

F-geometric mean labeling of graphs obtained by duplicating any edge of some graphs

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Abstract. A function f is called an F -geometric mean labeling of a graph $G = (V, E)$ with p vertices and q edges if $f : V \rightarrow \{1, 2, 3, \dots, q+1\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, 3, \dots, q\}$ defined as

$$f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor, \text{ for all } uv \in E,$$

is bijective. A graph that admits an F -geometric mean labeling is called an F -geometric mean graph. In this paper we discuss the F -geometric meanness of graphs obtained by duplicating any edge of some graphs.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let $G = (V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [4] and for a detailed survey on graph labeling, we refer [5].

Path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n . The graph $G \circ K_1$ is obtained from the graph G by attaching a new pendant vertex at each vertex of G . A ladder $L_n, n \geq 2$, is the graph $P_2 \times P_n$. Duplicating of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = (N(u) \cup \{v'\}) - \{v\}$ and $N(v') = (N(u) \cup \{u'\}) - \{u\}$ [6].

The concept of F -geometric mean labeling was first introduced by Durai Baskar et al. [1] and they studied the F -geometric mean labeling of some standard graphs [2, 3].

A function f is called an F-geometric mean labeling of a graph $G = (V, E)$ with p vertices and q edges if $f : V \rightarrow \{1, 2, 3, \dots, q + 1\}$ is injective and induced function $f^* : E \rightarrow \{1, 2, 3, \dots, q\}$ defined as $f^*(uv) = \lfloor \sqrt{f(u)f(v)} \rfloor$, for all $uv \in E$, is bijective. A graph that admits an F-geometric mean labeling is called an F-geometric mean graph. In this paper we discuss the F-geometric meanness of graphs obtained by duplicating any edge of some graphs.

§2. Main Results

Theorem 1. *Let G be a graph obtained by duplicating an edge e of a path $P_n, n \geq 3$. Then G is an F-geometric mean graph.*

Proof. Let v_1, v_2, \dots, v_n be the vertices of the path P_n . Let $e' = v'_i v'_{i+1}$ be the duplicating edge of $e = v_i v_{i+1}$ for some $i, 1 \leq i \leq n - 1$.

Case 1. $i = 1$ or $i = n - 1$.

Since the graph G is isomorphic when $i = 1$ or $i = n - 1$, we may take $i = 1$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2\}$ as follows:

$$f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq 3 \\ j + 2, & 4 \leq j \leq n \end{cases} \quad \text{and } f(v'_j) = 2j, \text{ for } 1 \leq j \leq 2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(v_j v_{j+1}) = \begin{cases} 2j - 1, & 1 \leq j \leq 3 \\ j + 2, & 4 \leq j \leq n - 1, \end{cases} \quad f^*(v'_1 v'_2) = 2 \text{ and } f^*(v'_2 v'_3) = 4.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Case 2. $i = 2$ and $n \geq 4$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 3\}$ as follows:

$$f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq 4 \\ j + 3, & 5 \leq j \leq n \end{cases} \quad \text{and } f(v'_j) = 2j, \text{ for } 2 \leq j \leq 3.$$

Then the induced edge labeling is obtained as follows:

$$f^*(v_j v_{j+1}) = \begin{cases} 2j - 1, & 1 \leq j \leq 4 \\ j + 3, & 5 \leq j \leq n - 1, \end{cases} \\ f^*(v_1 v'_2) = 2, f^*(v'_2 v'_3) = 4 \text{ and } f^*(v'_3 v_4) = 6.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Case 3. $3 \leq i \leq n - 2$ and $n \geq 5$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 3\}$ as follows:

$$f(v_j) = \begin{cases} j, & 1 \leq j \leq i - 1 \\ i + 2, & j = i \\ j + 3, & i + 1 \leq j \leq n, \end{cases} \quad f(v'_i) = i + 1 \text{ and } f(v'_{i+1}) = i + 3.$$

Then the induced edge labeling is obtained as follows:

$$f^*(v_j v_{j+1}) = \begin{cases} j, & 1 \leq j \leq i-2 \\ i, & j = i-1 \\ i+2, & j = i \\ j+3, & i+1 \leq j \leq n-1 \end{cases}$$

$f^*(v_{i-1} v'_i) = i-1, f^*(v'_i v'_{i+1}) = i+1$ and $f^*(v'_{i+1} v_{i+2}) = i+3$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph. \square

The F-geometric mean labeling of G in the above cases are shown in Figure 1.

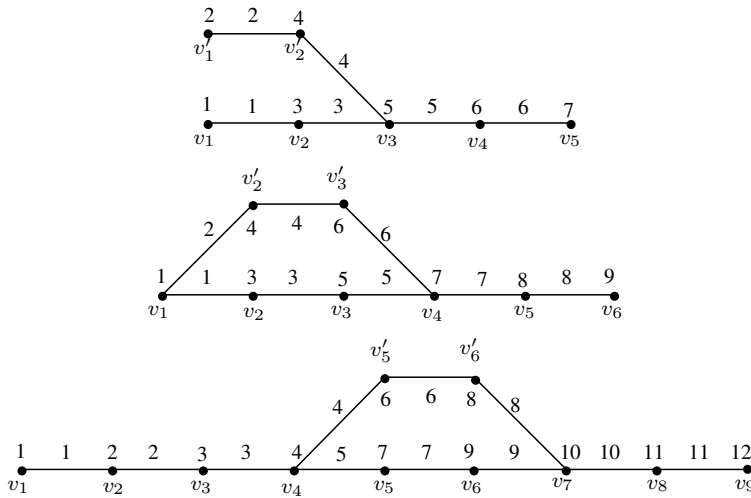


Figure 1.

Theorem 2. Let G be a graph obtained by duplicating an edge e of a graph $P_n \circ K_{1,n} \geq 2$. Then G is an F-geometric mean graph.

Proof. Let u_1, u_2, \dots, u_n be the vertices of the path P_n and v_i be a pendant vertex attached at u_i , for $1 \leq i \leq n$. When $n = 2$, an F-geometric mean labeling of G is shown in Figures 2 and 3 (Figure 2 is the case $e = u_1 v_1$ and Figure 3 is the case $e = u_1 u_2$). So we assume $n \geq 3$.

Case 1. $e = u_i v_i$, for $1 \leq i \leq n$

Let its duplication be $e' = u'_i v'_i$.

Subcase (i). $i = 1$ or $i = n$.

Since the graph G is isomorphic when $i = 1$ or $i = n$, we may take $i = 1$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 2\}$ as follows:

$$f(u_j) = \begin{cases} j+3, & 1 \leq j \leq 2 \\ 2j+2, & 3 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 4j-2, & 1 \leq j \leq 2 \\ 2j+1, & 3 \leq j \leq n, \end{cases}$$

$f(u'_1) = 3$ and $f(v'_1) = 1$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = 2j + 2, \text{ for } 1 \leq j \leq n - 1, f^*(u_j v_j) = \begin{cases} 2, & j = 1 \\ 2j + 1, & 2 \leq j \leq n, \end{cases}$$

$$f^*(u'_1 v'_1) = 1 \text{ and } f^*(u'_1 u_2) = 3.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 3\}$ as follows:

$$f(u_j) = \begin{cases} 3, & j = 1 \\ 4j - 4, & 2 \leq j \leq 3 \\ 2j + 3, & 4 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 1, & j = 1 \\ 7j - 12, & 2 \leq j \leq 3 \\ 2j + 2, & 4 \leq j \leq n \end{cases}$$

$$f(u'_2) = 7 \text{ and } f(v'_2) = 6.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2j + 1, & 1 \leq j \leq 2 \\ 2j + 3, & 3 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} j, & 1 \leq j \leq 2 \\ 2j + 2, & 3 \leq j \leq n, \end{cases}$$

$$f^*(u_1 u'_2) = 4, f^*(u'_2 u_3) = 7 \text{ and } f^*(u'_2 v'_2) = 6.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iii). $3 \leq i \leq n - 1$ and $n \geq 4$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 3\}$ as follows.

$$f(u_j) = \begin{cases} 2j, & 1 \leq j \leq i \\ 2i + 4, & j = i + 1 \\ 2j + 3, & i + 2 \leq j \leq n \end{cases} \quad f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq i \\ 2i + 5, & j = i + 1 \\ 2j + 2, & i + 2 \leq j \leq n, \end{cases}$$

$$f(u'_i) = 2i + 3 \text{ and } f(v'_i) = 2i + 2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2j, & 1 \leq j \leq i - 1 \\ 2i + 1, & j = i \\ 2j + 3, & i + 1 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq i \\ 2j + 2, & i + 1 \leq j \leq n, \end{cases}$$

$$f^*(u_{i-1} u'_i) = 2i, f^*(u'_i u_{i+1}) = 2i + 3 \text{ and } f^*(u'_i v'_i) = 2i + 2.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

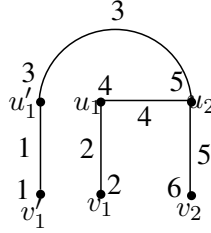


Figure 2

Case 2. $e = u_i u_{i+1}$, for $1 \leq i \leq n - 1$

Let its duplication be $e' = u'_i u'_{i+1}$

Subcase (i). $i = 1$ or $i = n - 1$.

Since the graph G is isomorphic when $i = 1$ or $i = n - 1$, we may take $i = 1$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 1, & j = 1 \\ 2j + 4, & 2 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 3j, & 1 \leq j \leq 3 \\ 2j + 3, & 4 \leq j \leq n, \end{cases}$$

$f(u'_1) = 4$ and $f(u'_2) = 5$.

Then the induced edge labeling obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2, & j = 1 \\ 2j + 4, & 2 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 5j - 4, & 1 \leq j \leq 2 \\ 2j + 3, & 3 \leq j \leq n, \end{cases}$$

$f^*(u'_1 v_1) = 3, f^*(u'_1 u'_2) = 4, f^*(u'_2 v_2) = 5$ and $f^*(u'_2 u_3) = 7$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$ and $n \geq 4$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 5\}$ as follows:

$$f(u_j) = \begin{cases} 5j - 3, & 1 \leq j \leq 2 \\ 4j - 4, & 3 \leq j \leq 4 \\ 2j + 5, & 5 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 5j - 4, & 1 \leq j \leq 2 \\ 3j + 1, & 3 \leq j \leq 4 \\ 2j + 4, & 5 \leq j \leq n, \end{cases}$$

$f(u'_2) = 3$ and $f(u'_3) = 11$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 3, & j = 1 \\ 2j + 3, & 2 \leq j \leq 3 \\ 2j + 5, & 4 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 1, & j = 1 \\ 2j + 2, & 2 \leq j \leq 3 \\ 2j + 4, & 4 \leq j \leq n, \end{cases}$$

$f^*(u_1 u'_2) = 2, f^*(u'_2 u'_3) = 5, f^*(u'_3 u_4) = 11, f^*(u'_2 v_2) = 4$ and $f^*(u'_3 v_3) = 10$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iii). $3 \leq i \leq n - 2$ and $n \geq 5$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 5\}$ as follows:

$$f(u_j) = \begin{cases} 2j, & 1 \leq j \leq i - 1 \\ 2i + 3, & j = i \\ 2i + 4, & j = i + 1 \\ 2i + 8, & j = i + 2 \\ 2j + 5, & i + 3 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq i - 1 \\ 2i, & j = i \\ 2i + 6, & j = i + 1 \\ 2i + 9, & j = i + 2 \\ 2j + 4, & i + 3 \leq j \leq n, \end{cases}$$

$$f(u'_i) = 2i - 1 \text{ and } f(u'_{i+1}) = 2i + 7.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2j, & 1 \leq j \leq i - 2 \\ 2i, & j = i - 1 \\ 2i + 3, & j = i \\ 2i + 5, & j = i + 1 \\ 2j + 5, & i + 2 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq i - 1 \\ 2i + 1, & j = i \\ 2i + 4, & j = i + 1 \\ 2j + 4, & i + 2 \leq j \leq n, \end{cases}$$

$$f^*(u_{i-1} u'_i) = 2i - 2, f^*(u'_i u'_{i+1}) = 2i + 2, f^*(u'_i u_{i+2}) = 2i + 7, f^*(u'_i v_i) = 2i - 1$$

and $f^*(u'_{i+1} v_{i+1}) = 2i + 6$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph. \square

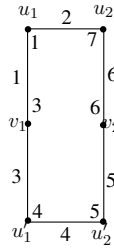
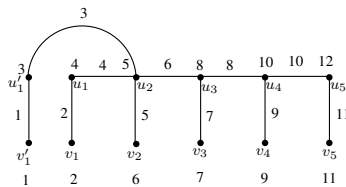


Figure 3

The F-geometric mean labeling of G in the above cases are shown in Figure 4.



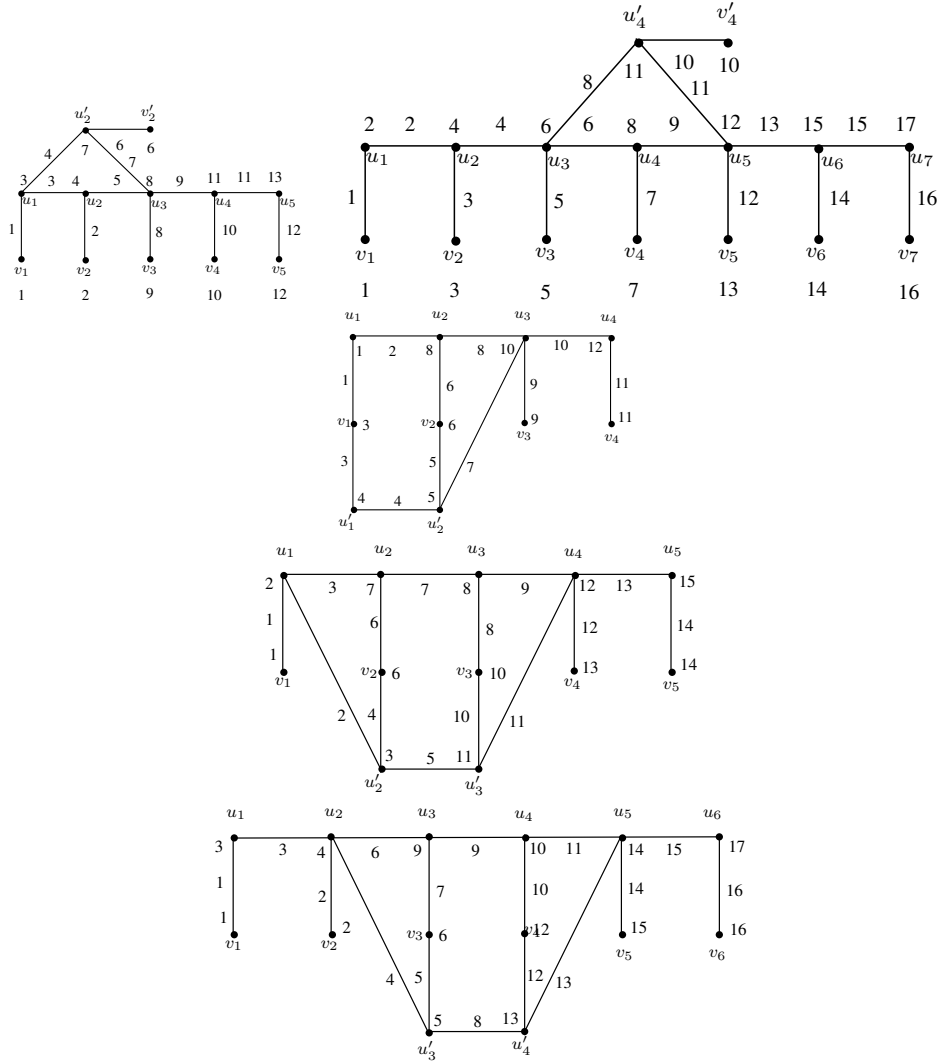


Figure 4

Theorem 3. *Let G be a graph obtained by duplicating an edge e of a graph $C_n \circ K_1, n \geq 3$. Then G is an F -geometric mean graph.*

Proof. Let u_1, u_2, \dots, u_n be the vertices of the cycle C_n and v_i be a pendant vertex attached at u_i , for $1 \leq i \leq n$. When $n = 3$, an F -geometric mean labeling of G is shown in Figures 5 and 6 (Figure 5 is the case $e = u_1v_1$ and Figure 6 is the case $e = u_1u_2$). So we assume $n \geq 4$.

Case 1. $e = u_iv_i$, for $1 \leq i \leq n$.

Let its duplication be $e' = u'_iv'_i$ and choose arbitrarily $i = 1$.

Subcase (i). n is odd

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j - 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 2n + 4, & j = \lfloor \frac{n}{2} \rfloor + 2 \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 2j + 3, & 1 \leq j \leq 2 \\ 4j, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j - 2, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 2n + 3, & j = \lfloor \frac{n}{2} \rfloor + 2 \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$f(u'_1) = 1$ and $f(v'_1) = 2$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ 2n + 1, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is odd} \\ 2n + 2, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is even} \\ 4n + 7 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_n u_1) = 6, f^*(u_j v_j) = \begin{cases} 3j + 1, & 1 \leq j \leq 2 \\ 4j - 2, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n, \end{cases}$$

$f^*(u'_1 v'_1) = 1, f^*(u_n u'_1) = 3$ and $f^*(u_2 u'_1) = 2$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

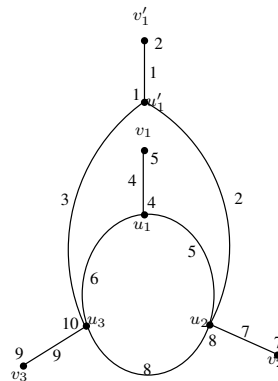


Figure 5

Subcase (ii). n is even.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j - 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 2j + 3, & 1 \leq j \leq 2 \\ 4j, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is odd} \\ 4j - 2, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is even} \\ 2n + 1, & j = \lfloor \frac{n}{2} \rfloor + 1 \text{ is odd} \\ 2n + 3, & j = \lfloor \frac{n}{2} \rfloor + 1 \text{ is even} \\ 2n + 2, & j = \lfloor \frac{n}{2} \rfloor + 2 \text{ is odd} \\ 2n + 4, & j = \lfloor \frac{n}{2} \rfloor + 2 \text{ is even} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$f(u'_1) = 1$ and $f(v'_1) = 2$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ 2n + 2, & j = \lfloor \frac{n}{2} \rfloor + 1 \\ 4n + 7 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 3j + 1, & 1 \leq j \leq 2 \\ 4j - 2, & 3 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ 2n + 1, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is odd} \\ 2n + 3, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is even} \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n, \end{cases}$$

$f^*(u_n u_1) = 6$, $f^*(u'_1 v'_1) = 1$, $f^*(u_n u'_1) = 3$ and $f^*(u_2 u'_1) = 2$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Case 2. $e = u_i u_{i+1}$, for $1 \leq i \leq n - 1$.

Let its duplication be $e' = u'_i u'_{i+1}$ and choose arbitrarily $i = 1$.

Subcase (i). n is odd

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 6\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 2n + 6, & j = \lfloor \frac{n}{2} \rfloor + 2 \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 13 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 2n + 5, & j = \lfloor \frac{n}{2} \rfloor + 2 \\ 4n + 13 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$f(u'_1) = 2$ and $f(u'_2) = 6$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ 2n + 4, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is odd} \\ 2n + 3, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is even} \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n - 1 \end{cases}$$

$$f^*(u_n u_1) = 6, f^*(u_j v_j) = \begin{cases} 2, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n, \end{cases}$$

$f^*(u_n u'_1) = 4, f^*(u'_1 u'_2) = 3, f^*(u'_2 u_3) = 9, f^*(u'_1 v_1) = 1$ and $f^*(u'_2 v_2) = 7$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

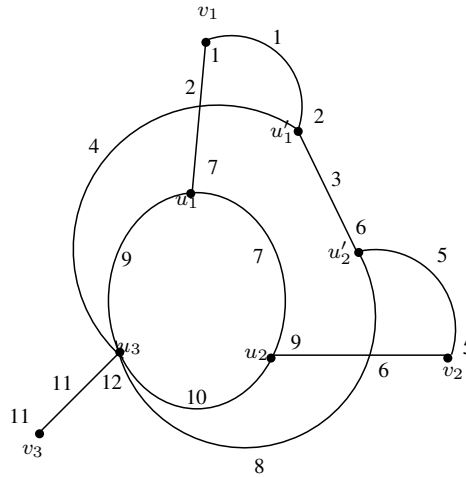


Figure 6

Subcase (ii). n is even.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 2n + 6\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1 \text{ and } j \text{ is even} \\ 4n + 13 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is odd} \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \text{ and } j \text{ is even} \\ 2n + 5, & j = \lfloor \frac{n}{2} \rfloor + 1 \text{ is odd} \\ 2n + 3, & j = \lfloor \frac{n}{2} \rfloor + 1 \text{ is even} \\ 2n + 6, & j = \lfloor \frac{n}{2} \rfloor + 2 \text{ is odd} \\ 2n + 4, & j = \lfloor \frac{n}{2} \rfloor + 2 \text{ is even} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is odd} \\ 4n + 13 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n \text{ and } j \text{ is even,} \end{cases}$$

$f(u'_1) = 2$ and $f(u'_2) = 6$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j + 2, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor - 2 \\ 2n - 2, & j = \lfloor \frac{n}{2} \rfloor - 1 \\ 2n + 2, & j = \lfloor \frac{n}{2} \rfloor \\ 2n + 4, & j = \lfloor \frac{n}{2} \rfloor + 1 \\ 4n + 9 - 4j, & \lfloor \frac{n}{2} \rfloor + 2 \leq j \leq n - 1, \end{cases}$$

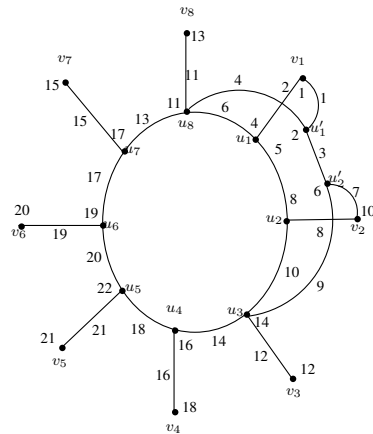
$$f^*(u_j v_j) = \begin{cases} 2, & j = 1 \\ 4j, & 2 \leq j \leq \lfloor \frac{n}{2} \rfloor \\ 2n + 5, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is odd} \\ 2n + 3, & \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 2 \text{ and } j \text{ is even} \\ 4n + 11 - 4j, & \lfloor \frac{n}{2} \rfloor + 3 \leq j \leq n, \end{cases}$$

$f^*(u_n u_1) = 6, f^*(u_n u'_1) = 4, f^*(u'_1 u'_2) = 3, f^*(u'_2 u_3) = 9,$

$f^*(u'_1 v_1) = 1$ and $f^*(u'_2 v_2) = 7.$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

The F-geometric mean labeling of G in the above cases are shown in Figure 7.



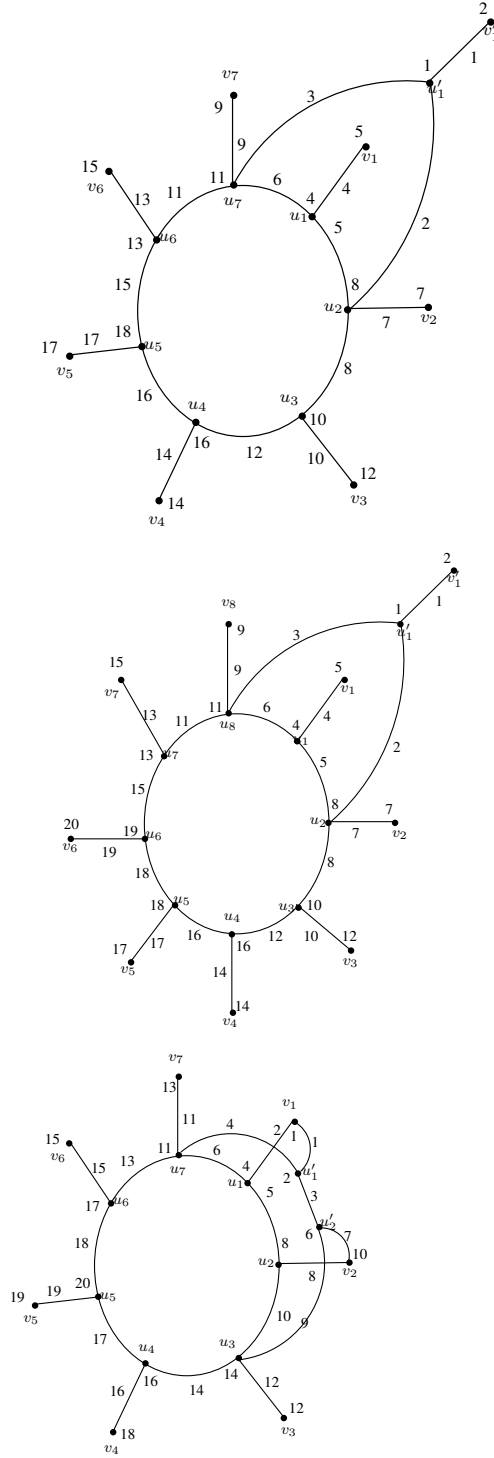


Figure 7

□

Theorem 4. *Let G be a graph obtained by duplicating an edge e of a graph $L_n, n \geq 2$, Then G is an F-geometric mean graph.*

Proof. Let u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_n be the vertices on the path of length $n - 1$ in the ladder L_n . When $n = 2$, an F-geometric mean labeling of G is shown in Figure 8 (Figure 8 is the case $e = u_1v_1$ and $e = u_1u_2$). So we assume $n \geq 3$.

Case 1. $e = u_i v_i$, for $1 \leq i \leq n$

Let its duplication be $e' = u'_i v'_i$.

Subcase (i). $i = 1$ or $i = n$.

Since the graph G is isomorphic when $i = 1$ or $i = n$, we may take $i = 1$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n + 2\}$ as follows:

$$f(u_j) = \begin{cases} 6, & j = 1 \\ 3j + 1, & 2 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 4, & j = 1 \\ 3j + 2, & 2 \leq j \leq n, \end{cases}$$

$f(u'_1) = 2$ and $f(v'_1) = 1$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 6, & j = 1 \\ 3j + 2, & 2 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = 3j + 1, \text{ for } 1 \leq j \leq n, \quad f^*(v_j v_{j+1}) = \begin{cases} 5, & j = 1 \\ 3j + 3, & 2 \leq j \leq n - 1, \end{cases}$$

$f^*(u'_1 v'_1) = 1, f^*(u'_1 u_2) = 3$ and $f^*(v'_1 v_2) = 2$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 3, & j = 1 \\ 1, & j = 2 \\ 3j + 4, & 3 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 8, & j = 1 \\ 5, & j = 2 \\ 3j + 3, & 3 \leq j \leq n, \end{cases}$$

$f(u'_2) = 10$ and $f(v'_2) = 9$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2j - 1, & 1 \leq j \leq 2 \\ 3j + 5, & 3 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 4, & j = 1 \\ 2, & j = 2 \\ 3j + 3, & 3 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} j + 5, & 1 \leq j \leq 2 \\ 3j + 4, & 3 \leq j \leq n - 1, \end{cases}$$

$f^*(u_1 u'_2) = 5, f^*(u'_2 u_3) = 11, f^*(v_1 v'_2) = 8, f^*(v'_2 v_3) = 10$ and $f^*(u'_2 v'_2) = 9$.

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iii). $i = 3$ and $n \geq 4$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leq j \leq 2 \\ 14, & j = 3 \\ 3j + 4, & 4 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 2j + 1, & 1 \leq j \leq 3 \\ 3j + 3, & 4 \leq j \leq n, \end{cases}$$

$$f(u'_3) = 10 \text{ and } f(v'_3) = 13.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5j - 3, & 1 \leq j \leq 2 \\ 3j + 5, & 3 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 1, & j = 1 \\ 5j - 6, & 2 \leq j \leq 3 \\ 3j + 3, & 4 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} 3, & j = 1 \\ 5j - 5, & 2 \leq j \leq 3 \\ 3j + 4, & 4 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_2 u'_3) = 6, f^*(u'_3 u_4) = 12, f^*(v_2 v'_3) = 8, f^*(v'_3 v_4) = 13 \text{ and}$$

$$f^*(u'_3 v'_3) = 11.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iv). $4 \leq i \leq n - 1$ and $n \geq 5$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3i + 3, & j = i \\ 3j + 4, & i + 1 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} 3j - 1, & 1 \leq j \leq i - 1 \\ 3i - 2, & j = i \text{ and } 4 \leq i \leq 6 \\ 3i - 3, & j = i \text{ and } 7 \leq i \leq n - 1 \\ 3j + 3, & i + 1 \leq j \leq n, \end{cases}$$

$$f(u'_i) = 3i + 1 \text{ and } f(v'_i) = 3i + 5.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 3j - 1, & 1 \leq j \leq i - 2 \\ 3i - 2, & j = i - 1 \\ 3i + 4, & j = i \\ 3j + 5, & i + 1 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3i, & j = i \text{ and } 4 \leq i \leq 6 \\ 3i - 1, & j = i \text{ and } 7 \leq i \leq n - 1 \\ 3j + 3, & i + 1 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} 3j, & 1 \leq j \leq i-2 \\ 3i-4, & j = i-1 \\ 3i+1, & j = i \\ 3j+4, & i+1 \leq j \leq n-1, \end{cases}$$

$$f^*(u_{i-1} u'_i) = 3i-3, f^*(u'_i u_{i+1}) = 3i+3, f^*(v'_i v_{i+1}) = 3i+5,$$

$$f^*(u'_i v'_i) = 3i+2 \text{ and } f^*(u_{i-1} v'_i) = \begin{cases} 3i-1, & 4 \leq i \leq 6 \\ 3i, & 7 \leq i \leq n-1. \end{cases}$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Case 2. $e = u_i u_{i+1}$, for $1 \leq i \leq n-1$

Let its duplication be $e' = u'_i u'_{i+1}$.

Subcase (i). $i = 1$ or $i = n-1$.

Since the graph G is isomorphic when $i = 1$ or $i = n-1$, we may take $i = 1$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n+3\}$ as follows:

$$f(u_j) = \begin{cases} 5j, & 1 \leq j \leq 2 \\ 3j+3, & 3 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 3, & j = 1 \\ 3j+2, & 2 \leq j \leq n, \end{cases}$$

$$f(u'_1) = 1 \text{ and } f(u'_2) = 4.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = 3j+4, \text{ for } 1 \leq j \leq n-1,$$

$$f^*(u_j v_j) = \begin{cases} 3, & j = 1 \\ 3j+2, & 2 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} 4, & j = 1 \\ 3j+3, & 2 \leq j \leq n-1, \end{cases}$$

$$f^*(u'_1 v_1) = 1, f^*(u'_1 u'_2) = 2, f^*(u'_2 u_3) = 6 \text{ and } f^*(u'_2 v_2) = 5.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$ and $n \geq 4$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n+4\}$ as follows:

$$f(u_j) = 3j+4, \text{ for } 1 \leq j \leq n, f(v_j) = 3j+3, \text{ for } 1 \leq j \leq n,$$

$$f(u'_2) = 1 \text{ and } f(u'_3) = 2.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = 3j+5, \text{ for } 1 \leq j \leq n-1, f^*(u_j v_j) = 3j+3, \text{ for } 1 \leq j \leq n,$$

$$f^*(v_j v_{j+1}) = 3j+4, \text{ for } 1 \leq j \leq n-1,$$

$$f^*(u_1 u'_2) = 2, f^*(u'_2 u'_3) = 1, f^*(u'_3 u_4) = 5, f^*(u'_2 v_2) = 3 \text{ and } f^*(u'_3 v_3) = 4.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iii). $i = 3$ and $n \geq 5$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n+4\}$ as follows:

$$f(u_j) = \begin{cases} j+3, & 1 \leq j \leq 2 \\ 4j-1, & 3 \leq j \leq 4 \\ 3j+4, & 5 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 2j-1, & 1 \leq j \leq 2 \\ 4j-3, & 3 \leq j \leq 4 \\ 3j+3, & 5 \leq j \leq n, \end{cases}$$

$$f(u'_3) = 8 \text{ and } f(u'_4) = 16.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 3j + 1, & 1 \leq j \leq 2 \\ 4j, & 3 \leq j \leq 4 \\ 3j + 5, & 5 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} j + 1, & 1 \leq j \leq 2 \\ 4j - 3, & 3 \leq j \leq 4 \\ 3j + 3, & 5 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} 1, & j = 1 \\ 5j - 5, & 2 \leq j \leq 4 \\ 3j + 4, & 5 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_2 u'_3) = 6, f^*(u'_3 u'_4) = 11, f^*(u'_4 u_5) = 17, f^*(u'_3 v_3) = 8 \text{ and } f^*(u'_4 v_4) = 14.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph.

Subcase (iv). $4 \leq i \leq n - 2$ and $n \geq 6$.

Define $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n + 4\}$ as follows:

$$f(u_j) = \begin{cases} 3j - 1, & 1 \leq j \leq i - 2 \\ 3i - 3, & j = i - 1 \\ 3i + 2, & j = i \\ 3i + 6, & j = i + 1 \\ 3j + 4, & i + 2 \leq j \leq n, \end{cases} \quad f(v_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3i - 1, & j = i \\ 3i + 4, & j = i + 1 \\ 3j + 3, & i + 2 \leq j \leq n, \end{cases}$$

$$f(u'_i) = 3i - 2 \text{ and } f(u'_{i+1}) = 3i + 8.$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 3j, & 1 \leq j \leq i - 2 \\ 3i - 1, & j = i - 1 \\ 3i + 3, & j = i \\ 3i + 7, & j = i + 1 \\ 3j + 5, & i + 2 \leq j \leq n - 1 \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 3j - 2, & 1 \leq j \leq i - 1 \\ 3i, & j = i \\ 3i + 4, & j = i + 1 \\ 3j + 3, & i + 2 \leq j \leq n, \end{cases}$$

$$f^*(v_j v_{j+1}) = \begin{cases} 3j - 1, & 1 \leq j \leq i - 1 \\ 3i + 1, & j = i \\ 3i + 6, & j = i + 1 \\ 3j + 4, & i + 2 \leq j \leq n - 1, \end{cases}$$

$$f^*(u_{i-1} u'_i) = 3i - 3, f^*(u'_i u'_{i+1}) = 3i + 2, f^*(u'_i u_{i+2}) = 3i + 8,$$

$$f^*(u'_i v_i) = 3i - 2 \text{ and } f^*(u'_{i+1} v_{i+1}) = 3i + 5.$$

Hence, f is an F-geometric mean labeling of G . Thus the graph G is an F-geometric mean graph. \square

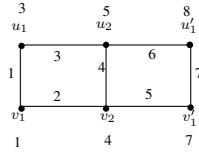
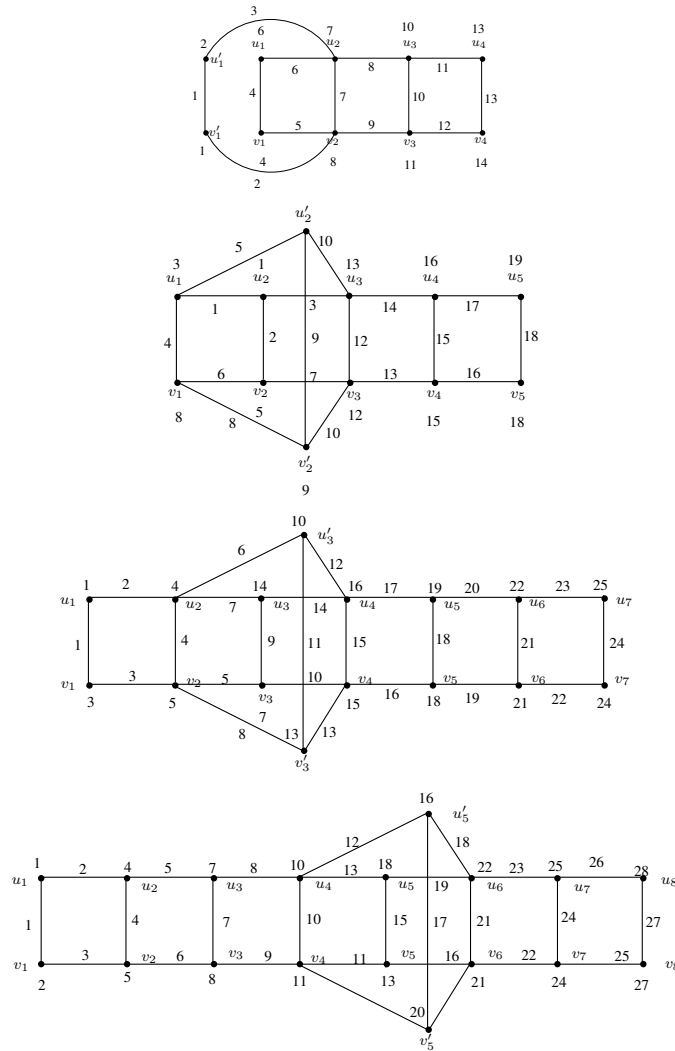


Figure 8

The F-geometric mean labeling of G in the above cases are shown in Figure 9.



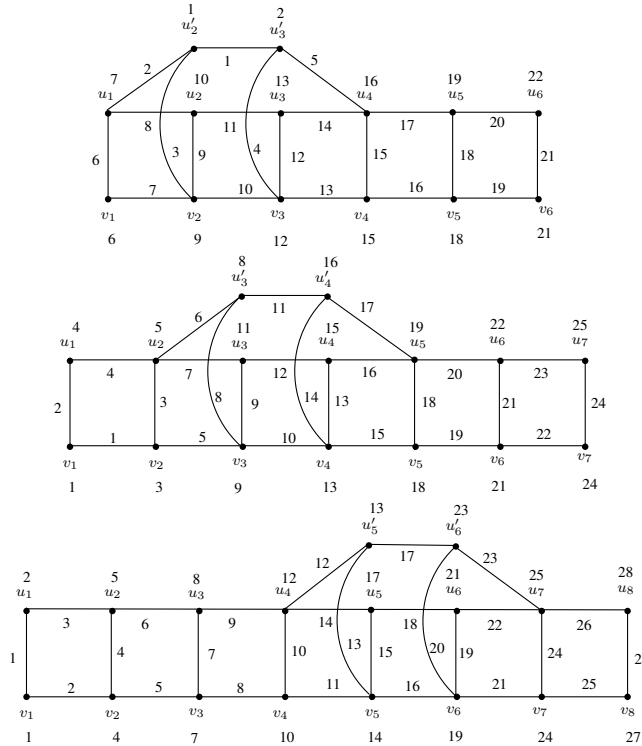


Figure 9

Acknowledgments

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References

- [1] A. Durai Baskar, S. Arockiaraj and B. Rajendran, Geometric Mean Labeling of Graphs Obtained from Some Graph Operations, *International J.Math. Combin.*, **1** (2013), 85–98.
- [2] A. Durai Baskar, S. Arockiaraj and B. Rajendran, F -Geometric mean labeling of some chain graphs and thorn graphs, *Kragujevac J. Math.*, **37**(1) (2013), 163–186.
- [3] A. Durai Baskar, S. Arockiaraj and B. Rajendran, Geometric meanness of graphs obtained from paths, *Util. Math.*, **101** (2016), 45-68.
- [4] F. Buckley and F. Harary, *Distance in graphs*, Addison-Wesley, Reading, 1990.
- [5] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, (2016), #DS6.

- [6] S. K. Vaidya and C. M. Barasara, Harmonic mean labeling in the context of duplication of graph elements, *Elixir Dist. Math.*, **48** (2012), 9482–9485.

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