) = z^n \bar{z}^m$ with $n + m = d$. The total number of roots of $f(z,\bar{z}) = 0$ with sign is denoted by $\beta(f)$. *Under the above assumption,* $\beta(f) = n - m$ by Theorem 20, [6].

1.2. Number of roots without the sign

We assume that roots of $f(z,\bar{z})=0$ are all simple. The number of roots without considering the sign is denoted by $\rho(f)$. Note that $\rho(f)$ is not described by the highest degree part f_d, which was the case for $\beta(f)$. Consider a mixed polynomial $f(z,\bar{z})=\sum_{\nu,\mu} a_{\nu,\mu} z^\nu \bar{z}^\mu$. We use the definitions

$$\begin{aligned}\deg_z\, f &:= \max\{\nu \,|\, a_{\nu,\mu}\neq 0\}\\ \deg_{\bar{z}}\, f &:= \max\{\mu \,|\, a_{\nu,\mu}\neq 0\}\\ \deg\, f &:= \max\{\mu+\nu \,|\, a_{\nu,\mu}\neq 0\}\end{aligned}$$

$\deg_z\, f$, $\deg_{\bar{z}}\, f$, $\deg\, f$ are called *the holomorphic degree* , *the anti-holomorphic degree* and *the mixed degree* of f respectively. We consider the following subclasses of mixed polynomials:

$$\begin{aligned}L(n+m;n,m) &:= \{f(z,\bar{z})=\bar{z}^m q(z)-p(z)\,|\,\deg_z q=n,\ \deg_z p\le n\},\\ L^{hs}(n+m;n,m) &:= \{f(z,\bar{z})=r(\bar{z})q(z)-p(z)\,|\,\deg_{\bar{z}} r(\bar{z})=m,\\ &\qquad\qquad \deg_z q=n,\ \deg_z p\le n\},\\ M(n+m;n,m) &:= \{f(z,\bar{z})\,|\,\deg\, f=n+m,\ \deg_z\, f=n,\ \deg_{\bar{z}}\, f=m\}.\end{aligned}$$

where $p(z),q(z)\in\mathbb{C}[z], r(\bar{z})\in\mathbb{C}[\bar{z}]$. Here z is an affine coordinate of $\mathbb{C}$ but we do not fix z. So, a mixed polynomial $f(u,\bar{u})$ is called a generalized Lens polynomial or a harmonically splitting Lens type polynomial if f takes the above form under some affine coordinate $u=z+c$. We have canonical inclusions:

$$L(n+m;n,m)\ \subset L^{hs}(n+m;n,m)\ \subset M(n+m;n,m).$$

The class $L(n+m;n,m), L^{hs}(n+m;n,m)$ corresponds to the numerators of harmonic functions

$$\bar{z}^m-\frac{p(z)}{q(z)},\ r(\bar{z})-\frac{p(z)}{q(z)}.$$

In particular, $L(n+1;n,1)$ corresponds to the lens equation. We call $\bar{z}^m q(z)-p(z)$ *a generalized lens polynomial* and and $r(\bar{z})q(z)-p(z)$ *a harmonically splitting lens type polynomial* respectively.

1.3. Lens equation

The following equation is known as the lens equation.

$$L(z,\bar{z})=\bar{z}-\sum_{i=1}^{n}\frac{\sigma_i}{z-\alpha_i}=0,\quad \sigma_i,\alpha_i\in\mathbb{C}^*. \tag{1.1}$$

We identify the left side rational function with the mixed polynomial given by its numerator

$$\tilde{L}(z,\bar{z}) := L(z,\bar{z})\prod_{i=1}^{n}(z-\alpha_i) \in M(n+1;n,1).$$

Theorem 1. (Khavinson-Neumann [2]) The number of roots of L or $\tilde{L}$ is bounded by $5n-5$ for $n \geq 2$.

Rhie gave an explicit polynomial which takes this bound $5n-5$ in [8]. Thus this bound is optimal. On the other hand, $\rho(L) \equiv n-1 \mod 2$ by Theorem 20, [6].

Theorem 2. (P. Bleher, Y. Homma, L. Ji and P. Roeder [1]) *The set of possible values of* $\rho(f)$ *for* $f \in L(n+1,n,1)$ *is equal to* $\{n-1,n+1,\cdots,5n-5\}$.

1.3.1. Motivation

The moduli space of smooth complex analytic projective hypersurfaces in $\mathbb{P}^n$ of a given degree is connected and thus the topology does not depend on a particular hypersurface. But the situation for mixed hypersutfaces is different. The moduli space of smooth mixed hypersurfaces of given polar radial degrees are not connected. Thus to know the number of connected components is very important. The moduli spaces $L(n+m;n,m), L^{hs}(n+m;n,m), M(n+m;n,m)$ corresponds to subspaces of the moduli spaces of mixed homogeneous polynomials of two variables by the correspondence

$$f(z,\bar{z}) \mapsto F(z_1,z_2,\bar{z}_1,\bar{z}_2) := f(z_1/z_2,\bar{z}_1/\bar{z}_2)z_2^n\bar{z}_2^m.$$

For further detail, we refer §2 of [4]. Two mixed polynomials of the same moduli space with different number of zeros belongs to different components of the moduli of mixed homogeneous polynomials and they have different topologies. Thus to know the possible number of zeros is very important from this point of view. Also for a given mixed homogeneous polynomial $F(\mathbf{z},\bar{\mathbf{z}})$ of two variables, we can take the following join operation for any integer $\ell > 0$,

$$G(\mathbf{z},\mathbf{w},\bar{\mathbf{z}},\bar{\mathbf{w}}) = F(\mathbf{z},\bar{\mathbf{w}}) + \sum_{i=1}^{\ell} w_i^n\bar{w}_i^m,\ \mathbf{w} = (w_1,\dots,w_\ell).$$

Thus we can construct mixed homogeneous polynomials of any number of variables with different topology and by the join theorem of [3]. The second motivation of this note is to give an counter example of a question in [4]. Our result shows the richness of the moduli space $L(n+m;n,m)$ and also it might be of some interest from astrophysicists, as the original Lens equations are first studied by them.

1.4. Bifurcation family

In [4], we have constructed a generalized Lens type polynomial which take $5(n-m)$-roots if $n > 3m$ and we have asked *if this is an optimal upper bound or not.* On the other hand, for the space of harmonically splitting Lens type polynomials $L^{hs}(n+m;n,m)$, we studied a bifurcation family $\psi_t(z,\bar z) := t\bar z^m + \ell_n(z,\bar z) \in L^{hs}(n+m;n,m)$ starting from a given Lens polynomial $\ell_n(z,\bar z)$ with $\rho(\ell_n) = k$. Let $\alpha_1,\dots,\alpha_k$ be the roots of ℓ_n. The main result of this note is the following.

Theorem 3. ([4]) *$\psi_t = 0$ has exactly $k+m-1$ roots for small t. Furthermore k roots of them are near each α_j with the same sign and $m-1$ roots are newly born roots bifurcated from $z=\infty$. These new roots are negative roots.*

§2. Proof of main result

2.1. Modification of the bifurcation family and the main result

In this note, we answer the above question negatively. In fact, we modify the above bifurcation family to prove the same assertion for generalized Lens polynomials. We start from an arbitrary Lens type polynomial with only simple roots:

$$\ell_n(z,\bar z) := \bar z q(z) - p(z), \quad \deg_z q = n,\, \deg_z p \le n,\, n \ge 2.$$

Put $k = \rho(\ell_n)$ and let $\alpha_1,\dots,\alpha_k$ be the roots of ℓ_n. Note that $n-1 \le k \le 5n-5$ and $k \equiv n-1 \mod 2$. Put γ be the coefficient of z^n in $q(z)$. Note that γ is non-zero, as $\deg_z q(z) = n$ by the assumption. Consider its small perturbation $\phi_t(z,\bar z)$ of $\ell_n(z)$ in the space of generalized Lens polynomials $L(n+m;n,m)$:

$$\phi_t(z,\bar z) := \frac{((t\bar z+\gamma)^m - \gamma^m)}{\gamma^{m-1}mt} q(z) - p(z),\ t \in \mathbb{C}. \tag{2.1}$$

Note that $\phi_0(z,\bar z) = \ell_n(z,\bar z)$ and for non-zero t, ϕ_t corresponds to the generalized Lens equation

$$(t\bar z+\gamma)^m q(z) = \gamma^{m-1} m t p(z) + \gamma^m q(z).$$

In fact, by the change of coordinate $u = \bar t z + \bar\gamma$, ϕ_t takes the expected form.

Theorem 4. *For sufficiently small $t \in \mathbb{C}$, $|t| \ll 1$, $\rho(\phi_t) = k+m-1$. Furthermore*

1. *k roots $\alpha_j(t), j = 1,\dots,k$ are small deformation of α_j and the sign of $\alpha_j(t)$ is the same as that of α_j.*

2. *$m-1$ new roots $\beta_a(t), a = 1, \ldots, m-1$ are born at infinity i.e., $\beta_a(0) = \infty$ and they are negative roots.*

Taking $\ell_n(z, \bar{z})$ to be a Rhie's polynomial, we get $\rho(\phi_t) = 5n + m - 6$.

Proof. For sufficiently small t and for each root α of ℓ_n, by the continuity of the roots, there exists a root $\alpha(t)$ of $\phi_t = 0$ in a neighborhood of α with $\alpha(0) = \alpha$ and $\alpha(t)$ has the same orientation as α. For $t \neq 0$, we know that $\beta(\phi_t) = n - m$ for $t \neq 0$ and $\beta(\ell_n) = n - 1$. Thus it is clear that we need at least $m - 1$ negative roots. Take a large $R > 0$ so that $|\alpha_j| \leq R/2$ for any $j = 1, \ldots, k$. For any small $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that ϕ_t has k roots near each $\alpha_j(t)$, $|t| \leq \delta(\varepsilon)$ with the same sign as α_j in the original equation $\ell_n = 0$. We may assume that $|\alpha_j(t) - \alpha_j| \leq \varepsilon$ for $j = 1, \ldots, k$ and there are no other roots of $\phi_t(z) = 0$ in the disk $D_R = \{z \,|\, |z| \leq R\}$. On the other hand, as $\beta(\phi_t) = n-m$, $t \neq 0$, we have the property $n-m \equiv k-(m-1)$, mod 2. Thus ϕ_t has at least $m - 1$ new negative roots outside of the disk D_R.

We assert that ϕ_t obtains exactly $m - 1$ new negative roots near infinity. To see this, we change the coordinate $u = 1/z$ and dividing (2.1) by $\bar{z}^m z^n$, we get

$$\tilde{\phi}_t(u) := \frac{(t + \gamma\bar{u})^m - \gamma^m \bar{u}^m}{\gamma^{m-1} m t} \tilde{q}(u) - \bar{u}^m \tilde{p}(u) \tag{2.2}$$

where $\tilde{q}, \tilde{p}$ are polynomials defined as $\tilde{q}(u) = u^n q(1/u)$, $\tilde{p}(u) = u^n p(1/u)$. By the asumption deg $q(z) = n$, we can write

$$\tilde{q}(u) = \gamma + \sum_{i=1}^{n} b_i u^i$$

We will show that for a sufficiently small $t > 0$, there exist exactly $m-1$ roots $u(t)$ which converges to 0 as $t \to 0$. Put $\tilde{\phi}_{t,1}, \tilde{\phi}_{t,2}$ be the first and the second term of (2.2). Putting $u = vt$ for $t \neq 0$, we can write $\tilde{\phi}_{t,1}$ as

$$\tilde{\phi}_{t,1}(v) \;:=\; t^{m-1} \frac{(1 + \bar{v})^m - \bar{v}^m}{\gamma^{m-1} m} \tilde{q}(vt) \tag{2.3}$$

$$= \; t^{m-1} h(v) \tilde{q}(vt) \tag{2.4}$$

where $h(v)$ is a polynomial with a non-zero constant. $h(v) = 0$ has $m - 1$ simple roots, and we put them as $v = \beta_1, \ldots, \beta_{m-1}$. Consider the disk at infinity and its subset W:

$$\Delta := \{u \,|\, |u| \leq 1/R\},\; W := \{u \in \Delta \,|\, |v - \beta_j| \geq \delta,\; j = 1, \ldots, m-1\}.$$

On Δ, we estimate $1/M \leq |\tilde{q}(vt)| \leq M$ for some $M > 0$. Taking a small number $\delta > 0$, we can make

$$\left| \frac{(1 + \bar{v})^m - \bar{v}^m}{\gamma^{m-1} m} \tilde{q}(vt) \right| \geq M'\delta,\;\; v \in W$$

with some constant $M' > 0$. Or equivalently,

$$|\phi_{t,1}(u)| \geq |t|^{m-1} M' \delta,\ u \in W.$$

Taking t small, we can make the second term of (2.2) as small as possible on Δ comparing with $|t|^{m-1}$. More precisely, there exists a positive number M'' such that

$$|\phi_{t,2}(u)| \leq M'' |t|^m.$$

Thus if $|t|$ is sufficiently small,

$$|\tilde{\psi}_t(v)| \geq \frac{M'}{2} |t|^{m-1},\ \text{for}\, v \in W$$

which implies $\tilde{\psi}_t(v) = 0$ has one simple negative root in $D_j := \{v \,|\, |v - \beta_j| \leq \delta\}$ for $j = 1, \dots, j$ and no root on W. The negativity of these $m-1$ new roots is clear as $\beta(\phi_t) = n - m$ and $\beta(\ell_n) = n - 1$. This completes the proof. □

2.2. Possible values of ρ

Assume that $n \geq m$. Combining Theorem 2, we can see that

Corollary 5. *$\rho(f)$ for $f \in L(n+m; n, m)$ can takes the values $\{n + m - 2, \dots, 5n + m - 6\}$.*

As for the lower values $\{n - m, \dots, n + m - 4\}$, we know that these values can be taken by some polynomials in $M(n+m; n, m)$. We do not know if these values can be taken in $L(n+m; n, m)$ or $L^{hs}(n+m; n, m)$ except $n - m$. For $n - m$, it can be taken by

$$f(z, \bar{z}) = \bar{z}^m z^n - 1.$$

2.3. Example

Consider

$$f(z, \bar{z}) = \left(\frac{\bar{z}}{100} + 1\right)^3 \left(z^3 - \frac{1}{8}\right) - z^3 - \frac{3z^2}{100} + \frac{12513}{100000}.$$

This is a bifurcation of a Rhie type polynomial $\ell_3(z, \bar{z})$ with $\rho(\ell_3) = 10$ where

$$\ell_3(z, \bar{z}) = \frac{3}{100} \bar{z}(z^3 - \frac{1}{8}) - z^3 - \frac{3z^2}{100} + \frac{12513}{100000} = 0.$$

Let $\Re f$ and $\Im f$ be the real and imaginary part of f. The roots of $f = 0$ can be read as the intersection of two real curves $\Re f = \Im f = 0$. In Fugure 1, the left graph is $\Im f(x,v)=0$ with $v=y+1$ and the right side is the graph of $\Re f \cdot \Im f = 0$. The new component is from $\Re f = 0$ and we see 10 roots in the graph. Actually there are 12 roots which are approximately given as follows.

$$
\begin{aligned}
(2.5)\quad & w_1=[-150.0,-86.59676649], w_2=[-150.0,86.59676649],\\
&[-0.5514593683,-0.9052203300],[-0.9351977551,0.0],[0.4816513190,\\
&-0.7932837118],[-0.1346963314,0.0],[0.06354099264,-0.1052498909],\\
&[0.007544143902,0.0],[0.06354099264,0.1052498909],[-0.5514593683,\\
&0.9052203300],[1.052348943,0.0],[0.4816513190,0.7932837118].
\end{aligned}
$$

where $[a,b]:=a+bi$ and two roots w_1,w_2 are not in the figure. These two roots w_1,w_2 are born from the infinity under the bifurcation family. The other 10 roots are deformations from the 10 roots of Rhie's polynomial.

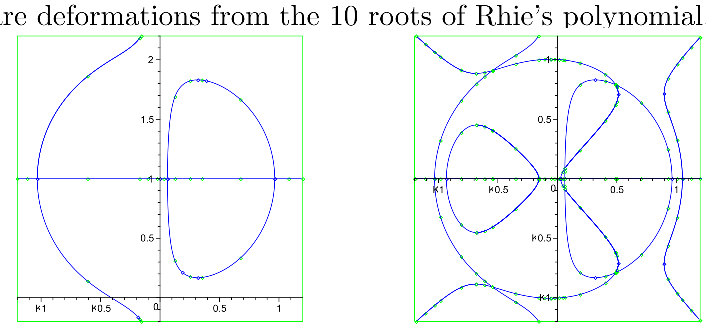

Figure 1: Left:$\Im f = 0$, Right: $\Re f \cdot \Im f = 0$

For the practical computation of the roots, we used the following program for maple which is kindly offered from Pho Duc Tai, Vietnam National University.

Pho's program to compute roots of mixed polynomial on Maple:

```
fsol3 := proc (f, z)
local aa, a, b, ff, f1, f2, h, i, j, k, s, temp;
print(Factorization_of_Input = factor(f));
ff := factors(f)[2];
temp := {};
for k to nops(ff) do
if 1 < ff[k][2] then
RETURN(printf("Input is not squarefree.
Please solve each factor.")) end if;
```

```
assume(a, real); assume(b, real);
h := expand(subs(z = a+I*b, ff[k][1]));
f1 := Re(h);
f2 := Im(h);
aa := RootFinding[Isolate](a[f1, f2], [a, b]);
temp := `union`(temp, seq([[op(aa[i][1])][2], [op(aa[i][2])][2]],
i = 1 ..  nops(aa))) end do;
RETURN([op(temp)]) end proc
```

References

[1] P. Bleher, Y. Homma, L. Ji and R. Roeder, *Counting roots of harmonic rational functions and its application to gravitational lensing*, Int. Math. Res. Not. IMRN (2014) No. 8, 2245–2264.

[2] D. Khavinson and G. Neumann, *On the number of roots of certain rational harmonic functions,* Proc. Amer. Math. Soc. **134** (2008), No. 6, 666–675.

[3] J. L. Cisneros-Molina, *Join theorem for polar weighted homogeneous singularities.* In Singularities II, *Contemp. Math.*, **475** (2008), 43–59.

[4] M. Oka, On the roots of an extended Lens equation and an application. arXiv:1505.03576, to appear in Singularities and Foliations. Geometry, Topology and Applications, Salvador, Brazil, 2015, Springer Proceedings in Mathematics & Statistics.

[5] M. Oka, *Topology of polar weighted homogeneous hypersurfaces.* Kodai Math. J., **31** (2008), no. 2, 163–182.

[6] M. Oka, *Intersection theory on mixed curves.* Kodai Math. J., **35** (2012), no. 2, 248–267.

[7] M. Oka, *Non-degenerate mixed functions.* Kodai Math. J., **33** (2010), no. 1, 1–62.

[8] S.H. Rhie, *n-point Gravitational Lenses with* $5(n-1)$ *Images.* arXiv:astro-ph/0305166, May 2003.

Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601
E-mail: oka@rs.kagu.tus.ac.jp