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Odd harmonious labeling of some new families of graphs

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Abstract. A graph $G(p, q)$ is said to be odd harmonious if there exists an injection $f : V(G) \rightarrow \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. A graph that admits odd harmonious labeling is called odd harmonious graph. In this paper, we prove that shadow and splitting of graph $K_{2,n}$, C_n for $n \equiv 0 \pmod 4$, the graph $H_{n,n}$, double quadrilateral snakes $DQ(n)$, $n \geq 2$, the graph $P_{r,m}$ if m is odd, banana tree and the path union of cycles C_n for $n \equiv 0 \pmod 4$ are odd harmonious.

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§1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [6]. A graph $G = (V, E)$ with p vertices and q edges is called a (p, q) – graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions. An extensive survey of various graph labeling problems is available in [3]. Labeled graphs serves as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [4] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v)) \pmod q$ is a bijection and f is called harmonious labeling of G. The concept of odd harmonious labeling was followed Liang and Bai [7]. A labeling is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. A graph that admits odd harmonious labeling is called odd harmonious graph. The odd harmoniousness of graph is useful for the solution of undetermined equations. The same authors have obtained necessary conditions for the existence of odd harmonious labeling of graphs:

1. If G is an odd harmonious graph, then G is a bipartite graph.
2. If a (p, q)-graph G is odd harmonious, then $2\sqrt{q} \le p \le 2q-1$.

Several results have been published on odd harmonious labeling see [1, 2, 5, 8, 9, 10]. Motivated by these results, in this paper we prove that the shadow and splitting of the graphs $K_{2,n}$, C_n for $n \equiv 0 \pmod 4$, the graph $H_{n,n}$, double quadrilateral snakes $DQ(n)$, $n \ge 2$, the graph $P_{r,m}$ if m is odd, banana tree and the path union of cycles C_n for $n \equiv 0 \pmod 4$ are odd harmonious.

Definition 1. *A function f is said to be a strongly odd harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to q such that the induced mapping $f^*(uv) = f(u) + f(v)$ from the edges of G to the odd integers between 1 to $2q-1$ is a bijection.*

Definition 2. *[9] The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say $G^{'}$ and $G^{''}$ and join each vertex $u^{'}$ in $G^{'}$ to the neighbours of the corresponding vertex $v^{'}$ in $G^{''}$.*

Definition 3. *[9]For a graph G the splitting graph $spl(G)$ of a graph G is obtained by adding a new vertex $v^{'}$ corresponding to each vertex v of G such that $N(v) = N(v^{'})$.*

Definition 4. *The graph $H_{n,n}$ has the vertex set $V(H_{n,n}) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_i u_j : 1 \le i \le n, n-i+1 \le j \le n\}$.*

Definition 5. *Let $Q(n)$ be the quadrilateral snake obtained from the path $v_1, v_2, v_3, \cdots, v_{n+1}$ by joining v_i and v_{i+1} to the new vertices u_i and w_i. That is, every edge of a path is replaced by a cycle C_4.*

Definition 6. *Let $Q(n)$ be the quadrilateral snake obtained from the path $v_1, v_2, v_3, \cdots, v_{n+1}$. The double quadrilateral snake $DQ(n)$ is obtained from $Q(n)$ by adding the vertices $s_1, s_2, s_3, \cdots, s_n; t_1, t_2, t_3, \cdots, t_n$ and the edges $v_i s_i, t_i v_{i+1}, s_i t_i$ for $1 \le i \le n$.*

Definition 7. *Let u and v be the fixed vertices and connect u and v by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. The resulting graph embedded in a plane is denoted by $P_{a,b}$. Let $v_0^i, v_1^i, v_2^i, \cdots, v_a^i$ be the vertices of the i^{th} copy of the path of length a where $i = 1, 2, \cdots, b, v_0^i = u$ and $v_a^i = v$ for all i. We observe that the graph $P_{a,b}$ has $(a-1)b+2$ vertices and ab edges.*

Definition 8. *For graphs $G_1, G_2, \cdots G_n (n \geq 2)$, we call a graph obtained by adding an edge from G_i to G_{i+1} for $i = 1, 2, \cdots, n-1$ a path-union of $G_1, G_2, \cdots G_n$ (the resulting graph may depends on how the edges are chosen).*

Definition 9. *[3] A banana tree is a graph obtained by connecting a vertex v to one leaf of each of any number of stars. Let $K_{1,n_1}, K_{1,n_2}, \cdots, K_{1,n_k}$ be a family of disjoint stars. Let v be a new vertex and the tree obtained by joining v to one pendant vertex of each star is called a banana tree. The class of all such trees is denoted by $BT(n_1, n_2, \cdots, n_k)$. If $n_1 = n_2 = \cdots = n_k$ we denote $BT(n_1, n_2, \cdots, n_k)$ as $BT_k(n)$*

§2. Main Results

Theorem 2.1. *The shadow graph $D_2(K_{2,n})$ is an odd harmonious graph.*

Proof. Consider the two copies of $K_{2,n}$. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the first copy of $K_{2,n}$ adjacent with u and v. Let $v_1', v_2', v_3', \cdots, v_n'$ be the vertices of the second copy of $K_{2,n}$ adjacent with u' and v'. Let $G = D_2(K_{2,n})$. Then $V(G) = \left\{u, v, u', v', v_i, v_i' | 1 \leq i \leq n\right\}$ and $E(G) = \left\{uv_i, vv_i, u'v_i', v'v_i', uv_i', vv_i', u'v_i, v'v_i | 1 \leq i \leq n\right\}$. Hence $|V(G)| = 2n+4$ and $|E(G)| = 8n$. We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 16n-1\}$ as follows:
$f(u) = 0$; $f(v) = 4n$; $f(u') = 8n$; $f(v') = 12n$; $f(v_i) = 2i-1$, $1 \leq i \leq n$; $f(v_i') = 2n-1+2i$, $1 \leq i \leq n$. The induced edge labelings are as follows: For $1 \leq i \leq n$

$f^*(uv_i) = 2i-1$; $f^*(uv_i') = 2n+2i-1$; $f^*(vv_i) = 4n+2i-1$; $f^*(vv_i') = 6n+2i-1$; $f^*(u'v_i) = 8n+2i-1$; $f^*(u'v_i') = 10n+2i-1$; $f^*(v'v_i) = 12n+2i-1$; $f^*(v'v_i') = 14n+2i-1$. Hence $f^*(E(G)) = \{1, 3, \cdots, 2n-1, 4n+1, 4n+3, \cdots, 6n-1, 10n+1, 10n+3, \cdots, 12n-1, 14n+1, 14n+3, \cdots, 16n-1, 2n+1, 2n+3, \cdots, 4n-1, 6n+1, 6n+3, \cdots, 8n-1, 8n+3, \cdots 10n-1, 12n+1, 12n+3, \cdots, 14n-1\} = \{1, 3, \cdots, 16n-1\}$

Thus f admits odd harmonious labeling on $D_2(K_{2,n})$. □

Illustration 1. *The odd harmonious labeling of the graph $D_2(K_{2,5})$ is given in Figure 1*

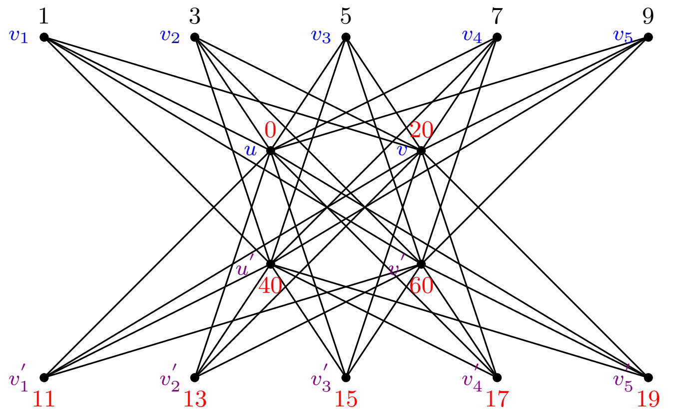

Figure 1: Odd harmonious labeling of $D_2(K_{2,5})$

Corollary 2.2. *The graph $spl(K_{2,n})$ is an odd harmonious graph.*

Proof. Let $G = spl(K_{2,n})$. Then G is isomorphic to a graph obtained from the graph $D_2(K_{2,n})$ by deleting the edges $u'v'_i, v'v'_i (1 \leq i \leq n)$. Hence $|V(G)| = 2n + 4$ and $|E(G)| = 6n$. By using the labeling given in Theorem 2.1, we get the induced edge labels as $\{1, 3, \cdots, 12n - 1\}$. Hence $spl(K_{2,n})$ is an odd harmonious graph. □

Illustration 2. *The odd harmonious labeling of the graph $spl(K_{2,4})$ is given in Figure 2*

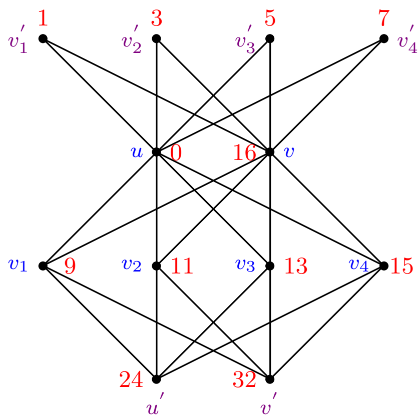

Figure 2: Odd harmonious labeling of $spl(K_{2,4})$

Theorem 2.3. *The shadow graph $D_2(C_n)$ is an odd harmonious graph if* $n \equiv 0 \pmod 4$.

Proof. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the first copy of C_n and $u_1, u_2, u_3, \cdots, u_n$ are the vertices of the second copy of C_n. Let $G = D_2(C_n)$ and $n \equiv 0 \pmod 4$. Then $V(G) = \{v_i, u_i | 1 \leq i \leq n\}$ and $E(G) = \{v_iv_{i+1}, u_iu_{i+1}, v_nv_1,$

$u_nu_1|1 \leq i \leq n-1\} \cup \{v_iu_{i+1}, v_nu_1|1 \leq i \leq n-1\} \cup \{v_1u_n, v_iu_{i-1}|2 \leq i \leq n\}$. Here $|V(G)| = 2n$ and $|E(G)| = 4n$. We define $f : V(G) \rightarrow \{0, 1, 2, \cdots, 8n-1\}$ as follows:

$$f(v_i) = \begin{cases} i-1 & \text{if} \quad 1 \leq i \leq \frac{n}{2}, \\ i+1 & \text{if} \quad i \text{ is odd and } \frac{n}{2}+1 \leq i \leq n \\ i-1 & \text{if} \quad i \text{ is even and } \frac{n}{2}+1 \leq i \leq n \end{cases}$$

For $1 \leq i \leq n$ $f(u_i) = \begin{cases} f(v_i) + 2n & \text{if } i \text{ is odd,} \\ f(v_i) + 4n & \text{if } i \text{ is even.} \end{cases}$

The induced edge labelings are as follows: $f^*(\{v_iv_{i+1}, u_iu_{i+1}, v_nv_1, u_nu_1|1 \leq i \leq n-1\}) = \{1, 3, \cdots, 2n-1\} \cup \{6n+1, 6n+3, \cdots, 8n-1\}$

$f^*(\{v_iu_{i+1}, v_nv_1|1 \leq i \leq n-1\}) = \{2n+3, 4n+5, 2n+7, \cdots, 5n-3, 3n+1, 5n+3, 3n+5, \cdots, 6n-1, 4n+1\}$

$f^*(\{v_1u_n, v_iu_{i-1}|2 \leq i \leq n\}) = \{5n-1, 2n+1, 4n+3, 2n+5, 4n-7, \cdots, 3n-3, 5n+1, 3n+3, \cdots, 6n-3, 4n-1\}$.

Hence $f^*(E(G)) = \{1, 3, \cdots, 8n-1\}$. Thus $D_2(C_n)$ is an odd harmonious graph.

□

Illustration 3. *The odd harmonious labeling of $D_2(C_8)$ is given in Figure 3*

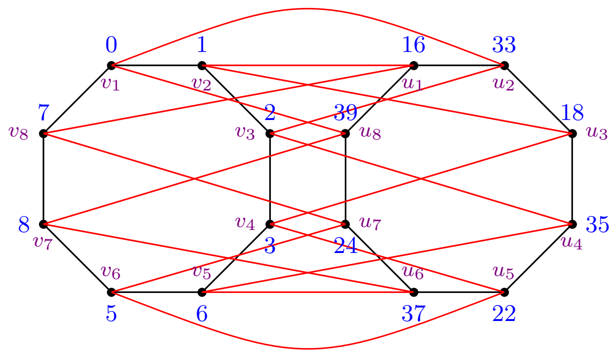

Figure 3: Odd harmonious labeling of $D_2(C_8)$

Theorem 2.4. *The graph $spl(C_n)$ if $n \equiv 0 \pmod 4$ is an odd harmonious graph .*

Proof. Let $G = spl(C_n)$.Then G is isomorphic to a graph obtained from the graph $D_2(C_n)$ by deleting the edges $u_iu_{i+1}, u_nu_1 (1 \leq i \leq n-1)$. Hence $|V(G)| = 2n$ and $|E(G)| = 3n$.By using the labeling given in Theorem 2.3, we get the induced edge labels as $\{1, 3, \cdots, 6n-1\}$. Hence $spl(C_n)$ is an odd harmonious graph. □

Illustration 4. *The odd harmonious labeling of $spl(C_8)$ is shown in Figure 4*

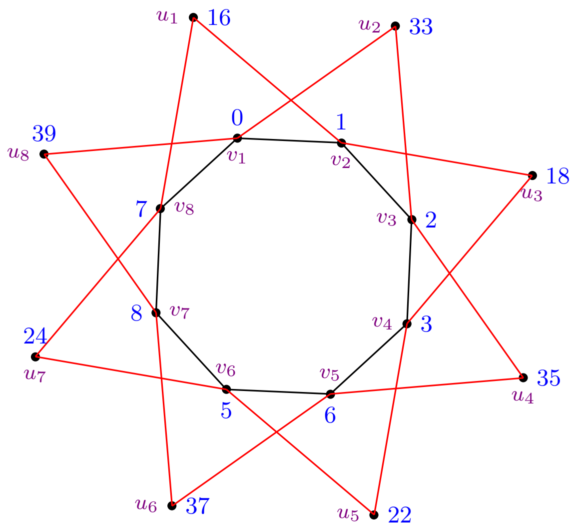

Figure 4: Odd harmonious labeling of $spl(C_8)$

Theorem 2.5. *The graph $D_2(H_{n,n})$ is an odd harmonious graph .*

Proof. Let $v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3 \cdots, u_n$ be the vertices of first copy of $H_{n,n}$ and $v_1^{'}, v_2^{'}, v_3^{'}, \cdots, v_n^{'}, u_1^{'}, u_2^{'}, u_3^{'} \cdots, u_n^{'}$ be the vertices of second copy of $H_{n,n}$. Let $G = D_2(H_{n,n})$. Then $|V(G)| = 4n$ and $|E(G)| = 2n(n+1)$. We define $f : V(G) \to \{0, 1, 2, \cdots, 4n(n+1) - 1\}$ as follows:

$f(v_i) = i(i-1),\ 1 \leq i \leq n$

$f(u_j) = (2n+1) - 2j,\ 1 \leq j \leq n$

$f(v_i^{'}) = i(i-1) + n(n+1),\ 1 \leq i \leq n$

$f(u_j^{'}) = (2n+1) - 2j + 2n(n+1),\ 1 \leq j \leq n$

The induced edge labelings are as follows:

$f^*(v_i u_j) = i(i-1) + (2n+1) - 2j$

$f^*(v_i^{'} u_j^{'}) = i(i-1) + 3n(n+1) + (2n+1) - 2j$

$f^*(v_i^{'} u_j) = i(i-1) + n(n+1) + (2n+1) - 2j$

$f^*(v_i u_j^{'}) = i(i-1) + (2n+1) - 2j + 2n(n+1).$

The induced edge labeling $f^*(E(G)) = \{1, 3, 5, \cdots, 4n(n+1) - 1\}$. Thus f is an odd harmonious labeling of G. Hence the graph $D_2(H_{n,n})$ is an odd harmonious graph. □

Illustration 5. *The odd harmonious labeling of $D_2(H_{3,3})$ is given in Figure 5*

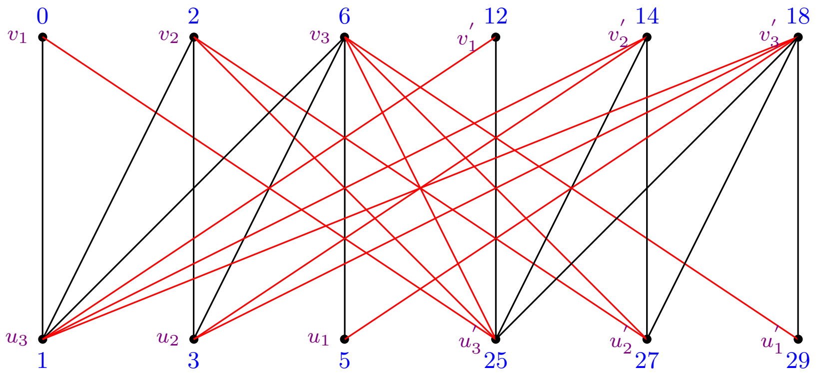

Figure 5: Odd harmonious labeling of $D_2(H_{3,3})$

Theorem 2.6. *The graph $spl(H_{n,n})$ is an odd harmonious graph.*

Proof. Let $v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3 \cdots, u_n$ be the vertices of first copy of $H_{n,n}$. Let $v_1^{'}, v_2^{'}, v_3^{'}, \cdots, v_n^{'}, u_1^{'}, u_2^{'}, u_3^{'} \cdots, u_n^{'}$ be the new vertices added to the corresponding vertices $v_1, v_2, v_3, \cdots, v_n, u_1, u_2, u_3 \cdots, u_n$ respectively. Let $G = spl(H_{n,n})$. Then $|V(G)|{=}4n$ and $|E(G)|{=}|E(H_{n-1,n-1})| + 3n$.

We define $f : V(G) \to \{0, 1, 2, \cdots, 2(|E(H_{n-1,n-1})| + 3n) - 1\}$ as follows:

$f(v_i) = i(i-1)$, $1 \leq i \leq n$

$f(u_j) = (2n+1) - 2j$, $1 \leq j \leq n$

$f(v_i^{'}) = i(i-1) + n(n+1)$, $1 \leq i \leq n$

$f(u_j^{'}) = (2n+1) - 2j + 2n(n+1)$, $1 \leq j \leq n$

The induced edge labelings are as follows:

$f^*(v_i u_j) = i(i-1) + (2n+1) - 2j$

$f^*(v_i^{'} u_j) = i(i-1) + n(n+1) + (2n+1) - 2j$

$f^*(v_i u_j^{'}) = i(i-1) + (2n+1) - 2j + 2n(n+1)$. The induced edge labeling is $f^*(E(G)) = \{1, 3, 5, \cdots, 2(|E(H_{n-1,n-1})| + 3n) - 1\}$. Thus f is an odd harmonious labeling of G. Hence the graph $spl(H_{n,n})$ is an odd harmonious graph. □

Illustration 6. *The odd harmonious labeling of $spl(H_{3,3})$ is shown in Figure 6*

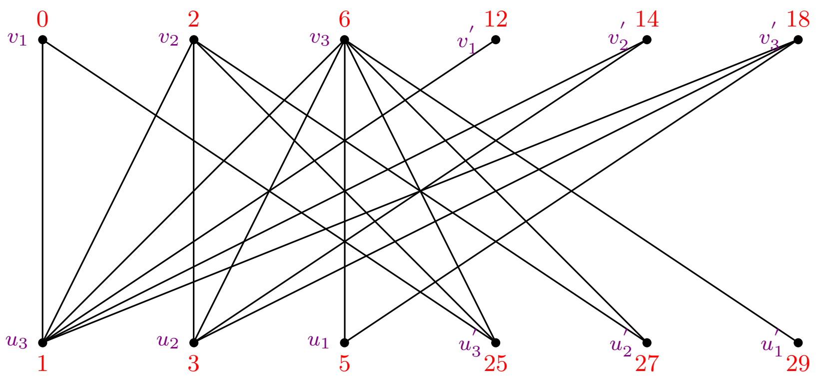

Figure 6: Odd harmonious labeling of $spl(H_{3,3})$

Theorem 2.7. *The double quadrilateral snake $DQ(n)$ is an odd harmonious graph for $n \geq 2$.*

Proof. Let $v_1, v_2, v_3, \cdots, v_n, v_{n+1}$; $u_1, u_2, u_3 \cdots, u_n$; $w_1, w_2, w_3, \cdots, w_n$; $s_1, s_2, s_3, \cdots, s_n$; $t_1, t_2, t_3, \cdots, t_n$ be the vertices of $DQ(n)$. Let $G = DQ(n)$. Hence $|V(G)| = 5n+1$ and $|E(G)| = 7n$. We define $f : V(G) \to \{0, 1, 2, \cdots, 14n-1\}$ as follows:

If n is even, then $f(v_1) = 0$ and $f(v_{n+1}) = 7n$

If n is odd, then $f(v_1) = 0$ and $f(v_{n+1}) = 7n - 4$

Also $f(v_i) = \begin{cases} f(v_{i-1}) + 3 & \text{if } i \text{ is even,} \\ f(v_{i-1}) + 11 & \text{if } i \text{ is odd} \end{cases}, 2 \leq i \leq n+1$

Now $f(u_i) = \begin{cases} f(v_i) + 1 & \text{if } i \text{ is odd,} \\ f(v_i) + 9 & \text{if } i \text{ is even} \end{cases}, 1 \leq i \leq n$

$f(w_i) = f(v_i) + 6$, $1 \leq i \leq n$

$f(s_i) = \begin{cases} f(v_i) + 5 & \text{if } i \text{ is odd,} \\ f(v_i) + 13 & \text{if } i \text{ is even} \end{cases}, 1 \leq i \leq n$

$f(t_i) = f(v_i) + 8$, $1 \leq i \leq n$

The induced edge labelings are as follows:

$f^*(v_i v_{i+1}) = 14i - 11$, $1 \leq i \leq n$

$f^*(v_i u_i) = 14i - 13$, $1 \leq i \leq n$

$f^*(u_i w_i) = 14i - 7$, $1 \leq i \leq n$

$f^*(v_i s_i) = 14i - 9$, $1 \leq i \leq n$

$f^*(s_i t_i) = 14i - 1$, $1 \leq i \leq n$

$f^*(w_i v_{i+1}) = 14i - 5$, $1 \leq i \leq n$

$f^*(t_i v_{i+1}) = 14i - 3$, $1 \leq i \leq n$

In view of the above defined labeling pattern, f admits odd harmonious labeling for $DQ(n)$. Hence $DQ(n)$, $n \geq 2$ is an odd harmonious graph. □

Illustration 7. *The odd harmonious labeling of $DQ(4)$ is shown in Figure 7*

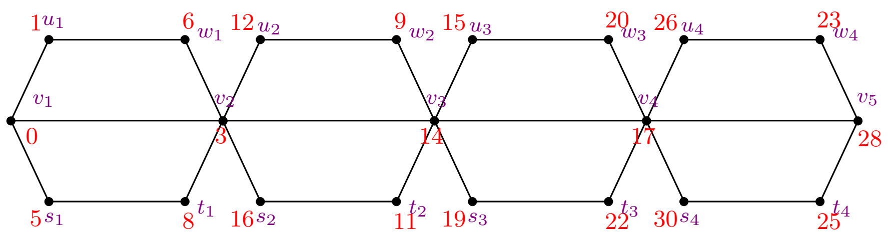

Figure 7: Odd harmonious labeling of $DQ(4)$

Theorem 2.8. *If m is odd, then $P_{r,m}$ is an odd harmonious graph for all the values of $r > 1$.*

Proof. let $v_0^i, v_1^i, v_2^i, \cdots, v_{2r}^i$ be the vertices of the i^{th} copy of the path of length r where $i = 1, 2, \cdots, m$, $v_0^i = u$ and $v_{2r}^i = v$ for all i. We observe that the number of vertices of the graph $P_{r,m}$ has $(r-1)m+2$ vertices and the number of edges of the graph is rm. We define $f : V(G) \to \{0, 1, 2, \cdots, 2rm-1\}$ as follows:

$f(u) = 0$; $f(v) = rm$; $f(v_1^1) = 1$

If $j = 1$, then $f(v_j^i) = f(v_j^{i-1}) + 2$, $i = 2, 3, \cdots, m$.

If $j = 2$, then $f(v_j^i) = \begin{cases} f(v_{j-1}^i) + 2m + 1, & i = 1, 2, \cdots, \left(\frac{m-1}{2}\right), \\ f(v_{j-1}^i) + 1 & \left(\frac{m-1}{2}\right) + 1 \le i \le m \end{cases}$

If $j = 3, 4, \cdots, r-1$, then $f(v_j^i) = f(v_{j-2}^i) + 2m$, $i = 1, 2, \cdots, m$. The induced edge labelings are as follows:

$f^*(uv_1^i) = 2i - 1$, $1 \le i \le m$.

Case i. $1 \le i \le \left(\frac{m-1}{2}\right)$.

$f^*(v_j^i v_{j+1}^i) = 2jm + 4i - 1$, $j = 1, 2, 3, 4, 5, \cdots (r-2)$

Case ii. $\left(\frac{m-1}{2}\right) + 1 \le i \le m$.

$f^*(v_j^i v_{j+1}^i) = 2(j-1)m + 4i - 1$, $j = 1, 2, 3, 4, 5, \cdots (r-2)$

If $j = r-1$ is even, then $f^*(v_{r-1}^i v) = \begin{cases} 4m + rm + 2i, & 1 \le i \le \left(\frac{m-1}{2}\right), \\ 2m + rm + 2i & \left(\frac{m-1}{2}\right) + 1 \le i \le m \end{cases}$

If $j = r-1$ is odd, then $f^*(v_{r-1}^i v) = 4m + rm + 2i - 1$, $1 \le i \le m$. In view of the above defined labeling pattern f is an odd harmonious labeling for $P_{r,m}$. Hence $P_{r,m}$ is an odd harmonious graph for all the values of $r > 1$. □

Illustration 8. *The odd harmonious labeling of $P_{5,5}$ is shown below:*

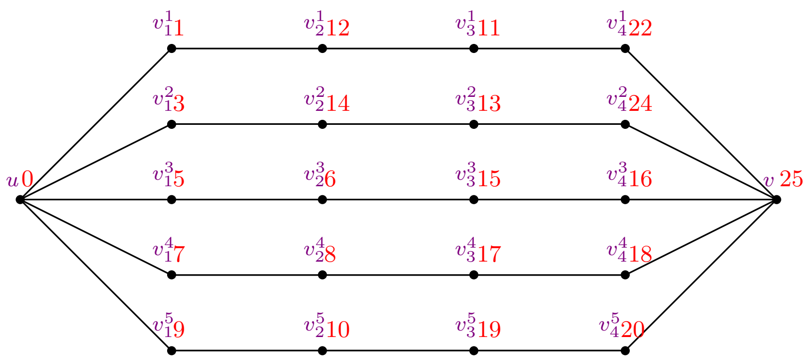

Figure 8: Odd harmonious labeling of $P_{5,5}$

Theorem 2.9. *Let $G_1(p_1, q_1), G_2(p_2, q_2), \cdots, G_m(p_m, q_m)$ be a strongly odd harmonious graphs and u_i and v_i be the vertices of G_i labeled with 0 and q_i $(1 \le i \le m)$ respectively. Then the path union graph G obtained by joining u_i with v_{i+1} by an edge for each i, $1 \le i \le m-1$ is a strongly odd harmonious graph.*

Proof. The path union graph G has $q = q_1+q_2+q_m+m-1$ edges. Let f_i be the strongly odd harmonious labeling of G_i and $V(G_i) = \{x_{ij} : j = 1, 2, \cdots, p_i\}$, $1 \le i \le m$. Then $V(G) = \bigcup_{i=1}^{m} V(G_i)$. Define a vertex labeling $f : V(G) \to \{0, 1, 2, \cdots, q\}$ as $f(x_{ij}) = f_i(x_{ij}) + \sum_{k=1}^{i-1} q_k + i - 1$. For each i, the vertex labels of G_i satisfy the inequality $\sum_{k=1}^{i-1} q_k+i-1 \le f(x_{ij}) \le \sum_{k=1}^{i} q_k+i-1$. Since f_i is an injection, f is also injection. The maximum vertex label of G is $\sum_{k=1}^{m} q_i + m - 1$. The set of edge labels of G_i is $\left\{2\sum_{k=1}^{i-1} q_k+2i-1, 2\sum_{k=1}^{i-1} q_k+2i+1, \cdots, 2\sum_{k=1}^{i} q_i+2i-3\right\}$ for $1 \le i \le m$. The label of the bridge between G_i and G_{i+1} is $2\sum_{k=1}^{i} q_k + 2i - 1$ for $1 \le i \le m-1$. Thus the set of edge labels of G is $\{1, 3, \cdots, 2q_1 - 1, 2q_1+1, 2q_1+3, \cdots, 2q_1+2q_2+1, 2q_1+2q_2+3, \cdots, 2\sum_{k=1}^{m} q_i+2m-3\}$. Hence f is a strongly harmonious odd labeling. □

Theorem 2.10. *Let $G_1(p_1, q_1)$ be a strongly odd harmonious graph and $G_2(p_2, q_2)$ be an odd harmonious graph. Then the graph G obtained by joining the vertex labeled by q_1 in $G_1$with the vertex labeled by 0 in G_2 is also odd harmonious graph.*

Proof. The graph G has $q = q_1 + q_2 + 1$ edges. Let f_1 be the strongly odd harmonious labeling of G_1 and f_2 be the odd harmonious labeling of G_2. Define a labeling $f : V(G) \to \{0, 1, 2, \cdots, q\}$ by $f(x)=\begin{cases} f_1(x) & \text{if} \quad x \in V(G_1) \\ f_2(x) + q_1 + 1 & \text{if} \quad x \in V(G_2). \end{cases}$

Since f_1 and f_2 are injective functions and $0 \le f_1(x) \le q_1$ and $0 \le f_2(x) \le 2q_2 - 1$, f is injective and $0 \le f(x) \le 2q_1 + 2q_2 - 1$. The edge labels of G_1 are remain constant and the edge labels G_2 are increased by $2q_1 + 2$. Hence the edge labels are $\{1, 3, \cdots, 2q_1 - 1, 2q_1 + 1, 2q_1 + 3, 2q_1 + 5 \cdots, 2q_1 + 2q_2 + 1\}$. Thus, f is an odd harmonious labeling of G. □

Illustration 9. *The odd harmonious labeling of path union of* 5 *cycles C_8 is shown in Figure 9*

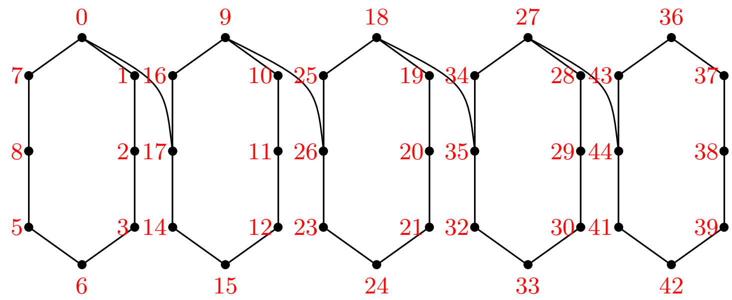

Figure 9: Odd harmonious labeling of path union of 5 cycles C_8

Theorem 2.11. *The banana tree $BT_k(n)$ is odd harmonious for all the values of $n \geq 1$ and $k \geq 1$*

Proof. Consider k disjoint stars $K_{1,n}$. Let v be a new vertex and $BT_k(n)$ the banana tree obtained by joining v to one pendant vertex of each star. Let $V(BT_k(n)) = \{a_i, v_{ij}, v | 1 \leq i \leq k$ and $1 \leq j \leq n\}$ and $E(BT_k(n)) = \{av_{ij} | 1 \leq j \leq k$ and $1 \leq j \leq n\} \cup \{vv_{in} | i = 1, 2, \cdots, k\}$.

We define $f : V(G) \to \{0, 1, 2, \cdots, 2(nk + k + 1) - 1\}$ as follows: $f(v) = 0$ and $f(v_{in}) = 2i - 1$, $i = 1, 2, 3, \cdots, k$

$f(a_i) = 2(2k - 2i + 1)$, $i = 1, 2, 3, \cdots, k$

$f(v_{ij}) = 2i - 1 + 2jk$, $i = 1, 2, 3, \cdots, k$ and $j = 1, 2, 3, \cdots, (n - 1)$.

The induced edge labelings are as follows: $f^*(vv_{in})$=$2i-1$, $i = 1, 2, 3, \cdots, k$

$f^*(a_i v_{in}) = 4k - 2i + 1$, $i = 1, 2, 3, \cdots, k$

$f^*(a_i v_{ij}) = 4k + 2jk - 2i + 1$, $i = 1, 2, 3, \cdots, k$ and $j = 1, 2, 3, \cdots, (n - 1)$.

In view of the above defined labeling pattern f is an odd harmonious labeling for $BT_k(n)$. Hence $BT_k(n)$ is an odd harmonious graph for all values of $n \geq 1$ and $k \geq 1$.

□

Illustration 10. *The odd harmonious labeling of $BT_3(5)$ is shown in Figure 10*

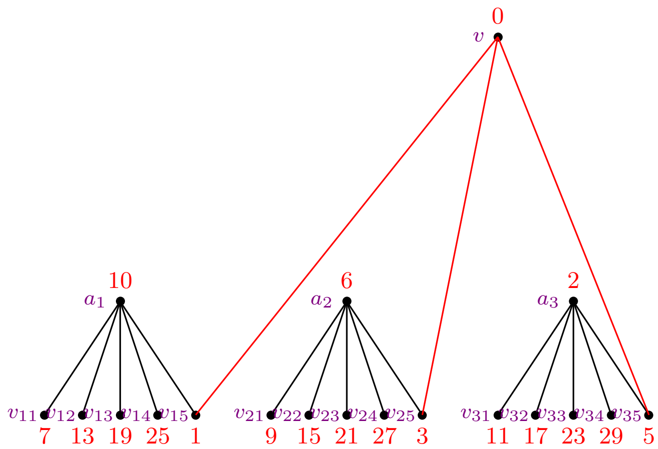

Figure 10: Odd harmonious labeling of $BT_3(5)$

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