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Double symmetry model and its orthogonal decomposition for multi-way tables

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Abstract. For two-way contingency tables, Tomizawa (1985b) gave a theorem that the double symmetry (DS) model holds if and only if both the quasi DS and the marginal DS models hold. The present paper proposes, for multi-way tables, the DS, some quasi DS and marginal DS models, and extends Tomizawa's theorem into multi-way tables. It also shows that for multi-way tables the likelihood ratio statistic for testing goodness of fit of the DS model is asymptotically equivalent to the sum of those for testing the quasi DS model with some order and the marginal DS with the corresponding order. An example is given.

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§1. Introduction

Consider an $r \times r$ square table. Bowker (1948) considered the symmetry (S) model, Caussinus (1965) proposed the quasi symmetry (Q) model, and Stuart (1955) considered the marginal homogeneity (M) model (see also, e.g., Bishop, Fienberg and Holland, 1975, p.282; van der Heijden, Falguerolles and Leeuw, 1989). Caussinus (1965) pointed out that the S model holds if and only if both the Q and M models hold.

For multi-way r^T tables, the complete symmetry (S^T), the hth-order quasi-symmetry (Q_h^T) and the hth-order marginal symmetry (M_h^T) models are considered; see, e.g., Bishop et al. (1975, p.299), Bhapkar and Darroch (1990), and Agresti (2002, p.440). Bhapkar and Darroch (1990) pointed out that for a fixed h ($h = 1, \ldots, T-1$), the S^T model holds if and only if both the Q_h^T and M_h^T models hold; see also Tomizawa and Tahata (2007).

Wall and Lienert (1976) considered the point symmetry (P^T) model. Tahata and Tomizawa (2008) proposed the hth-order quasi point-symmetry (QP_h^T) and the hth-order marginal point-symmetry (MP_h^T) models. Tahata and Tomizawa (2008) also pointed out that for a fixed h ($h = 1, \ldots, T-1$), the P^T model holds if and only if both the QP_h^T and MP_h^T models hold. Note that when $T = 2$, these were given by Tomizawa (1985a).

For an $r \times r$ square contingency table with the same row and column classifications, let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). One of our interests is also whether or not there is a structure of both symmetry and point symmetry (rather than independence) in the table. Tomizawa (1985b) considered the double symmetry (DS^2) model, defined by

$$p_{ij} = p_{ji} = p_{i^*j^*} = p_{j^*i^*} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where the symbol "$*$" denotes $i^* = r+1-i$. This model indicates a structure of double symmetry of the probabilities with respect to the center point and with respect to the main diagonal of the table. Tomizawa (1985b) considered the quasi double symmetry (QDS^2) and the marginal double symmetry (MDS^2) models. The QDS^2 model is defined by

$$p_{ij} = \mu\alpha_i\beta_j\psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji} = \psi_{i^*j^*} = \psi_{j^*i^*}$. The MDS^2 model is defined by

$$p_{i\cdot} = p_{\cdot i} = p_{i^*\cdot} = p_{\cdot i^*} \quad (i = 1, \ldots, r),$$

where $p_{i\cdot} = \sum_{t=1}^{r} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{r} p_{si}$. This indicates that the row marginal distribution is identical to the column marginal distribution and each marginal distribution is symmetric with respect to the midpoint of the categories.

Tomizawa (1985b) also gave the decomposition of the DS^2 model as follows:

Theorem 1. *For the $r \times r$ table, the DS^2 model holds if and only if both the QDS^2 and MDS^2 models hold.*

We are now interested in extending Theorem 1 into multi-way r^T contingency tables ($T \geq 3$).

The purpose of this paper is (i) to extend the DS^2, QDS^2 and MDS^2 models into multi-way r^T tables (say DS^T, QDS^T and MDS^T models), (ii) to extend Theorem 1 into multi-way tables, and (iii) to show that for multi-way tables the test statistic for the DS^T model is asymptotically equivalent to the sum of those for the QDS^T and MDS^T models. Section 2 proposes new models, i.e., the DS^T model, and some QDS^T and MDS^T models with the hth-order ($h = 1, \ldots, T-1$). Section 3 gives the decomposition of the DS^T model, and Section 4 shows the orthogonality of decomposition with respect to the goodness of fit test statistic. Section 5 gives an example.

§2. Models

2.1. Case of three-way tables

Consider an $r \times r \times r$ contingency table. Denote the kth variable by X_k ($k = 1, 2, 3$).

We shall now consider the double symmetry (DS^3) model as follows:

$$p_{ijk} = p_{ikj} = p_{jik} = p_{jki} = p_{kij} = p_{kji} = p_{i^*j^*k^*} \quad (1 \le i, j, k \le r).$$

This model indicates both of symmetry and point symmetry.

The DS^3 model may be expressed in a log-linear form

$$\log p_{ijk} = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)} + u_{123(ijk)} \quad (1 \le i, j, k \le r),$$

where

$$u_{1(i)} = u_{2(i)} = u_{3(i)} = u_{1(i^*)},$$

$$u_{12(ij)} = u_{13(ij)} = u_{23(ij)} = u_{12(ji)} = u_{12(i^*j^*)},$$

$$u_{123(ijk)} = u_{123(ikj)} = u_{123(jik)} = u_{123(jki)} = u_{123(kij)} = u_{123(kji)} = u_{123(i^*j^*k^*)}.$$

Next, consider two kinds of quasi double symmetry models. Consider a model defined by

$$\log p_{ijk} = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)} + u_{123(ijk)} \quad (1 \le i, j, k \le r),$$

where

$$u_{12(ij)} = u_{13(ij)} = u_{23(ij)} = u_{12(ji)} = u_{12(i^*j^*)},$$

$$u_{123(ijk)} = u_{123(ikj)} = u_{123(jik)} = u_{123(jki)} = u_{123(kij)} = u_{123(kji)} = u_{123(i^*j^*k^*)}.$$

We will refer to this model as the first-order quasi double symmetry (QDS_1^3) model. Note that the DS^3 model is a special case of the QDS_1^3 model.

Using the odds ratios, the QDS_1^3 model can also be expressed as

$$\theta^{1(i)}_{(j_1<j_2;k_1<k_2)} = \theta^{1(i)}_{(k_1<k_2;j_1<j_2)} = \theta^{1(i^*)}_{(j_2^*<j_1^*;k_2^*<k_1^*)} = \theta^{1(i^*)}_{(k_2^*<k_1^*;j_2^*<j_1^*)} \quad (1 \le i \le r; 1 \le j_1 < j_2 \le r; 1 \le k_1 < k_2 \le r),$$

$$\theta^{2(j)}_{(i_1<i_2;k_1<k_2)} = \theta^{2(j)}_{(k_1<k_2;i_1<i_2)} = \theta^{2(j^*)}_{(i_2^*<i_1^*;k_2^*<k_1^*)} = \theta^{2(j^*)}_{(k_2^*<k_1^*;i_2^*<i_1^*)}$$
$$(1 \le i_1 < i_2 \le r; 1 \le j \le r; 1 \le k_1 < k_2 \le r),$$

and

$$\theta^{3(k)}_{(i_1<i_2;j_1<j_2)} = \theta^{3(k)}_{(j_1<j_2;i_1<i_2)} = \theta^{3(k^*)}_{(i_2^*<i_1^*;j_2^*<j_1^*)} = \theta^{3(k^*)}_{(j_2^*<j_1^*;i_2^*<i_1^*)}$$
$$(1 \le i_1 < i_2 \le r; 1 \le j_1 < j_2 \le r; 1 \le k \le r),$$

where

$$\theta^{1(i)}_{(j_1<j_2;k_1<k_2)} = \frac{p_{ij_1k_1}p_{ij_2k_2}}{p_{ij_2k_1}p_{ij_1k_2}},$$

$$\theta^{2(j)}_{(i_1<i_2;k_1<k_2)} = \frac{p_{i_1jk_1}p_{i_2jk_2}}{p_{i_2jk_1}p_{i_1jk_2}},$$

$$\theta^{3(k)}_{(i_1<i_2;j_1<j_2)} = \frac{p_{i_1j_1k}p_{i_2j_2k}}{p_{i_2j_1k}p_{i_1j_2k}}.$$

Note that, for example, $\theta^{3(k)}_{(i_1<i_2;j_1<j_2)}$ indicates the odds ratio for i_1 and i_2 $(> i_1)$ of X_1, and j_1 and j_2 $(> j_1)$ of X_2 when $X_3 = k$. Therefore the QDS_1^3 model has its characterization in terms of double symmetry of odds ratios.

Consider another quasi double symmetry model defined by

$$\log p_{ijk} = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)} + u_{123(ijk)}$$
$$(1 \le i, j, k \le r),$$

where

$$u_{123(ijk)} = u_{123(ikj)} = u_{123(jik)} = u_{123(jki)} = u_{123(kij)} = u_{123(kji)} = u_{123(i^*j^*k^*)}.$$

We will refer to this model as the second-order quasi double symmetry (QDS_2^3) model. Note that the DS^3 and QDS_1^3 models are special cases of the QDS_2^3 model.

Using the odds ratios, the QDS_2^3 model can also be expressed as

$$\frac{\theta^{1(i_2)}_{(j_1<j_2;k_1<k_2)}}{\theta^{1(i_1)}_{(j_1<j_2;k_1<k_2)}} = \frac{\theta^{1(i_2)}_{(k_1<k_2;j_1<j_2)}}{\theta^{1(i_1)}_{(k_1<k_2;j_1<j_2)}} = \frac{\theta^{1(i_2^*)}_{(j_2^*<j_1^*;k_2^*<k_1^*)}}{\theta^{1(i_1^*)}_{(j_2^*<j_1^*;k_2^*<k_1^*)}} = \frac{\theta^{1(i_2^*)}_{(k_2^*<k_1^*;j_2^*<j_1^*)}}{\theta^{1(i_1^*)}_{(k_2^*<k_1^*;j_2^*<j_1^*)}},$$

or

$$\frac{\theta^{2(j_2)}_{(i_1<i_2;k_1<k_2)}}{\theta^{2(j_1)}_{(i_1<i_2;k_1<k_2)}} = \frac{\theta^{2(j_2)}_{(k_1<k_2;i_1<i_2)}}{\theta^{2(j_1)}_{(k_1<k_2;i_1<i_2)}} = \frac{\theta^{2(j_2^*)}_{(i_2^*<i_1^*;k_2^*<k_1^*)}}{\theta^{2(j_1^*)}_{(i_2^*<i_1^*;k_2^*<k_1^*)}} = \frac{\theta^{2(j_2^*)}_{(k_2^*<k_1^*;i_2^*<i_1^*)}}{\theta^{2(j_1^*)}_{(k_2^*<k_1^*;i_2^*<i_1^*)}},$$

or

$$\frac{\theta^{3(k_2)}_{(i_1<i_2;j_1<j_2)}}{\theta^{3(k_1)}_{(i_1<i_2;j_1<j_2)}} = \frac{\theta^{3(k_2)}_{(j_1<j_2;i_1<i_2)}}{\theta^{3(k_1)}_{(j_1<j_2;i_1<i_2)}} = \frac{\theta^{3(k_2^*)}_{(i_2^*<i_1^*;j_2^*<j_1^*)}}{\theta^{3(k_1^*)}_{(i_2^*<i_1^*;j_2^*<j_1^*)}} = \frac{\theta^{3(k_2^*)}_{(j_2^*<j_1^*;i_2^*<i_1^*)}}{\theta^{3(k_1^*)}_{(j_2^*<j_1^*;i_2^*<i_1^*)}},$$

for $1 \le i_1 < i_2 \le r; 1 \le j_1 < j_2 \le r; 1 \le k_1 < k_2 \le r$. Therefore the QDS_2^3 model has its characterization in terms of double symmetry of ratio of odds ratios.

Moreover, consider a model defined by

$$p_{i\cdot\cdot} = p_{\cdot i\cdot} = p_{\cdot\cdot i} = p_{i^*\cdot\cdot} \quad (i = 1, \ldots, r),$$

where $p_{i\cdot\cdot} = \sum_s \sum_t p_{ist}$, $p_{\cdot i\cdot} = \sum_s \sum_t p_{sit}$ and $p_{\cdot\cdot i} = \sum_s \sum_t p_{sti}$. This model indicates that the marginal distributions of X_k $(k = 1, 2, 3)$ are identical and each marginal distribution is point-symmetric with respect to the midpoint of the categories. We will refer to this model as the first-order marginal double symmetry (MDS_1^3) model.

Lastly, consider a model defined by

$$p_{ij\cdot} = p_{i\cdot j} = p_{\cdot ij} = p_{ji\cdot} = p_{i^*j^*\cdot} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $p_{ij\cdot} = \sum_s p_{ijs}$, $p_{i\cdot j} = \sum_s p_{isj}$ and $p_{\cdot ij} = \sum_s p_{sij}$. This model indicates that the marginal distributions of X_s and X_t $(1 \le s < t \le 3)$ are identical and symmetry, and also point-symmetric with respect to the center point in the marginal $r \times r$ table. We will refer to this model as the second-order marginal double symmetry (MDS_2^3) model.

2.2. Extension to multi-way tables

Consider an r^T table $(T \ge 2)$. Let $i = (i_1, \ldots, i_T)$ for $i_k = 1, \ldots, r$ $(k = 1, \ldots, T)$ and let p_i denote the probability that an observation will fall in the ith cell of the table. Let X_k $(k = 1, \ldots, T)$ denote the kth variable. We shall consider the double symmetry DS^T model as follows:

$$p_i = p_j = p_{i^*},$$

where $j = (j_1, \ldots, j_T)$ is any permutation of $i = (i_1, \ldots, i_T)$ and $i^* = (i_1^*, \ldots, i_T^*)$, $i_k^* = r + 1 - i_k$ $(k = 1, \ldots, T)$.

Next, for a fixed h $(h = 1, \ldots, T-1)$, consider a model defined by

$$\log p_i = u + \sum_{k=1}^{T} u_{k(i_k)} + \sum_{1\le k_1<k_2\le T} u_{k_1k_2(i_{k_1},i_{k_2})} + \cdots + \sum_{1\le k_1<\cdots<k_{T-1}\le T} \cdots \sum u_{k_1\ldots k_{T-1}(i_{k_1},\ldots,i_{k_{T-1}})} + u_{12\ldots T(i)},$$

for any $i = (i_1, \ldots, i_T)$, where

$$u_{k_1,\ldots,k_l(i_{k_1},\ldots,i_{k_l})} = u_{k_1,\ldots,k_l(j_{k_1},\ldots,j_{k_l})} = u_{m_1,\ldots,m_l(i_{k_1},\ldots,i_{k_l})}, \tag{2.1}$$

and

$$u_{k_1,\ldots,k_l(i_{k_1},\ldots,i_{k_l})} = u_{k_1,\ldots,k_l(i^*_{k_1},\ldots,i^*_{k_l})}, \tag{2.2}$$

for $l = h+1, \ldots, T; 1 \le k_1 < \cdots < k_l \le T; 1 \le m_1 < \cdots < m_l \le T$, and $(j_{k_1}, \ldots, j_{k_l})$ is any permutation of $(i_{k_1}, \ldots, i_{k_l})$ and $i^*_k = r + 1 - i_k$. We will refer to this model as the hth-order $(h = 1, \ldots, T-1)$ quasi double symmetry (QDS^T_h) model. Note that (2.1) indicates the structure of the Q^T_h model and (2.2) indicates the structure of the QP^T_h model.

Denote the hth-order $(h = 1, \ldots, T-1)$ marginal probability $\mathrm{P}(X_{s_1} = i_1, \ldots, X_{s_h} = i_h)$ by p^s_i, where $s = (s_1, \ldots, s_h)$ and $i = (i_1, \ldots, i_h)$ with $1 \le s_1 < \cdots < s_h \le T$ and $i_k = 1, \ldots, r$ $(k = 1, \ldots, h)$.

For a fixed h $(h = 1, \ldots, T-1)$, consider a model defined by

$$p^s_i = p^s_j = p^t_i, \tag{2.3}$$

and

$$p^s_i = p^s_{i^*}, \tag{2.4}$$

for any $s = (s_1, \ldots, s_h)$ and $t = (t_1, \ldots, t_h)$ with $1 \le t_1 < \cdots < t_h \le T$ and $s \ne t$, where $j = (j_1, \ldots, j_h)$ is any permutation of $i = (i_1, \ldots, i_h)$ and $i^* = (i^*_1, \ldots, i^*_h)$ with $i^*_k = r + 1 - i_k$. We will refer to this model as the hth-order $(h = 1, \ldots, T-1)$ marginal double symmetry (MDS^T_h) model. Note that (2.3) indicates the structure of the M^T_h model and (2.4) indicates the structure of the MP^T_h model.

§3. Decomposition of double symmetry model

Consider the r^T tables. As described in Section 1, for a fixed h $(h = 1, \ldots, T-1)$, the S^T model holds if and only if both the Q^T_h and M^T_h models hold. Also, for a fixed h $(h = 1, \ldots, T-1)$, the P^T model holds if and only if both the QP^T_h and MP^T_h models hold. We see from Section 2 that (1) the DS^T model indicates the structure of both the S^T and P^T, (2) the QDS^T_h model indicates the structure of both the Q^T_h and QP^T_h, and (3) the MDS^T_h model indicates the structure of both the M^T_h and MP^T_h. Therefore we obtain the following theorem.

Theorem 2. *For the r^T table and a fixed h $(h = 1, \ldots, T-1)$, the DS^T model holds if and only if both the QDS^T_h and MDS^T_h models hold.*

§4. Orthogonality of decomposition of double symmetry

Let $n_{i_1 \ldots i_T}$ denote the observed frequency in the $(i_1, \ldots, i_T)$th cell of the r^T $(T \geq 2)$ table $(i_k = 1, \ldots, r; k = 1, \ldots, T)$ with $n = \sum \cdots \sum n_{i_1 \ldots i_T}$ and let $m_{i_1 \ldots i_T}$ denote the corresponding expected frequency. Assume that $\{n_{i_1 \ldots i_T}\}$ have a multinomial distribution. The maximum likelihood estimates of expected frequencies $\{m_{i_1 \ldots i_T}\}$ under the DS^T model are given by solving a log-likelihood equation. Those under the QDS_h^T and MDS_h^T models could be obtained using the iterative procedures, for example, the general iterative procedure for log-linear models of Darroch and Ratcliff (1972) or using the Newton-Raphson method to the log-likelihood equations.

Each model can be tested for goodness of fit by, e.g., the likelihood ratio chi-squared statistic with the corresponding degrees of freedom (df). The numbers of df for the DS^T, QDS_h^T, MDS_h^T models are given in Table 1. Let $G^2(\Omega)$ denote the likelihood ratio statistic for testing goodness of fit of model Ω. Thus

$$G^2(\Omega) = 2 \sum \cdots \sum n_{i_1 \ldots i_T} \log \left(\frac{n_{i_1 \ldots i_T}}{\hat{m}_{i_1 \ldots i_T}} \right),$$

where $\hat{m}_{i_1 \ldots i_T}$ is the maximum likelihood estimate of expected frequency $m_{i_1 \ldots i_T}$ under model Ω.

For the analysis of contingency tables, Lang and Agresti (1994) and Lang (1996) considered the simultaneous modeling of the joint distribution and of the marginal distribution. Aitchison (1962) discussed the asymptotic separability, which is equivalent to the orthogonality in Read (1977) and the independence in Darroch and Silvey (1963), of test statistic for goodness of fit of two models (also see Lang and Agresti, 1994; Lang, 1996; Tomizawa and Tahata, 2007).

For the r^T table, we shall consider the orthogonality (i.e., separability or independence) of test statistics for decomposition of the DS^T model into the QDS_h^T and MDS_h^T models. Consider the case of $T = 3$.

Theorem 3. *For the $r \times r \times r$ table and a fixed h $(h = 1, 2)$, the following asymptotic equivalence holds:*

$$G^2(DS^3) \simeq G^2(QDS_h^3) + G^2(MDS_h^3). \tag{4.1}$$

The number of df for the DS^3 model equals the sum of numbers of df for the QDS_h^3 and MDS_h^3 models.

Proof. We shall consider the case of $h = 2$ when r is odd. The QDS_2^3 model is expressed as

$$\text{(4.2)} \qquad \log p_{ijk} = u + u_{1(i)} + u_{2(j)} + u_{3(k)} + u_{12(ij)} + u_{13(ik)} + u_{23(jk)} + u_{123(ijk)} \quad (i = 1, \ldots, r; j = 1, \ldots, r; k = 1, \ldots, r),$$

where

$$u_{123(ijk)} = u_{123(ikj)} = u_{123(jik)} = u_{123(jki)} = u_{123(kij)} = u_{123(kji)} = u_{123(i^*j^*k^*)}.$$

Without loss of generality, we set $\sum_i u_{m(i)} = 0$ $(m = 1, 2, 3)$, $\sum_i u_{st(ij)} = \sum_j u_{st(ij)} = 0$ $(1 \le s < t \le 3)$, and $\sum_i u_{123(ijk)} = \sum_j u_{123(ijk)} = \sum_k u_{123(ijk)} = 0$. Let

$$p = (p_{111}, \ldots, p_{1r1}, \ldots, p_{r11}, \ldots, p_{rr1}, p_{112}, \ldots, p_{1r2}, \ldots, p_{r12}, \ldots, p_{rr2}, \ldots, p_{11r}, \ldots, p_{1rr}, \ldots, p_{r1r}, \ldots, p_{rrr})',$$

$$\beta = (u, \beta_1, \beta_2, \beta_3, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{123})',$$

where "$\prime$" denotes the transpose, and where

$$\beta_m = (u_{m(1)}, \ldots, u_{m(r-1)}) \quad (m = 1, 2, 3),$$

$$\beta_{st} = (u_{st(11)}, \ldots, u_{st(1,r-1)}, u_{st(21)}, \ldots, u_{st(2,r-1)}, \ldots, u_{st(r-1,1)}, \ldots, u_{st(r-1,r-1)}) \quad (1 \le s < t \le 3),$$

and β_{123} is the $1 \times r(r-1)(r+1)/12$ vector of $u_{123(ijk)}$. Then the QDS_2^3 model is expressed as

$$\text{(4.3)} \qquad \log p = X\beta = (1_{r^3}, X_1, X_2, X_3, X_{12}, X_{13}, X_{23}, X_{123})\beta,$$

where X is the $r^3 \times K$ matrix with $K = (r^3 + 36r^2 - 37r + 12)/12$ and 1_s is the $s \times 1$ vector of 1 elements,

$$X_1 = 1_r \otimes \begin{bmatrix} I_{r-1} \otimes 1_r \\ -1_r 1'_{r-1} \end{bmatrix}; \text{the } r^3 \times (r-1) \text{ matrix},$$

$$X_2 = 1_{r^2} \otimes \begin{bmatrix} I_{r-1} \\ -1'_{r-1} \end{bmatrix}; \text{the } r^3 \times (r-1) \text{ matrix},$$

$$X_3 = \begin{bmatrix} I_{r-1} \otimes 1_{r^2} \\ -1_{r^2} 1'_{r-1} \end{bmatrix}; \text{the } r^3 \times (r-1) \text{ matrix},$$

$$X_{12} = 1_r \otimes \begin{bmatrix} I_{r-1} \otimes \begin{bmatrix} I_{r-1} \\ -1'_{r-1} \end{bmatrix} \\ \\ 1'_{r-1} \otimes \begin{bmatrix} -I_{r-1} \\ 1'_{r-1} \end{bmatrix} \end{bmatrix} ;\text{the } r^3 \times (r-1)^2 \text{ matrix},$$

$$X_{13} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{r-1} \\ -I_{r-1} \otimes 1_r 1'_{r-1} \\ 1_r 1'_{(r-1)^2} \end{bmatrix} ;\text{the } r^3 \times (r-1)^2 \text{ matrix},$$

where

$$C_i = \begin{bmatrix} I_{r-1} \otimes A_i \\ -1'_{r-1} \otimes A_i \end{bmatrix}$$

$$A_i = \left[O_{r,i-1}, 1_r, O_{r,r-1-i}\right] ;\text{the } r \times (r-1) \text{ matrix},$$

with

$$A_1 = \left[1_r, O_{r,r-2}\right],\ A_{r-1} = \left[O_{r,r-2}, 1_r\right],$$

and

$$X_{23} = \begin{bmatrix} D_{11} & \ldots & D_{1,r-1} \\ & \ldots & \\ D_{r-1,1} & \ldots & D_{r-1,r-1} \\ \\ \multicolumn{3}{l}{1_r \otimes \begin{bmatrix} -I_{r-1} \otimes 1'_{r-1} \\ 1'_{(r-1)^2} \end{bmatrix}} \end{bmatrix} ;\text{the } r^3 \times (r-1)^2 \text{ matrix},$$

where

$$D_{ij} = 1_r \otimes E_{ji}; \text{the } r^2 \times (r-1) \text{ matrix},$$

E_{ji} is the $r \times (r-1)$ matrix with 1 in the (j,i)th element, -1 in the (r,i)th element and 0's elsewhere, the elements of X_{123} which is the $r^3 \times r(r-1)(r+1)/12$ matrix are determined from (4.2) and I_{r-1} is the $(r-1)\times(r-1)$ identity matrix, O_{st} is the $s \times t$ zero matrix, and $\otimes$ denotes the Kronecker product. Note that the matrix X has full column rank which is K. In a similar manner to Haber (1985), Lang and Agresti (1994), and Tomizawa and Tahata (2007), we denote the linear space spanned by the columns of the matrix X by $S(X)$ with the dimension K. Let U be an $r^3 \times d_1$ full column rank matrix, where $d_1 = r^3 - K = (11r^3 - 36r^2 + 37r - 12)/12$, such that the linear space spanned by the columns of U, i.e., $S(U)$, is the orthogonal complement of the space $S(X)$. Thus, $U'X = O_{d_1,K}$. Therefore the QDS_2^3 model is expressed as

$$h_1(p) = 0_{d_1},$$

where 0_{d_1} is the $d_1 \times 1$ zero vector, and

$$h_1(p) = U' \log p.$$

The MDS_2^3 may be expressed as

$$p_{i\cdot\cdot} = p_{\cdot i\cdot} = p_{\cdot\cdot i} \quad (i = 1, \dots, r-1), \tag{4.4}$$

$$p_{ij\cdot} = p_{i\cdot j} \quad (i = 1, \dots, r-1; j = 1, \dots, r-1), \tag{4.5}$$

$$p_{ij\cdot} = p_{\cdot ij} \quad (i = 1, \dots, r-1; j = 1, \dots, r-1), \tag{4.6}$$

$$p_{ij\cdot} = p_{ji\cdot} \quad (i = 1, \dots, r-2; j = i+1, \dots, r-1), \tag{4.7}$$

$$p_{i\cdot\cdot} = p_{i^*\cdot\cdot} \quad (i = 1, \dots, (r-1)/2), \tag{4.8}$$

$$p_{ij\cdot} = p_{i^*j^*\cdot} \quad (i = 1, \dots, (r-1)/2; j = i, i+1, \dots, r-1-i). \tag{4.9}$$

The conditions given by the equations (4.4), (4.5) and (4.6) may be expressed as

$$p_{i\cdot\cdot} - p_{r\cdot\cdot} = p_{\cdot i\cdot} - p_{\cdot r\cdot} = p_{\cdot\cdot i} - p_{\cdot\cdot r} \quad (i = 1, \dots, r-1), \tag{4.10}$$

$$p_{ij\cdot} - p_{ir\cdot} - p_{rj\cdot} + p_{rr\cdot} = p_{i\cdot j} - p_{i\cdot r} - p_{r\cdot j} + p_{r\cdot r} \quad (i = 1, \dots, r-1; j = 1, \dots, r-1), \tag{4.11}$$

$$p_{ij\cdot} - p_{ir\cdot} - p_{rj\cdot} + p_{rr\cdot} = p_{\cdot ij} - p_{\cdot ir} - p_{\cdot rj} + p_{\cdot rr} \quad (i = 1, \dots, r-1; j = 1, \dots, r-1). \tag{4.12}$$

Thus the equations (4.10), (4.11) and (4.12) are expressed as

$$W_1 p = 0_{2r(r-1)},$$

where W_1 is the $2r(r-1) \times r^3$ matrix with

$$W_1 = \begin{bmatrix} X_1' - X_2' \\ X_1' - X_3' \\ X_{12}' - X_{13}' \\ X_{12}' - X_{23}' \end{bmatrix}.$$

The equation (4.7) is expressed as

$$W_2 p = 0_{(r-1)(r-2)/2},$$

where W_2 is the $(r-1)(r-2)/2 \times r^3$ matrix with

$$W_2 = A_1' - A_2',$$

$$A_1 = \left[a_{12}, a_{13}, \ldots, a_{1,r-1}, a_{23}, \ldots, a_{2,r-1}, \ldots, a_{r-2,r-1}\right],$$

$$A_2 = \left[a_{21}, a_{31}, \ldots, a_{r-1,1}, a_{32}, \ldots, a_{r-1,2}, \ldots, a_{r-1,r-2}\right],$$

and a_{ij} is the $r^3 \times 1$ vector, and

$$a_{ij} = \frac{1}{r}(x_{1(i)} + x_{2(j)}) + x_{12(ij)} - \frac{1}{r}\sum_{k=1}^{r-1}(x_{12(ik)} + x_{12(kj)}),$$

where $x_{t(i)}$ is the $r^3 \times 1$ column vector in X_t shouldering $u_{t(i)}$ $(t = 1, 2)$ and $x_{12(ij)}$ is the $r^3 \times 1$ column vector in X_{12} shouldering $u_{12(ij)}$.

The equation (4.8) is expressed as

$$W_3 p = 0_{(r-1)/2},$$

where W_3 is the $(r-1)/2 \times r^3$ matrix with

$$W_3 = \begin{bmatrix} x_{1(1)}' \\ x_{1(2)}' - x_{1(2^*)}' \\ \vdots \\ x_{1(\frac{r-1}{2})}' - x_{1((\frac{r-1}{2})^*)}' \end{bmatrix}.$$

The equation (4.9) is expressed as

$$W_4 p = 0_{(r-1)^2/4},$$

where W_4 is the $(r-1)^2/4 \times r^3$ matrix with

$$W_4 = B_1' - B_2',$$

$$B_1 = \left[a_{11}, a_{12}, \ldots, a_{1,r-2}, a_{22}, a_{23}, \ldots, a_{2,r-3}, \ldots, a_{\frac{r-1}{2},\frac{r-1}{2}}\right],$$

$$B_2 = \left[c_{11}, c_{12}, \ldots, c_{1,r-2}, c_{22}, c_{23}, \ldots, c_{2,r-3}, \ldots, c_{\frac{r-1}{2},\frac{r-1}{2}}\right],$$

$$c_{ij} = a_{i^*j^*} \quad (i = 1, \ldots, (r-1)/2; j = i, i+1, \ldots, r-1-i),$$

and where we set

$$x_{1(r)} = x_{2(r)} = x_{12(ir)} = x_{12(ri)} = 0_{r^3} \quad (i = 1, \ldots, r),$$

for convenience.

Thus, the MDS_2^3 model is expressed as

$$h_2(p) = 0_{d_2},$$

where $d_2 = (11r-3)(r-1)/4$,

$$h_2(p) = Wp = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} p.$$

All column vectors of W' belong to the space $S(X)$, i.e., $S(W') \subset S(X)$. Hence $WU = O_{d_2 d_1}$. From Theorem 2, the DS^3 model may be expressed as

$$h_3(p) = 0_{d_3},$$

where $d_3 = d_1 + d_2 = (11r^3 - 3r^2 - 5r - 3)/12$,

$$h_3 = (h_1', h_2')'.$$

Note that $h_s(p)$, $s = 1, 2, 3$, are the vectors of order $d_s \times 1$, and $d_s, s = 1, 2, 3$, are the numbers of degrees of freedoms for testing goodness of fit of the QDS_2^3, MDS_2^3 and DS^3 models, respectively.

Let $H_s(p)$, $s = 1, 2, 3$, denote the $d_s \times r^3$ matrix of partial derivatives of $h_s(p)$ with respect to p, i.e., $H_s(p) = \partial h_s(p)/\partial p'$. Let $\Sigma(p) = diag(p) - pp'$, where $diag(p)$ denotes a diagonal matrix with ith component of p as ith diagonal component. We see that

$$H_1(p)p = U' 1_{r^3} = 0_{d_1},$$

$$H_1(p) diag(p) = U',$$

$$H_2(p) = W.$$

Therefore we obtain

$$H_1(p)\Sigma(p)H_2(p)' = U'W' = O_{d_1 d_2}.$$

Thus we obtain $\Delta_3 = \Delta_1 + \Delta_2$, where

$$\Delta_s = h_s(p)'[H_s(p)\Sigma(p)H_s(p)']^{-1}h_s(p). \tag{4.13}$$

From the asymptotic equivalence of the Wald statistic and the likelihood ratio statistic (Rao, 1973, Sec. 6e. 3; Darroch and Silvey, 1963; Aitchison, 1962), and from (4.13), we obtain (4.1) when r is odd and $h = 2$. Also, in a similar way we obtain (4.1) when r is even and when $h = 1$. So, the proof is completed.

Next, for the r^T table, we obtain the following theorem.

Theorem 4. *For the r^T table and a fixed h ($h = 1, \ldots, T-1$), the following asymptotic equivalence holds:*

$$G^2(DS^T) \simeq G^2(QDS_h^T) + G^2(MDS_h^T).$$

The number of df for the DS^T model equals the sum of numbers of df for the QDS_h^T and MDS_h^T models.

The proof of Theorem 4 is omitted because it is obtained in a similar way to the proof of Theorem 3.

§5. Example

Consider the data in Table 2, taken directly from Tahata, Kobayashi and Tomizawa (2008). These data are obtained from the Meteorological Agency in Japan. These are obtained from the daily atmospheric temperatures at Hiroshima, Tokyo, Sapporo in Japan, 2003, using three levels, (1) low, (2) normal, and (3) high. The variables X_1, X_2 and X_3 mean the temperatures at Hiroshima, Tokyo and Sapporo, respectively.

Table 3 gives the values of likelihood ratio test statistic G^2 for various double symmetry models. The DS^3 model fits these data poorly, yielding $G^2 = 34.23$ with 21 df. By using the decompositions of the DS^3 model, we shall consider the reason why the DS^3 model fits these data poorly. The QDS_1^3 and QDS_2^3 models fit the data in Table 2 well, however, the MDS_1^3 and MDS_2^3 models fit these data poorly. Thus, it is seen from Theorem 3 that the poor fit of the DS^3 model is caused by the influence of the lack of structure of the MDS_1^3 and MDS_2^3 models. Also, we note that for these data the value of the test statistic for the DS^3 model is close to the sum of the values of those for the QDS_h^T and MDS_h^T ($h = 1, 2$) models.

According to the test based on the difference between the G^2 values for the QDS_1^3 and QDS_2^3 models, the QDS_1^3 model is preferable to the QDS_2^3 model.

Under the QDS_1^3 model, it is inferred that there is a structure of double symmetry of odds ratios for the data in Table 2, namely for $1 \leq i \leq 3$, $1 \leq j_1 < j_2 \leq 3$, $1 \leq k_1 < k_2 \leq 3$,

$$\theta^{1(i)}_{(j_1<j_2;k_1<k_2)} = \theta^{1(i)}_{(k_1<k_2;j_1<j_2)} = \theta^{1(i^*)}_{(j_2^*<j_1^*;k_2^*<k_1^*)} = \theta^{1(i^*)}_{(k_2^*<k_1^*;j_2^*<j_1^*)}.$$

Thus, under the QDS_1^3 model, for example, it is inferred that there is a structure of

$$\theta^{1(1)}_{(1<2;1<3)} = \theta^{1(1)}_{(1<3;1<2)} = \theta^{1(3)}_{(2<3;1<3)} = \theta^{1(3)}_{(1<3;2<3)}.$$

Therefore, the QDS_1^3 model provides that when the temperature at Hiroshima is “low”, (1) the odds that the temperature at Sapporo is “low” instead of “high” is estimated to be $\hat{\theta}^{1(1)}_{(1<2;1<3)} = 4.11$ times higher when “low” than when “normal” at Tokyo, and (2) the odds that it at Tokyo is “low” instead of “high” is estimated to be $\hat{\theta}^{1(1)}_{(1<3;1<2)} = 4.11$ times higher when “low” than when “normal” at Sapporo; and also when the temperature at Hiroshima is “high”, (3) the odds that it at Sapporo is “high” instead of “low” is estimated to be $\hat{\theta}^{1(3)}_{(2<3;1<3)} = 4.11$ times higher when “high” than when “normal” at Tokyo, and (4) the odds that it at Tokyo is “high” instead of “low” is estimated to be $\hat{\theta}^{1(3)}_{(1<3;2<3)} = 4.11$ times higher when “high” than when “normal” at Sapporo.

Also, under the QDS_1^3 model, it is inferred that there is the structure of

$$\theta^{1(2)}_{(1<2;1<3)} = \theta^{1(2)}_{(1<3;1<2)} = \theta^{1(2)}_{(2<3;1<3)} = \theta^{1(2)}_{(1<3;2<3)}.$$

Therefore, the QDS_1^3 model also provides that when the temperature at Hiroshima is “normal”, (1) the odds that the temperature at Sapporo is “low” instead of “high” is estimated to be $\hat{\theta}^{1(2)}_{(1<2;1<3)} = 2.08$ times higher when “low” than when “normal” at Tokyo, (2) the odds that it at Tokyo is “low” instead of “high” is estimated to be $\hat{\theta}^{1(2)}_{(1<3;1<2)} = 2.08$ times higher when “low” than when “normal” at Sapporo, (3) the odds that it at Sapporo is “high” instead of “low” is estimated to be $\hat{\theta}^{1(2)}_{(2<3;1<3)} = 2.08$ times higher when “high” than when “normal” at Tokyo, and (4) the odds that it at Tokyo is “high” instead of “low” is estimated to be $\hat{\theta}^{1(2)}_{(1<3;2<3)} = 2.08$ times higher when “high” than when “normal” at Sapporo.

Moreover, under the QDS_1^3 model, it is inferred that there is the structure of

$$\theta^{1(1)}_{(1<3;1<3)} = \theta^{1(3)}_{(1<3;1<3)}.$$

Therefore, the QDS_1^3 model provides that when the temperature at Hiroshima is "low" or when it is "high" (namely, when it at Hiroshima is not "normal"), the odds that the temperature at Sapporo is "low" instead of "high" is estimated to be $\hat{\theta}^{1(1)}_{(1<3;1<3)} = \hat{\theta}^{1(3)}_{(1<3;1<3)} = 8.54$ times higher when "low" than when "high" at Tokyo. Note that under the QDS_1^3 model, when the temperature at Hiroshima is "normal", the odds that the temperature at Sapporo is "low" instead of "high" is estimated to be $\hat{\theta}^{1(2)}_{(1<3;1<3)} = 4.32$ times higher when "low" than when "high" at Tokyo.

§6. Concluding remarks

For two-way contingency tables, Tomizawa (1985b) considered the DS^2, QDS^2 and MDS^2 models, and gave a decomposition theorem of the DS^2 model into the QDS^2 and MDS^2 models. [Note that in Tomizawa (1985b) the orthogonality of test statistics was not shown.] In the present paper, we extended them into multi-way r^T tables and showed the orthogonality of test statistics.

We point out from Theorem 4 that for a fixed h ($h = 1, \dots, T-1$), the likelihood ratio statistic for testing goodness of fit of the DS^T model assuming that the QDS_h^T model holds true is $G^2(DS^T) - G^2(QDS_h^T)$ and this is asymptotically equivalent to the likelihood ratio statistic for testing goodness of fit of the MDS_h^T model, i.e., $G^2(MDS_h^T)$. Namely, $G^2(MDS_h^T)$ can be utilized for testing goodness of fit of the MDS_h^T model and also for testing goodness of fit of the DS^T model assuming that the QDS_h^T model holds true.

Generally suppose that model Ω_3 holds if and only if both models Ω_1 and Ω_2 hold, where the number of df for Ω_3 equals the sum of numbers of df for Ω_1 and Ω_2. Darroch and Silvey (1963) described that (i) when the asymptotic equivalence,

$$G^2(\Omega_3) \simeq G^2(\Omega_1) + G^2(\Omega_2), \tag{6.1}$$

holds, if both Ω_1 and Ω_2 are accepted (at the α significance level) with high probability, then Ω_3 would be accepted; however (ii) when (6.1) does not hold, such an incompatible situation that both Ω_1 and Ω_2 are accepted with high probability but Ω_3 is rejected with high probability is quite possible [in fact, Darroch and Silvey (1963) showed such an interesting example]. For the orthogonal decomposition of the DS^T model into the QDS_h^T and MDS_h^T models, such an incompatible situation would not arise in terms of Theorem 4.

When the DS^T model fits the data poorly, the decomposition of the DS^T model may be useful to observe the reason for its poor fit. Indeed, for the data in Table 2, the poor fit of the DS^3 model is caused by the poor fit of the MDS_1^3 (MDS_2^3) model rather than the QDS_1^3 (QDS_2^3) model.

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Table 1

Numbers of degrees of freedom for various double symmetry models applied to the r^T table.

Models	When r is even Degrees of freedom	When r is odd Degrees of freedom
DS^T	$r^T - \frac{1}{2}A_T$	$r^T - \frac{1}{2}B_T$
QDS_h^T	$r^T - \frac{1}{2}A_T - C_{T(h)}$	$r^T - \frac{1}{2}B_T - D_{T(h)}$
MDS_h^T	$C_{T(h)}$	$D_{T(h)}$

Note that

$$A_T = \begin{cases} \binom{T-1+r}{T} + \binom{\frac{T+r-2}{2}}{\frac{T}{2}} & (T : \text{even}), \\ \binom{T-1+r}{T} & (T : \text{odd}), \end{cases}$$

$$B_T = \begin{cases} \binom{T-1+r}{T} + \binom{\frac{T+r-1}{2}}{\frac{T}{2}} & (T : \text{even}), \\ \binom{T-1+r}{T} + \binom{\frac{T+r-2}{2}}{\frac{T-1}{2}} & (T : \text{odd}), \end{cases}$$

$$C_{T(h)} = 1 + \sum_{u=1}^{h} \left\{ \binom{T}{u} (r-1)^u - \frac{1}{2}E_u \right\},$$

$$D_{T(h)} = \sum_{u=1}^{h} \left\{ \binom{T}{u} (r-1)^u - \frac{1}{2}F_u \right\},$$

where

$$E_u = \begin{cases} \binom{u+r-2}{u} + \binom{\frac{u+r-2}{2}}{\frac{u}{2}} & (u : \text{even}), \\ \binom{u+r-2}{u} + \binom{\frac{u+r-3}{2}}{\frac{u-1}{2}} & (u : \text{odd}), \end{cases}$$

$$F_u = \begin{cases} \binom{u+r-2}{u} + \binom{\frac{u+r-3}{2}}{\frac{u}{2}} & (u : \text{even}), \\ \binom{u+r-2}{u} & (u : \text{odd}). \end{cases}$$

Table 2

The daily atmospheric temperatures at Hiroshima, Tokyo and Sapporo in Japan, 2003. (The upper and lower parenthesized values are the maximum likelihood estimates of expected frequencies under the QDS_1^3 and QDS_2^3 models, respectively.)

Hiroshima	Tokyo	Sapporo		
		(1) low	(2) normal	(3) high
(1) low	(1) low	37	13	3
		(36.52)	(13.46)	(2.28)
		(35.15)	(13.99)	(3.86)
(1) low	(2) normal	21	17	5
		(19.74)	(13.96)	(5.06)
		(21.51)	(15.02)	(6.47)
(1) low	(3) high	4	4	5
		(5.90)	(8.93)	(3.14)
		(5.34)	(4.99)	(2.67)
(2) normal	(1) low	19	15	5
		(20.08)	(14.20)	(5.14)
		(19.58)	(14.75)	(4.67)
(2) normal	(2) normal	20	29	8
		(20.82)	(29.00)	(11.09)
		(20.15)	(29.00)	(7.85)
(2) normal	(3) high	20	20	12
		(13.32)	(19.59)	(14.75)
		(19.27)	(20.25)	(12.48)
(3) high	(1) low	2	8	4
		(4.70)	(7.11)	(2.50)
		(3.27)	(7.26)	(3.47)
(3) high	(2) normal	8	15	14
		(10.43)	(15.34)	(11.55)
		(7.34)	(16.98)	(12.68)
(3) high	(3) high	7	21	29
		(6.48)	(20.40)	(29.48)
		(6.39)	(19.76)	(30.85)

Table 3

Likelihood ratio chi-squared values G^2 for models applied to Table 2.

Models	Degrees of freedom	G^2
DS^3	21	34.23*
QDS_1^3	16	14.55
QDS_2^3	6	4.69
MDS_1^3	5	18.22*
MDS_2^3	15	28.31*

* means significant at the 0.05 level.

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