

Super (a, d) -edge antimagic total labeling of some classes of graphs

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Abstract. A graph $G(V, E)$ is (a, d) -edge antimagic total if there exists a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that the edge-weights $\Lambda(uv) = f(u) + f(uv) + f(v)$, $uv \in E(G)$ form an arithmetic progression with first term a and common difference d . It is said to be a *super (a, d) -edge antimagic total* if $f(V(G)) = \{1, 2, \dots, |V(G)|\}$. In this paper, we have obtained a relation between a super $(a, 0)$ -edge antimagic total labeling and a super $(a, 2)$ -edge antimagic total labeling of any graph. Also we study the super (a, d) -edge antimagic total labeling of fan graphs and two special classes of star graphs namely bi-star and extended bi-star.

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§1. Introduction

By a *graph* $G(V, E)$ we mean a finite, undirected, connected graph without loops or multiple edges. The order and size of $G(V, E)$ are denoted by p and q respectively. For graph theoretic terminologies we refer to Harary [7].

A *labeling* of a graph is an assignment of numbers (usually positive or non-negative integers) to the vertices (a *vertex labeling*) or to the edges (an *edge labeling*) or to the combined set of vertices and edges (a *total labeling*) of the graph. There are many types of labelings and a detailed survey of many of them can be found in the dynamic survey of graph labeling by J.A. Gallian [6].

The *edge weight* of an edge uv , denoted by $\Lambda(uv)$, is defined as the sum of labels of the graph elements associated with uv . That is, if f is an edge labeling, then $\Lambda(uv) = f(uv)$; if f is a vertex labeling, then $\Lambda(uv) = f(u) + f(v)$; and if f is a total labeling, then $\Lambda(uv) = f(u) + f(uv) + f(v)$. Similarly

the *vertex weight* of a vertex v , denoted by $\Lambda(v)$, is defined as the sum of labels of the graph elements associated with v . That is, if f is a vertex labeling, then $\Lambda(v) = \sum_{u \in N(v)} f(u)$; if f is an edge labeling, then $\Lambda(v) = \sum_{uv \in E} f(uv)$; and if f is a total labeling, then $\Lambda(v) = f(v) + \sum_{uv \in E} f(uv)$.

In 1970 Kotzig and Rosa [9] defined an *edge-magic total labeling* of a graph $G(V, E)$ as a bijection f from $V \cup E$ to the set $\{1, 2, \dots, |V| + |E|\}$ such that for each edge $uv \in E$, the edge weight $f(u) + f(uv) + f(v)$ is a constant.

Enomoto et al. [4] defined a *super edge magic labeling* as an edge-magic total labeling such that the vertex labels are $\{1, 2, \dots, |V|\}$ and the edge labels are $\{|V| + 1, |V| + 2, \dots, |V| + |E|\}$. They have proved that if a graph with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$. They also conjectured that every tree is super edge-magic.

As a natural extension of the notion of edge-magic total labeling, Simanjuntak et al. [10] defined an (a, d) -*edge antimagic total labeling* of a graph $G(V, E)$ as an injective mapping f from $V \cup E$ onto the set $\{1, 2, \dots, |V| + |E|\}$ such that the set $\{f(u) + f(uv) + f(v) | uv \in E\}$ is $\{a, a + d, a + 2d, \dots, a + (|E| - 1)d\}$ for any two integers $a > 0$ and $d \geq 0$.

An (a, d) -edge antimagic total labeling of a graph $G(V, E)$ is called a *super (a, d) -edge antimagic total* if the vertex labels are $\{1, 2, \dots, |V|\}$ and the edge labels are $\{|V| + 1, |V| + 2, \dots, |V| + |E|\}$. The super $(a, 0)$ -edge antimagic total labelings are usually called as super edge magic in the literature (see [4, 5]).

Many researchers investigated different forms of antimagic labelings [8]. Bača et al. [1, 2] proved several results on antimagic labelings. Also in [3] Bača and Barrientos presented some relationships between (a, d) -edge antimagic vertex labelings and super (a, d) -edge antimagic total labelings.

In this paper, we prove that a graph is super $(a_1, 0)$ -edge antimagic total, then it is super $(a_2, 2)$ -edge antimagic total. Also we study the super (a, d) -edge antimagic total labeling of fan graphs and two special classes of star graphs namely bi-star and extended bi-star.

§2. Super (a, d) -edge antimagic total labeling

The following theorem gives a relation between a super $(a_1, 0)$ -edge antimagic total labeling and a super $(a_2, 2)$ -edge antimagic total labeling of any graph.

Theorem 1. *If a graph $G(V, E)$ is super $(a_1, 0)$ -edge antimagic total, then it is super $(a_2, 2)$ -edge antimagic total.*

Proof. Suppose the graph $G(V, E)$ is super $(a_1, 0)$ -edge antimagic total, then by definition, there exists a bijection $f : V \cup E \rightarrow \{1, 2, \dots, p + q\}$ such that

- (i) $\{f(v) | v \in V\} = \{1, 2, \dots, p\}$
- (ii) $\{f(uv) | uv \in E\} = \{p + 1, p + 2, \dots, p + q\}$ and
- (iii) for all $uv \in E$, $f(u) + f(uv) + f(v) = a_1$.

In order to prove $G(V, E)$ has a super $(a_2, 2)$ -edge antimagic total labeling, we define an induced map g_f as follows:

Let $g_f : V \cup E \rightarrow \{1, 2, \dots, p + q\}$ such that

- (i) for all $u \in V$, $g_f(u) = f(u)$ and
- (ii) for all $uv \in E$, $g_f(uv) = 2p + q + 1 - f(uv)$.

Then we see that $\{g_f(v) | v \in V\} = \{1, 2, \dots, p\}$ and $\{g_f(uv) | uv \in E\} = \{p + 1, p + 2, \dots, p + q\}$.

Also for all $uv \in E$ we have

$$\begin{aligned} g_f(u) + g_f(uv) + g_f(v) &= f(u) + 2p + q + 1 - f(uv) + f(v) \\ &= 2p + q + 1 + a_1 - 2f(uv) \\ &= a_1 - q + 1 + 2(p + q) - 2f(uv). \end{aligned}$$

Thus the set of edge-weights is in arithmetic progression with first term $(a_1 - q + 1)$ and common difference 2.

Hence $G(V, E)$ is super $(a_2, 2)$ -edge antimagic total with $a_2 = (a_1 - q + 1)$. \square

§3. Fan graph

A fan graph $F_{m,2}$ is defined as the graph join $\bar{K}_m + P_2$, where \bar{K}_m is an empty graph with m vertices and P_2 is a path with 2 vertices. Let the vertices be $u_1, u_2, \dots, u_m, v_1, v_2$ and the edges be v_1v_2 and u_iv_j , $1 \leq i \leq m$, $1 \leq j \leq 2$.

Theorem 2. *If the fan graph $F_{m,2}$, $m \geq 2$, is super (a, d) -edge antimagic total, then $d \leq 2$.*

Proof. Assume that $F_{m,2}$, $m \geq 2$ has a super (a, d) -edge antimagic total labeling $f : V(F_{m,2}) \cup E(F_{m,2}) \rightarrow \{1, 2, \dots, 3m + 3\}$ such that the set of edge-weights is given by $\{a, a + d, \dots, a + 2md\}$.

It is easy to see that the minimum possible edge-weight in a super (a, d) -edge antimagic total labeling is at least $|V| + 4$. On the other hand, the

maximum edge weight is no more than $3|V| + |E| - 1$.

Thus we have,

$$(3.1) \quad a \geq m + 6$$

and

$$(3.2) \quad a + 2md \leq 5m + 6.$$

From the inequalities (3.1) and (3.2), we get $d \leq 2$. \square

Theorem 3. *Every fan graph $F_{m,2}$, $m \geq 2$ has a super $(a, 0)$ -edge antimagic total labeling.*

Proof. Let us define the vertex labeling $f_1 : V(F_{m,2}) \rightarrow \{1, 2, \dots, m + 2\}$ and the edge labeling $f_2 : E(F_{m,2}) \rightarrow \{m + 3, m + 4, \dots, 3m + 3\}$ as follows:

$$f_1(v_1) = 1; \quad f_1(v_2) = m + 2 \quad \text{and} \quad f_2(v_1v_2) = 2m + 3.$$

For $1 \leq i \leq m$,

$$f_1(u_i) = i + 1; \quad f_2(u_iv_1) = 3m + 4 - i \quad \text{and} \quad f_2(u_iv_2) = 2m + 3 - i.$$

It is easy to verify that $\{\Lambda(uv) | uv \in E(F_{m,2})\} = 3(m + 2)$.

Thus the labelings f_1 and f_2 are super $(a, 0)$ -edge antimagic total labeling of $F_{m,2}$, $m \geq 2$ with $a = 3(m + 2)$. \square

In view of Theorem 1, it is clear that the fan graph $F_{m,2}$, $m \geq 2$ has a super $(a, 2)$ -edge antimagic total labeling with $a = m + 6$.

Theorem 4. *Every fan graph $F_{m,2}$, $m \geq 2$ has a super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let the vertex labeling f_1 be defined as in Theorem 3.

We define the edge labeling $f_3 : E(F_{m,2}) \rightarrow \{m + 3, m + 4, \dots, 3m + 3\}$ as follows:

Case (i) m is odd:

$$f_3(v_1v_2) = 2(m + 1) - \left(\frac{m - 1}{2}\right)$$

For $1 \leq i \leq m$

$$f_3(u_iv_1) = \begin{cases} 3(m + 1) - \frac{i-1}{2}, & \text{if } i \text{ is odd} \\ 2m + 3 - \frac{i}{2}, & \text{if } i \text{ is even} \end{cases}$$

and

$$f_3(u_i v_2) = \begin{cases} 3(m+1) - \frac{m+i}{2}, & \text{if } i \text{ is odd} \\ 2(m+1) - \frac{m-1+i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

Case (ii) m is even:

$$f_3(v_1 v_2) = 3(m+1) - \frac{m}{2}$$

For $1 \leq i \leq m$

$$f_3(u_i v_1) = \begin{cases} 3(m+1) - \frac{i-1}{2}, & \text{if } i \text{ is odd} \\ 2m+3 - \frac{i}{2}, & \text{if } i \text{ is even} \end{cases}$$

and

$$f_3(u_i v_2) = \begin{cases} 2m+3 - \frac{m+i+1}{2}, & \text{if } i \text{ is odd} \\ 3(m+1) - \frac{m+i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

In both the cases, it is easy to see that $\{\Lambda(uv) | uv \in E(F_{m,2})\} = \{(2m+6), (2m+7), \dots, (4m+6)\}$.

Thus the labelings f_1 and f_3 are super $(a, 1)$ -edge antimagic total labeling of $F_{m,2}$, $m \geq 2$ with $a = 2m+6$. \square

§4. Bistar

A *bistar* $B_{m,n}$ is defined as the graph obtained by attaching an edge with the center vertices of two stars $K_{1,m}$ and $K_{1,n}$. Let the vertices be $c_1, c_2, u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ and the edges be $c_1 c_2, c_1 u_i, 1 \leq i \leq m$ and $c_2 v_j, 1 \leq j \leq n$.

Theorem 5. *If the bistar $B_{m,n}$, $m \geq 2$, $n \geq 2$ is super (a, d) -edge antimagic total, then $d \leq 3$.*

Proof. Assume that $B_{m,n}$, $m \geq 2$, $n \geq 2$ has a super (a, d) -edge antimagic total labeling $f : V(B_{m,n}) \cup E(B_{m,n}) \rightarrow \{1, 2, \dots, 2m+2n+3\}$ such that the set of edge-weights is given by $\{a, a+d, \dots, a+(m+n)d\}$.

Clearly the maximum edge-weight is no more than

$$(m+n+1) + (m+n+2) + (2m+2n+3).$$

Thus,

$$(4.1) \quad a + (m+n)d \leq 4m+4n+6.$$

On the other hand, the minimum possible edge-weight is at least $1 + 2 + (m + n + 3)$.

Thus,

$$(4.2) \quad a \geq m + n + 6.$$

From the inequalities (4.1) and (4.2), we get $d \leq 3$. \square

Theorem 6. *Every bistar $B_{m,n}$, $m \geq 2$, $n \geq 2$ has a super $(a, 0)$ -edge antimagic total labeling.*

Proof. Let us define the vertex labeling $f_4 : V(B_{m,n}) \rightarrow \{1, 2, \dots, m + n + 2\}$ and the edge labeling $f_5 : E(B_{m,n}) \rightarrow \{m + n + 3, m + n + 4, \dots, 2m + 2n + 3\}$ as follows:

$$f_4(c_1) = 1; \quad f_4(c_2) = m + n + 2 \quad \text{and} \quad f_5(c_1c_2) = m + 2n + 3.$$

For $1 \leq i \leq m$

$$f_4(u_i) = n + i + 1; \quad f_5(c_1u_i) = 2m + 2n + 4 - i$$

and for $1 \leq j \leq n$

$$f_4(v_j) = j + 1; \quad f_5(c_2v_j) = m + 2n + 3 - j.$$

By direct computation we obtain that $\{\Lambda(uv) | uv \in E(B_{m,n})\} = 2m + 3n + 6$.

Thus the labelings f_4 and f_5 are super $(a, 0)$ -edge antimagic total labeling of $B_{m,n}$, $m \geq 2$, $n \geq 2$ with $a = 2m + 3n + 6$. \square

In view of Theorem 1, it is clear that the bistar $B_{m,n}$, $m \geq 2$, $n \geq 2$ has a super $(a, 2)$ -edge antimagic total labeling with $a = m + 2n + 6$.

Theorem 7. *For $n \in \{m - 1, m, m + 1\}$ or $(m + n) \equiv 0 \pmod{2}$, the bistar $B_{m,n}$, $m \geq 2$, $n \geq 2$ has a super $(a, 1)$ -edge antimagic total labeling.*

In order to prove the theorem, we prove the following lemmas.

Lemma 1. *For $n \in \{m - 1, m, m + 1\}$, $m \geq 2$, the bistar $B_{m,n}$, has a super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let us define the vertex labeling $g_1 : V(B_{m,n}) \rightarrow \{1, 2, \dots, m + n + 2\}$ and the edge labeling $g_2 : E(B_{m,n}) \rightarrow \{m + n + 3, m + n + 4, \dots, 2m + 2n + 3\}$ as follows:

Case (i) $n = m - 1$:

$$\begin{aligned} g_1(c_1) &= 2, \\ g_1(c_2) &= m + n + 2, \\ g_1(u_i) &= 2i - 1, \quad 1 \leq i \leq m, \\ g_1(v_j) &= 2(j + 1), \quad 1 \leq j \leq n, \\ g_2(c_1c_2) &= m + 2n + 3, \\ g_2(c_1u_i) &= 2(m + n + 2) - i, \quad 1 \leq i \leq m, \\ g_2(c_2v_j) &= m + 2n + 3 - j, \quad 1 \leq j \leq n. \end{aligned}$$

Case (ii) $n = m$:

$$\begin{aligned} g_1(c_1) &= 2, \\ g_1(c_2) &= m + n + 1, \\ g_1(u_i) &= 2i - 1, \quad 1 \leq i \leq m, \\ g_1(v_j) &= 2(j + 1), \quad 1 \leq j \leq n \text{ and } g_2 \text{ same as in Case (i)}. \end{aligned}$$

Case (iii) $n = m + 1$:

$$\begin{aligned} g_1(c_1) &= m + n + 2, \\ g_1(c_2) &= 2, \\ g_1(u_i) &= 2(i + 1), \quad 1 \leq i \leq m, \\ g_1(v_j) &= 2j - 1, \quad 1 \leq j \leq n, \\ g_2(c_1c_2) &= 2m + n + 3, \\ g_2(c_1u_i) &= 2m + n + 3 - i, \quad 1 \leq i \leq m, \\ g_2(c_2v_j) &= 2m + 2n + 4 - j, \quad 1 \leq j \leq n. \end{aligned}$$

In all the above three cases, it is easy to verify that $\{\Lambda(uv) | uv \in E(B_{m,n})\} = \{2m + 2n + 6, 2m + 2n + 7, \dots, 3m + 3n + 6\}$.

Thus the labelings g_1 and g_2 are super $(a, 1)$ -edge antimagic total labeling of $B_{m,n}$, $n \in \{m - 1, m, m + 1\}$, $m \geq 2$ with $a = 2m + 2n + 6$. \square

Lemma 2. For $(m + n) \equiv 0 \pmod{2}$, the bistar $B_{m,n}$ has a super $(a, 1)$ -edge antimagic total labeling.

Proof. Let the vertex labeling f_4 be defined as in Theorem 6.

We define the edge labeling $g_3 : E(B_{m,n}) \rightarrow \{m + n + 3, m + n + 4, \dots, 2m + 2n + 3\}$ as follows:

Case (i) m and n are even:

$$g_3(c_1c_2) = 2m + 2n + 3 - \frac{m}{2},$$

For $1 \leq i \leq m$,

$$g_3(c_1u_i) = \begin{cases} 2m + 2n + 3 - \frac{i-1}{2}, & \text{if } i \text{ is odd} \\ 2m + 2n + 3 - \frac{m+n+i}{2}, & \text{if } i \text{ is even} \end{cases}$$

and for $1 \leq j \leq n$,

$$g_3(c_2v_j) = \begin{cases} 2m + 2n + 3 - \frac{2m+n+j+1}{2}, & \text{if } j \text{ is odd} \\ 2m + 2n + 3 - \frac{m+j}{2}, & \text{if } j \text{ is even.} \end{cases}$$

Case (ii) m and n are odd:

$$g_3(c_1c_2) = 2m + 2n + 3 - \frac{2m + n + 1}{2},$$

For $1 \leq i \leq m$,

$$g_3(c_1u_i) = \begin{cases} 2m + 2n + 3 - \frac{i-1}{2}, & \text{if } i \text{ is odd} \\ 2m + 2n + 3 - \frac{m+n+i}{2}, & \text{if } i \text{ is even} \end{cases}$$

and for $1 \leq j \leq n$,

$$g_3(c_2v_j) = \begin{cases} 2m + 2n + 3 - \frac{m+j}{2}, & \text{if } j \text{ is odd} \\ 2m + 2n + 3 - \frac{2m+n+j+1}{2}, & \text{if } j \text{ is even.} \end{cases}$$

In both the cases, we see that the bistar $B_{m,n}$ is super $(a, 1)$ -edge antimagic total with $a = 2m + 3n - \frac{m+n}{2} + 6$. \square

Proof of Theorem 7, directly follows from Lemmas 1 and 2.

Theorem 8. *For $n \in \{m-1, m, m+1\}$, $m \geq 2$, the bistar $B_{m,n}$ has a super $(a, 3)$ -edge antimagic total labeling.*

Proof. Let the vertex labeling g_1 be defined as in Lemma 1.

We define the edge labeling $f_6 : E(B_{m,n}) \rightarrow \{m+n+3, m+n+4, \dots, 2m+2n+3\}$ as follows:

Case (i) $n = m-1$ or $n = m$:

$$\begin{aligned} f_6(c_1c_2) &= 2m + n + 3, \\ f_6(c_1u_i) &= m + n + 2 + i, \quad 1 \leq i \leq m, \\ f_6(c_2v_j) &= 2m + n + 3 + j, \quad 1 \leq j \leq n. \end{aligned}$$

Case (ii) $n = m+1$:

$$f_6(c_1c_2) = m + 2n + 3,$$

$$\begin{aligned} f_6(c_1u_i) &= m + 2n + 3 + i, \quad 1 \leq i \leq m, \\ f_6(c_2v_j) &= m + n + 2 + j, \quad 1 \leq j \leq n. \end{aligned}$$

In both the cases, we see that $\{\Lambda(uv) | uv \in E(B_{m,n})\} = \{m + n + 6, m + n + 6 + 3, \dots, 4m + 4n + 6\}$.

Thus the bistar $B_{m,n}$, $n \in \{m - 1, m, m + 1\}$, $m \geq 2$ is super $(a, 3)$ -edge antimagic total with $a = m + n + 6$. \square

§5. Extended Bistar

An *extended bistar* $\langle K_{1,m} : n \rangle$ is defined as the graph obtained by attaching a path of length n with the centre vertices of two copies of the star graph $K_{1,m}$. Let the vertices be $c_1, c_2, \dots, c_{n+1}, u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m$ and the edges be $c_1u_i, c_{n+1}v_i$, $1 \leq i \leq m$ and c_jc_{j+1} , $1 \leq j \leq n$.

Theorem 9. *If the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ is super (a, d) -edge antimagic total, then $d \leq 3$.*

Proof. Assume that $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ has a super (a, d) -edge antimagic total labeling $f : V(\langle K_{1,m} : n \rangle) \cup E(\langle K_{1,m} : n \rangle) \rightarrow \{1, 2, \dots, 4m + 2n + 1\}$ such that the set of edge-weights is given by $\{a, a + d, \dots, a + (2m + n - 1)d\}$.

Clearly the maximum edge-weight is no more than

$$(2m + n) + (2m + n + 1) + (4m + 2n + 1).$$

Thus,

$$(5.1) \quad a + (2m + n - 1)d \leq 8m + 4n + 2.$$

On the other hand, the minimum possible edge-weight is at least $1 + 2 + (2m + n + 2)$.

Thus,

$$(5.2) \quad a \geq 2m + n + 5.$$

From the inequalities (5.1) and (5.2), we get $d \leq 3$. \square

Theorem 10. *Every extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ has a super $(a, 0)$ -edge antimagic total labeling.*

Proof. Let us define the vertex labeling $f_7 : V(\langle K_{1,m} : n \rangle) \rightarrow \{1, 2, \dots, 2m + n + 1\}$ and the edge labeling $f_8 : E(\langle K_{1,m} : n \rangle) \rightarrow \{2m + n + 2, 2m + n + 3, \dots, 4m + 2n + 1\}$ as follows:

Case (i) n is odd:

For $1 \leq i \leq m$,

$$f_7(u_i) = \left(\frac{n+1}{2}\right) + m + i; \quad f_7(v_i) = \left(\frac{n+1}{2}\right) + i$$

and for $1 \leq j \leq n+1$,

$$f_7(c_j) = \begin{cases} \left(\frac{j+1}{2}\right), & \text{if } j \text{ is odd} \\ 2m + \left(\frac{n+j+1}{2}\right), & \text{if } j \text{ is even.} \end{cases}$$

Case (ii) n is even:

For $1 \leq i \leq m$,

$$f_7(u_i) = \frac{n}{2} + i + 1; \quad f_7(v_i) = m + n + i + 1$$

and for $1 \leq j \leq n+1$,

$$f_7(c_j) = \begin{cases} \left(\frac{j+1}{2}\right), & \text{if } j \text{ is odd} \\ m + 1 + \left(\frac{n+j}{2}\right), & \text{if } j \text{ is even.} \end{cases}$$

For any n , we define

$$\begin{aligned} f_8(c_1u_i) &= 4m + 2(n+1) - i, \quad 1 \leq i \leq m, \\ f_8(c_jc_{j+1}) &= 3m + 2(n+1) - j, \quad 1 \leq j \leq n, \\ f_8(c_{n+1}v_i) &= 3m + (n+2) - i, \quad 1 \leq i \leq m. \end{aligned}$$

By direct computation, we get

$$\{\Lambda(uv) | uv \in E\} = \begin{cases} 5\left(m + \frac{n+1}{2}\right) + 1, & \text{if } n \text{ is odd} \\ 4(m+1) + \frac{5n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Thus the labelings f_7 and f_8 are super $(a, 0)$ -edge antimagic total labeling of $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ with

$$a = \begin{cases} 5\left(m + \frac{n+1}{2}\right) + 1, & \text{if } n \text{ is odd} \\ 4(m+1) + \frac{5n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

□

In view of Theorem 1, it is clear that the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ has a super $(a, 2)$ -edge antimagic total labeling with

$$a = \begin{cases} 3\left(m + \frac{n+3}{2}\right), & \text{if } n \text{ is odd} \\ 2(m+2) + \frac{3n}{2} + 1, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 11. *For odd n , the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ has a super $(a, 1)$ -edge antimagic total labeling.*

Proof. Let us define the vertex labeling $f_9 : V(\langle K_{1,m} : n \rangle) \rightarrow \{1, 2, \dots, 2m + n + 1\}$ as follows:

$$f_9(u_i) = 2i, \quad 1 \leq i \leq m,$$

$$f_9(v_i) = n + 2i, \quad 1 \leq i \leq m, \text{ (since } n \text{ is odd, } f_9 \text{ is bijective)}$$

and for $1 \leq j \leq n + 1$,

$$f_9(c_j) = \begin{cases} j, & \text{if } j \text{ is odd} \\ 2m + j, & \text{if } j \text{ is even.} \end{cases}$$

Let the edge labeling f_8 be as defined in Theorem 10.

Then we see that $\{\Lambda(uv) | uv \in E\} = \{4m + 2n + 4, 4m + 2n + 5, \dots, 6m + 3n + 3\}$.

Hence, when n is odd, the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ is super $(a, 1)$ -edge antimagic total with $a = 4m + 2n + 4$. \square

Theorem 12. *For odd n , the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ has a super $(a, 3)$ -edge antimagic total labeling.*

Proof. Let the vertex labeling f_9 be as defined in Theorem 11.

We define the edge labeling $f_{10} : E(\langle K_{1,m} : n \rangle) \rightarrow \{2m + n + 2, 2m + n + 3, \dots, 4m + 2n + 1\}$ as follows:

$$f_{10}(c_1u_i) = 2m + n + 1 + i, \quad 1 \leq i \leq m,$$

$$f_{10}(c_jc_{j+1}) = 3m + n + 1 + j, \quad 1 \leq j \leq n,$$

$$f_{10}(c_{n+1}v_i) = 3m + 2n + 1 + i, \quad 1 \leq i \leq m.$$

Then we see that $\{\Lambda(uv) | uv \in E\} = \{2m + n + 5, 2m + n + 5 + 3, \dots, 2m + n + 5 + (2m + n - 1)3\}$.

Hence, when n is odd, the extended bistar $\langle K_{1,m} : n \rangle$, $m, n \geq 2$ is super $(a, 3)$ -edge antimagic total with $a = 2m + n + 5$. \square

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References

- [1] M. Bača, Y. Lin, M. Miller, M.Z. Youssef, *Edge antimagic graphs*, Discrete Mathematics, **307** (2007), 1232–1244.
- [2] M. Bača, Barrientos, *On super edge antimagic total labelings of mK_n* , Discrete Mathematics, **308** (2008), 5032–5037.
- [3] M. Bača, Barrientos, *Graceful and edge antimagic labelings*, Ars Combinatoria, to appear.
- [4] H. Enomoto, A.S. Llado, T. Nakanigawa, G. Ringel, *Super edge magic graphs*, SUT J. of Math. **34** (1998), 105–109.
- [5] R.M. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, *The place of super edge-magic labelings among other classes of labelings*, Discrete Mathematics, **231** (2001), 153–168.
- [6] J.A. Gallian, *A dynamic survey of graph labeling*, Electronic Journal of Combinatorics 18 (2011), #DS6.
- [7] F. Harary, *Graph Theory* (Addison-Wesley, 1994).
- [8] N. Hartsfield, G. Ringel, *Pearls in Graph Theory*, (Academic Press, Boston, San Diego, New York, London, 1990).
- [9] A. Kotzig, A. Rosa, *Magic valuation of finite graphs*, Canad. Math. Bull., **13** (1970), 451–561.
- [10] R. Simanjuntak, F. Bertault, M. Miller, *Two new (a, d) -antimagic graph labelings*, Proc. Eleventh Australian Workshop Combin. Algor., Hunrer Valley, Australia (2000), 179–189.

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