SUT Journal of Mathematics
Vol. 44, No. 1 (2008), 55–88

Moyal algebra: relevant properties, projective limits and applications in noncommutative field theory

Joseph Ben Geloun, Mahouton Norbert Hounkonnou
and
Fortuné Massamba

(Received September 7, 2007)

Abstract. From the definition of the Moyal $\star$-product in terms of projective limits of the ring of polynomials of vector fields, the Moyal configuration space of Schwartzian functions, equipped with the $\star$-product, is built as a formal power series ring with elements assimilated to free indeterminates. We then define the projector on the ideal depending on a fixed indeterminate, which allows to use the definition of algebraic derivations with respect to any order of field derivative. As a consequence and in a direct manner, Euler-Lagrange equations of motion, in the framework of both the noncommutative scalar and gauge induced Dirac fields, are deduced from the nonlocal Lagrange function. A connection of this theory to a generalized Ostrogradski's formalism is also discussed here.

AMS 2000 *Mathematics Subject Classification.* 53D55, 46L65, 81T75, 13B25.

Key words and phrases. Noncommutative field theory, Moyal algebra.

§1. Introduction

One of the main features inherent to the noncommutativity in field theories is the nonlocality, i.e. the existence of an infinite dimensional phase space [5]. These field theories, where infinite order of time and spatial derivatives occurs, are referred to as nonlocal field theories ([6] and references therein). Historically, even if the noncommutative field theory (NCFT) was born in the early days of quantum mechanics [24, 19], and have evolved through the years with close entanglement with deformation quantization [1] and advanced Weyl calculus [11], one must wait the 90's with the development of the noncommutative (NC) geometry, first applied to the Yang Mills fields [3, 4], before observing a real infatuation of the theoretician community for this topic. Still more recently, NCFTs became the focus of intense research activities in general quantum field theory since the advent of a class of renormalizable NCFTs hightlighted by Grosse and Wulkenhaar [9][22] with particular translational broken symmetry features [2].

The most common way to realize a NC spacetime in field theory consists in defining a $\star$-product on its standard space of functions. Thus, the $\star$-product appears as a source of the nonlocality of the theory. The nonlocality is itself a source of additional difficulties in the computation of physical quantities of the field theory such that the NC momentums and related properties. In particular, the Hamiltonian formalism is highly affected by the higher order time derivative dependence in the expressions of the conjugate momentums and thus, obviously cannot be applied in the usual manner [6][21]. Indeed, instead of the standard Legendre transformation giving the momentum relations, one deals here, in the framework of NCFT, with a θ-deformed Legendre correspondence mapping the tangent bundle into the cotangent one over the Minkowski manifold. However, one notes that the occurrence of time derivatives of any order in the interaction Hamiltonian is not forbidden in nonlocal theories, property which is not shared by local theories [6].

In this work, starting from basics, we investigate a new algebraic structure of the Moyal algebra admitting the definition of formal power series, so that the $\star$-product remains still defined, and such that functional derivatives are viewed as algebraic derivations. We deduce, using the definition of the projective limit algebra of differential operators, some rules which render more convenient the computation of physical quantities. The Ostrogradski formulas for momentums and Euler-Lagrange equations can be computed for this nonlocal algebraic theory.

The paper is organized as follows. In Section 2, we present the theoretical framework. Starting from the main known properties of the $\star$-product, we deduce new relevant one, useful for the description of the deformed field theory in the NC spacetime. Then, the nonlocal operators, induced by the $\star$-product, are investigated in the framework of the projective limit algebra of differential operators. In the ring $\mathcal{R}$ of such projective limits, we define a projector $i_{\mathbf{a}_0}$ on the principal ideal generated by a given indeterminate $\mathbf{a}_0$ and the corresponding equivalent class. Then, we derive some interesting properties, useful for the computation in NCFT. Section 3 provides a conjectured NC Euler-Lagrange equation from pure Lagrangian formulation. Illustrations of this claim follow from examples of the derivation of equations of motion in the case of both the NC scalar and Dirac Lagrangian densities. The computation of Ostrogradski quantities is also performed. Finally, in Section 4, we end with some concluding remarks.

§2. Theory

In this section, we discuss the main properties of the $\star$-algebra and the algebra of polynomials and series. Then, we define the $\star$-product as a projective limit and deduce some relevant computational rules.

2.1. Main properties of the $\star$-algebra

The theoretical tools, the target spaces and objects are hereafter developed.

Consider M a differentiable manifold which can be viewed as $\mathbb{R}^N$, the set of N-tuples of real numbers equipped with the Euclidean metric $\delta^{\mu\nu}$, or the usual Minkowski spacetime $\mathbb{R}^{1,N-1}$ endowed by the diagonal Lorentz metric $\eta^{\mu\nu}$ with mostly minus signs. It is worth noticing that the following developments can be generalized to any (pseudo) Riemannian spaces. Throughout the text, the term function refers to as any C^∞ Schwartzian function (roughly speaking, smooth function with rapid decay) defined on M with complex values. The Latin letters f, g, h etc..., refer to such functions. Finally, Einstein summation convention is assumed.

Definition 1. Let f and g be two complex valued functions defined on M. Then, the $\star$-product of f and g is defined by:

$$\forall x \in M,\ (f \star g)(x) = e^{\frac{\sqrt{-1}}{2}\theta^{\mu\nu}\left(\partial_{x^\mu}\partial_{y^\nu}\right)} f(x)g(y)\ |_{x=y}, \tag{2.1}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric tensor.

The multiplication (2.1) can be expanded as follows:

$$\begin{aligned}(f \star g)(x) &= f(x) \star g(x) = f(x)g(x) \\ &+ \sum_{n=1}^{\infty} \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!}\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\partial_{\mu_1}\dots\partial_{\mu_n} f(x)\partial_{\nu_1}\dots\partial_{\nu_n} g(x).\end{aligned} \tag{2.2}$$

Remark 1. The deformation tensor $\theta^{\mu\nu}$ may be chosen as $\theta\epsilon^{\mu\nu}$, where $\epsilon^{\mu\nu}$ is the absolute antisymmetric tensor and θ is a constant real deformation parameter which can be fixed in order to ensure the convergence of the Moyal product. In the context of NC symplectic geometry (the spacetime coordinates x^μ may be then replaced by phase space conjugate coordinates (q^i, p_i)), the deformation parameter notation is $\hbar$ holding of course reminiscent ideas of quantum mechanics [1, 11]. In a physical background, the parameter θ has a dimension of length square. Clearly, as $\theta \to 0$, the $\star$-product collapse to the ordinary multiplication of functions, canceling the NC character of the theory.

Changing θ by $-\theta$, one obtains the law $\star^{-1}$-product. As a formal mathematical property and excluding any $C^\star$-algebra consideration, if we introduce a complex $\theta^c = re^{\sqrt{-1}\lambda} \in \mathbb{C}$, then, the following law is obtained

$$f \star_{\theta^c} g = f \star'_{r\cos\lambda} g + \sqrt{-1} f \star'_{r\sin\lambda} g,$$

with

$$f \star'_{r(\cos\leftrightarrow\sin)\lambda} g = \sum_{n=0}^{\infty} \frac{r^n}{n!}(\cos\leftrightarrow\sin)(n\lambda)(\frac{\sqrt{-1}}{2})^n \times \epsilon^{\mu_1\nu_1}\epsilon^{\mu_2\nu_2}\dots\epsilon^{\mu_n\nu_n}\partial_{\mu_1\mu_2\dots\mu_n} f \partial_{\nu_1\nu_2\dots\nu_n} g.$$

The usual $\star$-product is recovered as $\lambda \to 0$ with real deformation parameter $\theta = r$.

Definition 2. The Moyal brackets, also called $\star$-commutator, denoted by $[.,.]_\star$ are defined by

$$\forall f, g, \quad [f, g]_\star = f \star g - g \star f. \tag{2.3}$$

Applying (2.3) to the local coordinates x^μ, one then finds:

$$[x^\mu, x^\nu]_\star = \sqrt{-1}\theta^{\mu\nu} \tag{2.4}$$

which specifies the noncommutative geometry of the space M, in the sense that the coordinates do not anymore commute, in opposite to the commutative theory. The relation (2.4) confirms that θ has length square dimension. A relevant identity follows

$$[x^\mu, \partial_\nu(\cdot)]_\star = \sqrt{-1}\theta^{\mu\rho}\partial_\rho\partial_\nu(\cdot). \tag{2.5}$$

To prove that the $\star$-product is associative we need a technical lemma.

Lemma 1. *Consider $k^\mu, q^\mu, x^\mu \in M$ and the notation $kx = k_\mu x^\mu$. Then, we have*

$$e^{\sqrt{-1}kx} \star e^{\sqrt{-1}qx} = e^{\sqrt{-1}(k+q)x} e^{-\frac{\sqrt{-1}}{2}(k\theta q)}, \tag{2.6}$$

with $k\theta q = k_\mu \theta^{\mu\nu} q_\nu$.

Proof. Since x^μ and y^ν commute in the usual sense ($[x^\mu, y^\nu] = 0$), we get the following relations:

$$\begin{aligned}
&\left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!}\theta^{i_1j_1}\dots\theta^{i_nj_n}\partial_{i_1}\dots\partial_{i_n} e^{\sqrt{-1}kx}\partial_{j_1}\dots\partial_{j_1} e^{\sqrt{-1}qx} \\
&= \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!}\left((\sqrt{-1})^2 k_{i_1}\theta^{i_1j_1}q_{j_1}\right)\dots\left((\sqrt{-1})^2 k_{i_n}\theta^{i_nj_n}q_{j_n}\right) e^{\sqrt{-1}(k+q)x} \\
&= \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!}((\sqrt{-1})^2 k\theta q)^n e^{\sqrt{-1}(k+q)x}.
\end{aligned} \tag{2.7}$$

Now summing over all n, the result follows. There is, of course, an alternative proof, using the Baker-Campbell-Hausdorf formula. $\square$

The following statement holds.

Proposition 2. *The $\star$-product is associative, i.e.*

$$\forall f, g, h, \quad f \star g \star h = (f \star g) \star h = f \star (g \star h)\,.$$

Proof. Let us introduce the momentum space by the usual inverse Fourier transform with a normalized measure:

$$f(x) = \int d^N k\, \tilde{f}(k)\, e^{\sqrt{-1}kx}$$

and observe that, with (2.6), the $\star$-product of two functions f and g can be written as:

$$\begin{aligned} (f \star g)(x) &= \int d^N k\, d^N q\, \tilde{f}(k)\, \tilde{g}(q)\, e^{\sqrt{-1}kx} \star e^{\sqrt{-1}qx} \\ &= \int d^N k\, d^N q\, \tilde{f}(k)\, \tilde{g}(q)\, e^{-\frac{\sqrt{-1}}{2}(k\theta q)} e^{\sqrt{-1}(k+q)x}. \end{aligned} \tag{2.8}$$

Keeping in mind (2.8), one has

$$\begin{aligned} &(f \star g)\,(x) \star h(x) \\ &= \left[\int d^N k\, d^N q\ \tilde{f}(k)\, \tilde{g}(q) e^{-\frac{\sqrt{-1}}{2}(k\theta q)} e^{\sqrt{-1}(k+q)x}\right] \star \left[\int d^N p\ \ \tilde{h}(p)\, e^{\sqrt{-1}px}\right] \\ &= \int d^N k\, d^N q\, d^N p\ \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p) \left[e^{-\frac{\sqrt{-1}}{2}(k\theta q)} e^{\sqrt{-1}(k+q)x}\right] \star \left[e^{\sqrt{-1}px}\right] \\ &= \int d^N k\, d^N q\, d^N p\ \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p)\, e^{-\frac{\sqrt{-1}}{2}(k\theta q)} e^{-\frac{\sqrt{-1}}{2}((k+q)\theta p)} e^{\sqrt{-1}(k+q+p)x} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &f(x) \star (g \star h)\,(x) \\ &= \left[\int d^N k\ \tilde{f}(k) e^{\sqrt{-1}kx}\right] \star \left[\int d^N q\, d^N p\ \tilde{g}(q) \tilde{h}(p) e^{-\frac{\sqrt{-1}}{2}(q\theta p)} e^{\sqrt{-1}(q+p)x}\right] \\ &= \int d^N k\, d^N q\, d^N p\ \tilde{f}(k)\, \tilde{g}(q)\, \tilde{h}(p) e^{-\frac{\sqrt{-1}}{2}(k\theta(q+p))} e^{-\frac{\sqrt{-1}}{2}(q\theta p)} e^{\sqrt{-1}(k+q+p)x}. \end{aligned} \tag{2.10}$$

The results (2.9) and (2.10) are identical. $\square$

Remark 2. This property leads to the comparison of the $\star$-product with the product of matrices. Similar properties based on the associativity of matrices are the same as in M-theory and non-Abelian Yang-Mills gauge theories.

Proposition 3. *The Moyal-brackets* $[(.),(.)]_\star$ *are Lie brackets.*

Proof. The bilinearity and the antisymmetry property of the $\star$-product are immediate. The Jacobi identity is a consequence of the associativity of the $\star$-product. □

Under integral, the behavior of the $\star$-product is now specified.

Proposition 4. *The $\star$-product under integral has the following property, for any f and g,*

$$\begin{aligned} \int d^N x\ (f\star g)(x) &= \int d^N x\, f(x)\, g(x) = \int d^N x\ (g\star f)(x) \\ &\Leftrightarrow \int d^N x\, [f,g]_\star = 0. \end{aligned} \tag{2.11}$$

Proof. The middle integral of the first line is taken over the usual product of functions. This identity is obtained by integrating (2.8). We have:

$$\begin{aligned} \int d^N x\ (f\star g)(x) &= \int d^N x\, d^N k\, d^N q\, \tilde f(k)\, \tilde g(q)\, e^{-\frac{\sqrt{-1}}{2}(k\theta q)} e^{\sqrt{-1}(k+q)x} \\ &= \int d^N k\, d^N q\, \tilde f(k)\, \tilde g(q)\, \delta(k+q) e^{-\frac{\sqrt{-1}}{2}(k\theta q)} \\ &= \int d^N k\, \tilde f(k)\, \tilde g(-k)\, e^{\frac{\sqrt{-1}}{2}(k\theta k)}, \end{aligned}$$

where use has been made of $\int d^N x e^{\sqrt{-1}kx} = \delta(k)$. But θ is antisymmetric then $k\theta k = k_\mu \theta^{\mu\nu} k_\nu = k_\nu \theta^{\mu\nu} k_\mu = -k_\nu \theta^{\nu\mu} k_\mu = -k_\mu \theta^{\mu\nu} k_\nu = -k\theta k$, so $k\theta k = 0$. This shows that $\int d^N x\, (f\star g)(x)$ does not depend on the $\star$-product but only on the Fourier components of functions. In the same manner, it can be observed that:

$$\int d^N x\, f(x)\, g(x) = \int d^N k\, \tilde f(k)\, \tilde g(-k) = \int d^N x\, g(x)\, f(x)$$

which ends the proof. □

Remark 3. This integral property turns out to be crucial in NCFT. Indeed, when all interaction terms are neglected, the free NC field theory reduces to its free commutative counterpart. Such a reduction is prohibited when interaction terms appear at least in cubic $\star$-product factor, for instance in the presence of the terms $A\star B\star C$. The typical examples of the non applicability of such a reduction concern integral actions involving the interaction terms of the form $\phi^{\star 3}$ or $\phi^{\star 4}$ in NC scalar field theory [18][2]. Other 'irreducible' NCFTs count the models with gauge interactions involving the spinors $\psi, \psi^\dagger$ and a gauge field A_μ for instance $\psi\star\gamma^\mu A_\mu\star\psi$ and the $\star$-product of $[A_\mu, A_\nu]_\star$ which occur in the NC electrodynamics model [10][12].

Proposition 5. *Let* $\{f_i\}_{1\le i\le n}$*,* $n \in \mathbb{N}$*, be a set of functions, then,* $\forall 1 \le k \le n$*,*

$$\int d^N x\,(f_1 \star f_2 \star \ldots \star f_k \star f_{k+1} \star \ldots \star f_n)\,(x) = \int d^N x\,(f_{k+1} \star \ldots \star f_n \star f_1 \star f_2 \ldots \star f_k)\,(x). \tag{2.12}$$

Proof. A consequence of Proposition 4 is the cyclicity property of a product of $\star$-factors under the integral in the presence of more than two factors. □

Corollary 1. *For any three functions* f, g *and* h*,*

$$\int d^N x\ \{f,g\}_{\pm,\star} \star h = \int d^N x\, f \star \{g,h\}_{\pm,\star}\,, \tag{2.13}$$

where $\{A,B\}_{+,\star} := A\star B + B\star A$ *and* $\{A,B\}_{-,\star} := [A,B]_\star$.

Proof. We use (2.12) in order to obtain:

$$\begin{aligned}\int d^N x\ \{f,g\}_{\pm,\star} \star h &= \int d^N x\ (f\star g \pm g\star f)\star h \\ &= \int d^N x\ (f\star g\star h \pm f\star h\star g)\,,\end{aligned}$$

which is the expected relation. □

Proposition 6. *Let* f *and* g *be two complex valued functions, then* $(f\star g)^* = g^*\star f^*$*. Moreover, if* f *is real valued* $f\star f$ *is still real valued.*

Proof. This can be immediately deduced from the definition of the $\star$-product. □

Remark 4. Proposition 6 implies that the $\star$-product defines a $\star$-algebra. Note that to choose θ as a complex scalar explicitly conflicts with the existence of such a $\star$-algebra. However, mathematically, a complex θ leads to new interesting bi-parametrized (r and λ) sub-laws which should be investigated on their own (See Remark 1).

Proposition 7. *For any functions* f *and* g*,* $\partial_\mu(f\star g) = \partial_\mu(f)\star g + f\star\partial_\mu(g)$.

Proof. The last equality can be obtained by computations from the expansion of $f\star g$ and the usual Leibniz derivation rule on product of functions. □

We consider the Moyal algebra of functions in the following sense.

Definition 3. The Moyal algebra $\mathcal{M}$ of functions over $M = \mathbb{R}^N$ is a subset of the Schwartz class of complex valued functions of $C^\infty(M)$, which is an associative, involutive under complex conjugation, noncommutative algebra equipped with the $\star$-Moyal product, endowed with a differential calculus satisfying a Leibniz chain rule and an integration calculus in the ordinary sense.

Remark 5. As defined as in Definition 3, the Moyal algebra $\mathcal{M}$ is not unital since nonvanishing constant functions do not belong to the Schwartz space which entails that the latter space is not large enough to be physically interesting ([26] and references therein). This statement can be improved by extending, thanks to duality brackets and smoothening and integral properties of the $\star$-product, the Moyal algebra as an intersection of two tempered distribution subspaces such that the unit constant function could belong to $\mathcal{M}$ [26].

2.2. Algebra of polynomials and series

In this subsection, we first develop the projective limit of a ring of polynomials, namely the ring of formal series. The connection to a new structure of the Moyal field space follows. Within this framework, a new definition of the $\star$-product in terms of a projective limit as well as the projections onto the principal ideals of the ring of series are provided.

Let us mention that the following treatment lies in pure algebra where the notion of convergence and topology do not make obligatory a sense. For instance, the ring of formal series and projective limits are well defined object in abstract algebra, however the actual limit and convergence domain of their elements are never studied from the point of view of topology. We will not prospect in the direction of the convergence (this actually deserves a deep study, of course worthy of interest on its own) of the series and quantities in the subsequent developments, and will adopt the usual algebra formalism. Finally, the applications of our study concern the notion of algebraic functional differentiation, and clearly do not need in any case the notion of convergence.

2.2.1. Projective limit of a ring of polynomials

We extend here some of the developments available in Refs.[17][14].

Proposition 8. *Let $\mathcal{R}_n = \mathbb{C}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$ be the ring of polynomials in the indeterminates $\{\mathbf{x}_i\}_{1\leq i\leq n}$ with complex coefficients. Then, $\mathcal{R}_n$ is a graded ring and $\mathcal{R}_n = \bigoplus_{r\geq 0} \mathcal{R}_n^r$, where $\mathcal{R}_n^r$ is the additive group of homogeneous polynomials of degree r in the same indeterminates $\{\mathbf{x}_i\}_{1\leq i\leq n}$.*

Proof. See Refs.[17][14]. □

Adding another indeterminate $\mathbf{x}_{n+1}$ to the set of indeterminates $\{\mathbf{x}_i\}_{0\leq i\leq n}$, we can naturally form the extended ring $\mathcal{R}_{n+1} = \mathbb{C}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}]$ of polynomials in the indeterminates $\{\mathbf{x}_i\}_{1\leq i\leq n+1}$. There is a projection π_{n+1} from $\mathcal{R}_{n+1}$ onto $\mathcal{R}_n$ defined by setting $\mathbf{x}_{n+1} = 0$.

Lemma 9. *π_{n+1} is a surjective homomorphism of graded rings, i.e.*

(2.14) $\pi_{n+1} : \mathcal{R}_{n+1} \to \mathcal{R}_n$ *is a surjective homomorphism of rings*

(2.15) $\forall r \in \mathbb{N}, \quad \pi^r_{n+1} := \pi_{n+1}\,|_{\mathcal{R}^r_{n+1}} : \mathcal{R}^r_{n+1} \to \mathcal{R}^r_n$ *is a group homomorphism.*

Moreover, $\forall\, r \in \mathbb{N}$, π^r_{n+1} (2.15) is surjective.

Proof. The canonical injection $\mathcal{R}_n \subset \mathcal{R}_{n+1}$, $(\{\mathbf{x}_i\}_{1\leq i\leq n} \subset \{\mathbf{x}_i\}_{1\leq i\leq n+1})$, is naturally graded. The surjectivity (of the group homomorphism π^r_{n+1} and thus of ring homomorphism π_{n+1}) comes from the fact that any polynomial of a given degree r, expressed in n indeterminates, is also a polynomial in $n+1$ indeterminates of degree r where the indeterminate $\mathbf{x}_{n+1}$ does not occur. So, there is also a canonical injection $\mathcal{R}^r_n \subset \mathcal{R}^r_{n+1}$. □

Lemma 10. *Let p and n be a two nonnegative integers. Then, the mapping $\pi_{n+p} : \mathcal{R}_{n+p} \to \mathcal{R}_n$, defined by setting $\mathbf{x}_k = 0$, $n+1 \leq k \leq n+p$, is a surjective homomorphism of graded rings, i.e.*

(2.16) $\pi_{n+p} : \mathcal{R}_{n+p} \to \mathcal{R}_n$ *is a surjective homomorphism of rings*

(2.17) $\forall r \in \mathbb{N}, \quad \pi^r_{n+p} = \pi_{n+p}\,|_{\mathcal{R}^r_{n+p}} : \mathcal{R}^r_{n+p} \to \mathcal{R}^r_n$ *is a group homomorphism.*

Moreover, $\forall\, r \in \mathbb{N}$, π^r_{n+p} (2.17) is surjective.

Proof. This is trivial by induction from Lemma 9. □

By convention, $\forall r \in \mathbb{N}, r \neq 0, \mathcal{R}^r_0 = \emptyset$ and $\forall n \geq 0, \mathcal{R}^0_n = \mathbb{C}$. As a matter of notation, any derivative of order $k \geq 0$ relatively to the indeterminates is denoted by ∂^k in general discussion. The following statements are immediate

$$\begin{aligned} &\forall p, q \in \mathbb{N}, \quad \mathbf{x}^q_{n+p}\mathcal{R}^r_n \subset \mathcal{R}^{r+q}_{n+p}, \\ &\forall k \in \mathbb{N}, \quad \partial^k\mathcal{R}^r_n := \left\{\partial^k P, P \in \mathcal{R}^r_n\right\} \subset \mathcal{R}^{r-k}_n. \end{aligned}$$

Definition 4. Let r a nonnegative integer. The projective or inverse limit of the sequence $(\mathcal{R}^r_n)_{n\in\mathbb{N}}$, as n tends to infinity, is the additive group denoted by $\mathcal{R}^r = \varprojlim_n \mathcal{R}^r_n$ such that: $\forall f^r \in \mathcal{R}^r$, $f^r = (f^r_1, f^r_2, \ldots, f^r_n, \ldots)$, where $f^r_n = \pi_{n+1}(f^r_{n+1})$ and $f^r_n \in \mathcal{R}^r_n$. The general term f^r_n of the sequence is the partial sum of f^r. f^r thus appears as the limit $\lim_{n\to\infty} f^r_n$. Taking the sum over all degrees, we obtain the graded ring

$$\mathcal{R} = \bigoplus_{r\geq 0} \mathcal{R}^r.$$

The elements of $\mathcal{R}$ are called *series*. We have, $\forall f \in \mathcal{R}$, $\; f = \sum_{r\geq 0} \varprojlim_n f^r_n$.

Remark 6. Let us pay attention to the fact that the elements of $\mathcal{R}$ are no longer polynomials as they are expressed in terms of infinite sums. One usually calls $(\mathcal{R}_n^r, \pi_n^r)_{n\in\mathbb{N}}$ a system. Besides, the ring structure of $\mathcal{R}$ is consistent as the projective limit acts with respect to the ring structure, namely, the addition and product of series are well defined through the inverse limit by the direct sum

$$f^r + g^r = (f_1^r + g_1^r,\, f_2^r + g_2^r, \dots, f_n^r + g_n^r, \dots)$$

and the direct product

$$f^r \cdot g^r = (f_1^r g_1^r,\, f_2^r g_2^r, \dots, f_n^r g_n^r, \dots)$$

of groups. Note that $(f^r \cdot g^r)_n = f_n^r\, g_n^r$ is not an element of $\mathcal{R}_n^r$ but belongs to $\mathcal{R}_n^{2r}$ and the series is still well defined by the homomorphism π_n of graded ring.

The following proposition is then straightforward from Lemma 10

Proposition 11. *Let n be a nonnegative integer. Then, the mapping Π_n : $\mathcal{R} \to \mathcal{R}_n$ defined by setting $\mathbf{x}_p = 0$, $p \geq n+1$, is a surjective homomorphism of graded ring.*

2.2.2. Ring of series and phase space

Suppose an infinite dimensional phase space spanned by an infinite many degrees of freedom (dofs) generated by a family of complex valued fields (scalar functions with rapid decay at infinity) $\{\phi_i\}_{i\in I}$ acting on the Minkowski space-time $(\mathbb{R}^{1,3}, \eta)$. This is a conventional choice even though the space manifold dimension could be D in general, without altering the following results.

The following maps

$$\begin{array}{lll}
\forall i \in I, & & \phi_i : M \to \mathbb{R} \\
\forall i \in I, & \forall 0 \leq \mu \leq 3, & \partial_\mu \phi_i : M \to \mathbb{R} \\
\forall i \in I, & \forall 0 \leq \mu_1, \mu_2 \leq 3, & \partial_{\mu_1}\partial_{\mu_2}\phi_i : M \to \mathbb{R} \\
\vdots & \vdots & \vdots \\
\forall i \in I, & \forall 0 \leq \mu_1, \mu_2, \dots, \mu_k \leq 3, & \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_k}\phi_i : M \to \mathbb{R} \\
\forall i \in I, & \forall 0 \leq \mu_1, \mu_2, \dots, \mu_k, \dots \leq 3, & \partial_{\mu_1}\partial_{\mu_2}\dots\partial_{\mu_k}\dots\phi_i : M \to \mathbb{R}
\end{array}$$

define the suitable dofs of the configuration space. For the sake of simplicity, let us adopt the following notation. Let $\mathbf{a}$ be a scalar field, then

(i) any derivative of order k of a field $\mathbf{a}$ with respect to the variables $x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_k}$ is denoted by $\partial_{\mu_k\dots\mu_2\mu_1}\mathbf{a}$ or by $\partial_{[\mu]_k}\mathbf{a}$. Moreover, given a nonnegative integer k, we write, when confusion does not arise,

$$\epsilon^{\mu_1\nu_1}\epsilon^{\mu_2\nu_2}\dots\epsilon^{\mu_k\nu_k}\partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a} = \epsilon^{[\mu\nu]_k}\partial_{[\mu]_k}\mathbf{a}; \tag{2.18}$$

(ii) $\partial\mathbf{a}$ denotes an undetermined derivative of any order of $\mathbf{a}$ in general discussion; otherwise specifications are given;

(iii) $\mathcal{D}$ denotes the infinite set of all dofs spanning the field space:

$$\begin{aligned}\mathcal{D} &= \Big\{\{\mathbf{a}_i\}, \{\partial_\mu \mathbf{a}_i\}_{\mu=0,\dots,3}, \dots, \{\partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_i\}_{\mu_1,\mu_2,\dots,\mu_k=0,\dots,3}, \dots\Big\}_{i\in I} \\ &= \{\mathbf{a}_i, \partial\mathbf{a}_i, \dots\}_{i\in I}\end{aligned}$$

Note that each $\partial_{[\mu]_k}\mathbf{a}_i$ is regarded both as a dof and a k^{th} order derivative of $\mathbf{a}_i$.

In this section, I is a countable set and $|I|$ denotes the cardinal of I. $|I|$ may be infinite. Following the set of indeterminates $\mathcal{D}$, we can define the following subsets of $\mathcal{D}$:

$$\begin{aligned}\mathcal{D}^0 &= \{\mathbf{a}_i\}_{i\in I} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \dots\}, \\ \mathcal{D}^1 &= \{\partial_\mu\mathbf{a}_i\}_{\{i\in I;\ \mu=0,\dots,3\}}, \\ \mathcal{D}^k &= \{\partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_i\}_{\{i\in I;\ \mu_1,\mu_2,\dots,\mu_k=0,\dots,3\}}.\end{aligned}$$

By convention, a derivative of order 0 is the identity. Given n, a nonnegative integer, we also define the finite subsets $\mathcal{D}^k_n$ of $\mathcal{D}^k$ for any $0\le k$, by

$$\begin{aligned}\mathcal{D}^0_n &= \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}, \\ \mathcal{D}^1_n &= \{\partial_\mu\mathbf{a}_i\}_{\{0\le i\le n;\ \mu=0,\dots,3\}}, \\ \mathcal{D}^k_n &= \{\partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_i\}_{\{0\le i\le n;\ \mu_1,\mu_2,\dots,\mu_k=0,\dots,3\}}.\end{aligned}$$

Let us organize the sets of indeterminates as follows. Through the section, the index n is reserved for indexing the fields, namely $\mathbf{a}_n$, the coordinate indices $\mu,\nu,\dots$ denote the indices of variables relative to the derivations while the index k displays the order of derivation. For instance, $\partial_{[\mu]_k}\mathbf{a}_n$ is a k^{th}-order derivative taken on $\mathbf{a}_n$ relatively to the x^μ's. Given a nonnegative integer i such that $1\le i\le n$, setting the singleton set $D^0_i = \{\mathbf{a}_i\}$, then

$$\mathcal{D}^0_n = \bigcup_{1\le i\le n} D^0_i \quad \text{and} \quad \mathcal{D}^0 = \bigcup_{i\in I} D^0_i.$$

Given $k\neq 0$, setting $D^k_i = \bigcup_{0\le\mu_1,\mu_2,\dots,\mu_k\le 3}\{\partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_i\}$, it can be introduced

$$\mathcal{D}^k_n = \bigcup_{1\le i\le n} D^k_i \quad \text{and} \quad \mathcal{D}^k = \bigcup_{i\in I} D^k_i$$

and defined the finite set

$$\mathcal{D}_{n,k} = \bigcup_{1\le i\le n}\ \bigcup_{0\le d\le k} D^d_i. \tag{2.19}$$

Remark 7. It is noteworthy to point out the close relation between dof spaces and jet (or prolongation)-spaces in differential geometry language. In the above formalism, let $\mathbf{a}_i$ be a field, i.e. a function from $\mathbb{R}^{1,3}$ to $\mathbb{R}$, and let n be a nonegative integer, the n^{th} jet of $\mathbf{a}_i$, say $\mathsf{pr}^n\mathbf{a}_i$ (in notations of [20]), is a vector function mapping a point of $\mathbb{R}^{1,3}$ to the vector

$$(\mathbf{a}_i, \partial_\mu \mathbf{a}_i, \dots, \partial_{\mu_1\mu_2\dots\mu_n}\mathbf{a}_i) \\ = (\mathbf{a}_i, \partial_0\mathbf{a}_i, \partial_1\mathbf{a}_i, \dots, \partial_3\mathbf{a}_i, \dots, \partial_{00\dots0}\mathbf{a}_i, \partial_{00\dots01}\mathbf{a}_i, \dots, \partial_{33\dots3}\mathbf{a}_i) \tag{2.20}$$

evaluated at that point. Note that the vector (2.20) belongs to the target jet space

$$U_i^n \subset D_i^0 \times \{\times_{\mu=0,1,2,3} D_i^1\} \times \cdots \times \{\times_{\mu_1,\mu_2,\dots,\mu_n=0,1,2,3} D_i^n\}.$$

Definition 5. Given two positive integers n and k, the polynomial ring

$$\mathbb{C}\,[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n, \partial_\mu\mathbf{a}_1, \partial_\mu\mathbf{a}_2, \dots, \partial_\mu\mathbf{a}_n, \partial_{\mu_1\mu_2}\mathbf{a}_1, \partial_{\mu_1\mu_2}\mathbf{a}_2, \dots, \partial_{\mu_1\mu_2}\mathbf{a}_n, \dots, \\ \partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_1, \partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_2, \dots, \partial_{\mu_1\mu_2\dots\mu_k}\mathbf{a}_n\,],$$

where $\mu, \mu_1, \mu_2, \mu_k \in \{0,1,2,3\}$, is denoted by $\mathcal{R}_{n,k}$. $\mathcal{R}_{n,k}$ is a graded commutative unitary ring. We denote also the additive group of homogeneous polynomials of degree r by $\mathcal{R}^r_{n,k}$.

Lemma 12. *Let r, n and k be three positive integers, and let us denote the cardinal of the set $\mathcal{D}_{n,k}$ by $|\mathcal{D}_{n,k}|$. There is a group isomorphism $\mathcal{R}^r_{n,k} \equiv \mathcal{R}^r_{|\mathcal{D}_{n,k}|}$ leading to a graded ring isomorphism $\mathcal{R}_{n,k} \equiv \mathcal{R}_{|\mathcal{D}_{n,k}|}$.*

Proof. $\mathcal{D}_{n,k}$ is in bijection with $\left\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{|\mathcal{D}_{n,k}|}\right\}$. The independence of indeterminates requires that each of the $\partial_{\mu_1\mu_2\dots\mu_p}\mathbf{a}_n$ should correspond to a unique $\mathbf{x}_i$. One can build a well defined one-to-one homomorphism between the generators of $\mathcal{R}^r_{n,k}$ and those of $\mathcal{R}^r_{|\mathcal{D}_{n,k}|}$. □

Let us introduce the definition.

Definition 6. Let n, k and r be nonnegative integers. Then,

(i) the group homomorphism $v^r_{n+1,k} : \mathcal{R}^r_{n+1,k} \to \mathcal{R}^r_{n,k}$ defined by setting $\mathbf{a}_{n+1} = 0$, (and consequently, we have $\partial_\mu\mathbf{a}_{n+1} = 0, \dots, \partial_{\mu_1\mu_2,\dots,\mu_k}\mathbf{a}_{n+1} = 0$,) is called the $(n+1,k)$ vertical projection of degree r or simply the v-projection when no confusion occurs.

(ii) The group homomorphism $h^r_{n,k+1} : \mathcal{R}^r_{n,k+1} \to \mathcal{R}^r_{n,k}$ defined by setting

$$\partial_{\mu_1\mu_2,\dots,\mu_{k+1}}\mathbf{a}_1 = 0, \;\; \partial_{\mu_1\mu_2,\dots,\mu_{k+1}}\mathbf{a}_2 = 0, \dots, \partial_{\mu_1\mu_2,\dots,\mu_{k+1}}\mathbf{a}_n = 0$$

is called $(n, k+1)$ horizontal projection of degree r or simply the h-projection when no confusion occurs.

(iii) The group homomorphism $\pi^r_{n+1,k+1} : \mathcal{R}^r_{n+1,k+1} \to \mathcal{R}^r_{n,k}$ defined by setting:

$$\mathbf{a}_{n+1} = 0,\ \partial_\mu \mathbf{a}_{n+1} = 0,\ \ldots, \partial_{\mu_1\mu_2,\ldots,\mu_k}\mathbf{a}_{n+1} = 0,\ \partial_{\mu_1\mu_2,\ldots,\mu_{k+1}}\mathbf{a}_1 = 0,$$
$$\partial_{\mu_1\mu_2,\ldots,\mu_{k+1}}\mathbf{a}_2 = 0,\ \partial_{\mu_1\mu_2,\ldots,\mu_{k+1}}\mathbf{a}_n = 0, \text{and}\ \ \partial_{\mu_1\mu_2\ldots\mu_{k+1}}\mathbf{a}_{n+1} = 0$$

is called the $(n+1, k+1)$ projection of degree r.

Remark 8. A simple observation of the above defined projections proves that their definition does not depend on the degree r of their domain $\mathcal{R}^r_{n,k}$. We can admit henceforth that the way by which $v^r_{n,k}$, $h^r_{n,k}$ and $\pi^r_{n,k}$ act on $\mathcal{R}^r_{n,k}$ is the same as that by which $v^{r+p}_{n,k}$, $h^{r+p}_{n,k}$ and $\pi^{r+p}_{n,k}$ act on $\mathcal{R}^r_{n,k}$, for all $p \in \mathbb{N}$.

The following Lemma is satisfied.

Lemma 13. *Given a positive integer n, the $(n+1, 1)$ v-projection*

$$v^r_{n+1,1} : \mathcal{R}^r_{n+1,1} \to \mathcal{R}^r_{n,1}$$

is surjective for all $r \geq 0$.

Proof. We have from Lemma 12, $\mathcal{R}^r_{n+1,1} \equiv \mathcal{R}^r_{|\mathcal{D}_{n+1,1}|}$ and $\mathcal{R}^r_{n,1} \equiv \mathcal{R}^r_{|\mathcal{D}_{n,1}|}$. □

Proposition 14. (i) *Given a positive integer n, $\forall k \in \mathbb{N}$, the $(n+1, k)$ v-projection of degree r is surjective for any $r \geq 0$.*

(ii) *Given a positive integer k, $\forall n \in \mathbb{N}$, the $(n, k+1)$ h-projection is surjective for any $r \geq 0$.*

(iii) $\forall k \in \mathbb{N}$, $\forall n \in \mathbb{N}$, *the $(n+1, k+1)$ projection is surjective for any $r \geq 0$.*

Proof. The proofs of the surjections are immediate by induction on k using Lemmas 12 and 13. Indeed, consider Lemma 13 for the order $k = 1$. (i) becomes obvious. The points (ii) and (iii) can be shown in a similar way: one has just to give the analogues of Lemma 13 (for $k = 1$) for $h^r_{1,k+1}$, $\pi_{n+1,1}$ and $\pi_{1,k+1}$. □

The following statement holds.

Theorem 1. *Let k be a nonnegative integer and*

$$\mathcal{R}_{n,k} = \bigoplus_{r\geq 0} \mathcal{R}^r_{n,k} = \mathbb{C}\left[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n, \partial_\mu \mathbf{a}_1, \partial_\mu \mathbf{a}_2 \ldots, \partial_\mu \mathbf{a}_n, \ldots, \partial_{\mu_1\mu_2\ldots\mu_k}\mathbf{a}_1,\right.$$
$$\left.\partial_{\mu_1\mu_2\ldots\mu_k}\mathbf{a}_2, \ldots, \partial_{\mu_1\mu_2\ldots\mu_k}\mathbf{a}_n\right],$$

be the graded ring of polynomials in the indeterminate elements of $\mathcal{D}_{n,k}$ over $\mathbb{C}$. Then, the $(n, k+1)$ h-projection

$$h_{n,k+1} : \mathcal{R}_{n,k+1} \to \mathcal{R}_{n,k} \tag{2.21}$$

defined by setting, for $1 \leq p \leq n$, $\partial_{\mu_1\mu_2\ldots\mu_{k+1}}\mathbf{a}_p = 0$, *defines the graded ring* $\mathcal{R}_{n,\cdot} = \bigoplus_{r\geq 0} \lim_{\overleftarrow{k}} \mathcal{R}^r_{n,k}$ *of series in the infinite number of indeterminates of* $\bigcup_{k=0}^{\infty} \mathcal{D}_{n,k}$.

Proof. The projection $h_{n,k+1}$ is a surjection from $\mathcal{R}_{n,k+1}$ onto $\mathcal{R}_{n,k}$ as established by Proposition 14. It readily defines a system having for projective limit, a ring of series. □

Remark 9. Of course, projective limits relatively to vertical and diagonal projections, namely $v_{n+1,k}$ and $\pi_{n+1,k+1}$, respectively, could be defined in a similar way with adapted considerations. However, it turns out that the relevant aspects useful in NCFTs are provided only by horizontal projection.

Lemma 15. *Given three positive integers* r, k *and* n, $P^r_{\cdot,k}$*- an element of* $\mathcal{R}^r_{\cdot,k}$ *and* $Q^r_{n,\cdot}$*- an element of* $\mathcal{R}^r_{n,\cdot}$ *then,*

$$\forall p \in \mathbb{N}, \quad v^{r+p}_{n+1,k}(P^r_{n+1,k}) = P^r_{n,k}, \quad h^{r+p}_{n,k+1}(Q^r_{n,k+1}) = Q^r_{n,k}. \tag{2.22}$$

Proof. See Lemma 13 and Proposition 11. □

There are two equivalent ways to realize any monomial as a series. This depends on the ring where the projective limit is built, namely, $\mathcal{R}_{\cdot,k}$ or $\mathcal{R}_{n,\cdot}$. One has to use the good *representation* to realize a relevant and easiest analysis. Given the positive integers m, n, k, the following realizations of $\partial_{[\mu]_k}\mathbf{a}_n$ are valid

$$\partial_{[\mu]_k}\mathbf{a}_n = (0, \ldots, 0, \partial_{[\mu]_k}\mathbf{a}_n, \partial_{[\mu]_k}\mathbf{a}_n, \ldots), \tag{2.23}$$

where $\partial_{[\mu]_k}\mathbf{a}_n$ is at the k^{th} position and the remainder of the terms are constant and equal to $\partial_{[\mu]_k}\mathbf{a}_n$. This series is used when the projective limit is done with respect to $\mathcal{R}_{n,\cdot}$. The second way is denoted by a similar expression but, in this case, $\partial_{[\mu]_k}\mathbf{a}_n$ appears at the n^{th} position and the remainder of the terms are constant and equal to $\partial_{[\mu]_k}\mathbf{a}_n$. This series is used when calculations are done in $\mathcal{R}_{\cdot,k}$. Following this realization as series, $\partial_{[\mu]_k}\mathbf{a}_n$ belongs to $\mathcal{R}^1_{n,\cdot}$ or to $\mathcal{R}^1_{\cdot,k}$. Besides, $\mathbf{a}^m_n = (0, \ldots, 0, \mathbf{a}^m_n, \ldots)$ (the first $\mathbf{a}^m_n$ appearing at the n^{th} position) is viewed as an element of $\mathcal{R}^m_{\cdot,0}$, while written as $(\mathbf{a}^m_n, \mathbf{a}^m_n, \ldots, \mathbf{a}^m_n, \ldots)$, $\mathbf{a}^m_n$ is viewed as an element of $\mathcal{R}^m_{n,\cdot}$.

Let us define the double sequence $(q^n_m)_{n,m\in\mathbb{N}}$ of positive integers such that: $\forall n, m \in N$, such that $q^n_m = 0$ for $m > n$, i.e. explicitly

$$\begin{array}{l} q^n_0 = 0, \quad \forall n \geq 0, \\ q^1_1 \in \mathbb{N}, \quad q^1_2 = 0, \quad \ldots \quad , q^1_n = 0, \ldots; \\ q^2_1 \in \mathbb{N}, \quad q^2_2 \in \mathbb{N}, \quad q^2_3 = 0, \quad \ldots \quad , q^2_n = 0, \quad \ldots; \\ \vdots \end{array}$$

$$q_1^n \in \mathbb{N}, \;\; q_2^n \in \mathbb{N}, \;\; \dots, \;\; q_n^n \in \mathbb{N}; \;\; q_{n+1}^n = 0 \;\; \dots$$

In the following, we do not take care of the index 0 in the first term of the inverse limit. We assume that it is null. The following statement holds.

Lemma 16. *Given n, r, k positive integers, $P^r_{n,.} \in \mathcal{R}^r_{n,.}$ and $P^r_{.,k} \in \mathcal{R}^r_{.,k}$. Then, for any positive integers p and q, we have*

$$\mathbf{a}^q_{n+p} P^r_{n,.} \in \mathcal{R}^{r+q}_{(n+p),.}; \qquad \mathbf{a}^q_{n+p} P^r_{.,k} \in \mathcal{R}^{r+q}_{.,k}. \tag{2.24}$$

Furthermore, for any positive integers m and p

$$\partial_{[\mu]_m} \mathbf{a}_{n+p} P^r_{n,.} \in \mathcal{R}^{r+1}_{n+p,.}; \qquad \partial_{[\mu]_m} \mathbf{a}_n P^r_{.,k} \in \mathcal{R}^{r+1}_{.,sup(m,k)}. \tag{2.25}$$

Considering the following element of $\bigoplus_{n,m\geq 1} \mathcal{R}^{\sum_{i=1}^n q^i_m}_{.,0} \subset \mathcal{R}_{.,0}$ defined by

$$\tilde{P}_{.,0} = \left(\tilde{P}_{1,0} = \mathbf{a}_1^{q_1^1}, \;\; \tilde{P}_{2,0} = \mathbf{a}_1^{q_1^1} + \mathbf{a}_1^{q_1^2} \mathbf{a}_2^{q_2^2}, \dots, \tilde{P}_{n,0} = \sum_{i=1}^n \mathbf{a}_1^{q_1^i} \mathbf{a}_2^{q_2^i} \dots \mathbf{a}_i^{q_i^i}, \dots \right),$$

if $\sum_{i=1}^n q^i_m = q^1_1$, $\forall m \in \mathbb{N}$, i.e. the degree of each monomial term realizes a partition of q^1_1, then,

$$\tilde{P}_{.,0} \mathcal{R}^r_{.,k} \subset \mathcal{R}^{r+q_1^1}_{.,k}. \tag{2.26}$$

Proof. Given $P^r_{n,.} \in \mathcal{R}^r_{n,.}$, then, for any q, let us define $\mathcal{P}^{q+r}_{n+p,k} := \mathbf{a}^q_{n+p} P^r_{n,k} \in \mathcal{R}^{r+q}_{(n+p),k}$. The direct product of $\mathbf{a}^q_{n+p}$ (realized as series) by $P^r_{n,.}$ lies in $\mathcal{R}^{r+q}_{(n+p),.}$. Indeed, one gets

$$\mathbf{a}^q_{n+p} P^r_{n,.} = (\mathbf{a}^q_{n+p} P^r_{n,0}, \dots, \mathbf{a}^q_{n+p} P^r_{n,k}, \mathbf{a}^q_{n+p} P^r_{n,(k+1)}, \dots).$$

From Lemma 15, any h-projection $h^{r+q}_{(n+p),(k+1)}$ has the same action as the projection $h^r_{(n+p),(k+1)}$. We get

$$\begin{aligned} h^{r+q}_{(n+p),(k+1)} \mathcal{P}^{q+r}_{(n+p),(k+1)} &= h^r_{(n+p),(k+1)} (\mathbf{a}^q_{n+p} P^r_{n,(k+1)}) \\ &= \mathbf{a}^q_{n+p} h^r_{(n+p),(k+1)} (P^r_{n,(k+1)}) = \mathbf{a}^q_{n+p} P^r_{n,k} = \mathcal{P}^{q+r}_{n+p,k}. \end{aligned}$$

This ends the proof of the l.h.s. relation of (2.24). The proof of the r.h.s relation can be deduced by a similar procedure. Given $P^r_{.,k} \in \mathcal{R}^r_{.,k}$, for any q, we set

$$\mathcal{P}^{r+q}_{n+p,k} := \mathbf{a}^q_{n+p} P^r_{n,k} \in \mathcal{R}^{r+q}_{n+p,k}.$$

We have the direct product

$$\mathcal{P}^{r+q}_{.,k} = (0, \dots, \mathbf{a}^q_{n+p}, \dots)(P^r_{1,k}, P^r_{2,k}, \dots, P^r_{n,k}, \dots)$$

$$= \quad (0, \dots, \mathbf{a}^q_{n+p} P^r_{(n+p),k}, \mathbf{a}^q_{n+p} P^r_{(n+p+1),k}, \dots).$$

Then, for any integer $m \geq 0$, applying the v-projection $v^{r+q}_{(m+1),k}(\mathcal{P}^{r+q}_{(m+1),k}) = 0$ if $(m+1) \leq n+p$, due to $\mathcal{P}^{r+q}_{(m\leq n+p),k} = 0$. Meanwhile, for any positive integer l,

$$v^{r+q}_{(n+p+l+1),k} \mathcal{P}^{r+q}_{(n+p+l+1),k} = v^{r+q}_{(n+p+l+1),k}(\mathbf{a}^q_{n+p} P^r_{(n+p+l+1),k})$$

which gives, using Lemma 15,

$$\mathbf{a}^q_{n+p} v^{r+q}_{(n+p+l+1),k}(P^r_{(n+p+l+1),k}) = \mathbf{a}^q_{n+p} P^r_{(n+p+l),k}.$$

This is simply $\mathcal{P}^{r+q}_{(n+p+l),k} = \mathcal{P}^{r+q}_{m,k}$ if $(m = n+p+l) \geq n+p$. This ends the proof of (2.24).

The proof of (2.25) is given by the following. The direct product of the realization of $\partial_{[\mu]_k} \mathbf{a}_{n+p}$ (See (2.23)) by the series $P^r_{n,\cdot} \in \mathcal{R}^r_{n,\cdot}$ can be written

$$\mathcal{P}^{r+1}_{(n+p),\cdot} := \partial_{[\mu]_m} \mathbf{a}_{n+p} P^r_{n,\cdot} = (0, \dots, \partial_{[\mu]_m} \mathbf{a}_{n+p} P^r_{n,m}, \partial_{[\mu]_m} \mathbf{a}_{n+p} P^r_{n,m+1}, \dots)$$

with $\mathcal{P}^{r+1}_{(n+p),k} \in \mathcal{R}^{r+1}_{(n+p),k}$. The h-projection thus acts as follows:

$$h^{r+1}_{(n+p),(k+1)} \mathcal{P}^{r+1}_{(n+p),(k+1)} = h^{r+1}_{n+p,k+1}(\partial_{[\mu]_m} \mathbf{a}^q_{n+p} P^r_{n,k+1}).$$

Suppose $m < k+1$. Using Lemma (15), we obtain

$$h^{r+1}_{(n+p),(k+1)}(\mathcal{P}^{r+1}_{(n+p),(k+1)}) = \partial_{[\mu]_m} \mathbf{a}^q_{n+p} h^{r+1}_{(n+p),(k+1)}(P^r_{n,k+1}) = \partial_{[\mu]_m} \mathbf{a}^q_{n+p} P^r_{n,k}.$$

Now, assume $m \geq k+1$. We get $h^{r+1}_{(n+p),(k+1)} \mathcal{P}^{r+1}_{(n+p),(k+1)} = 0$. This ends the proof of the l.h.s. relation of (2.25). For the r.h.s. relation, given m and k, let us set $k_0 = max(m,k)$. The direct product of the series (2.23) by $P^r_{\cdot,k}$ is defined by

$$\mathcal{P}^{r+1}_{\cdot,k_0} = \partial_{[\mu]_m} \mathbf{a}_n P^r_{\cdot,k} = (0, \dots, \partial_{[\mu]_m} \mathbf{a}_n P^r_{p,k}, \partial_{[\mu]_m} \mathbf{a}_n P^r_{(p+1),k}, \dots),$$

where the general term of the sequence is given by

$$\mathcal{P}^{r+1}_{p,k_0} = 0, \ \forall p < n \ \text{ and } \ \mathcal{P}^{r+1}_{p,k} = \partial_{[\mu]_m} \mathbf{a}_n P^{r+1}_{p,k}, \ \forall p \geq n.$$

Then, $\mathcal{P}^{r+1}_{p,k_0} \in \mathcal{R}^{r+1}_{p,k_0}$. The v-projection acts as follows on $\mathcal{R}^{r+1}_{(p+1),k_0}$:

$$v^{r+1}_{(p+1),k_0}(\mathcal{P}^{r+1}_{(p+1),k_0}) = 0, \ \ \forall\, p+1 \leq n$$

and

$$v^{r+1}_{(p+1),k_0}(\mathcal{P}^{r+1}_{(p+1),k_0}) = \partial_{[\mu]_m} \mathbf{a}_n v^{r+1}_{(p+1),k_0}(P^r_{(p+1),k}) = \partial_{[\mu]_m} \mathbf{a}_n P^r_{p,k}, \ \ \forall\, p+1 > n.$$

This ends the proof of (2.25). The relation (2.26) is a corollary of the first relation of (2.24) and its proof proceeds from the same manner. The requirement that any monomial must be of the same degree q_1^1, i.e. $q_1^n + \cdots + q_m^n = q_1^1$, insures the stability of the direct product in the group $\mathcal{R}^{r+q_1^1}_{.,k}$. □

Theorem 2. *The direct product of series of* $\mathcal{R}^r_{n,.}$ *(resp.*$\mathcal{R}^r_{.,k}$*) by series of* $\mathcal{R}^d_{n,.}$ *(resp.*$\mathcal{R}^d_{.,k}$*) belongs to* $\mathcal{R}^{r+d}_{n,.}$ *(resp.*$\mathcal{R}^{r+d}_{.,k}$*). The direct product of series of* $\mathcal{R}^r_{n,.}$ *(resp.*$\mathcal{R}^r_{.,k}$*) by series of* $\mathcal{R}^d_{m,.}$ *(resp.*$\mathcal{R}^d_{.,q}$*) belongs to* $\mathcal{R}^{r+d}_{sup(n,m),.}$ *(resp.*$\mathcal{R}^{r+d}_{.,sup(k,q)}$*).*

Proof. This can be derived from the definition of direct product of series and from Lemmas 15 and 16, considering that the double sequence $P^r_{n,k}$ for $n, k \in \mathbb{N}$ can be viewed as an ordinary sequence when the second index is fixed. Indeed, let $P^r_{n.} = (P^r_{n,1}, P^r_{n,2}, \ldots, P^r_{n,k}, \ldots)$ and $Q^d_{n.} = (Q^d_{n,1}, Q^d_{n,2}, \ldots, Q^d_{n,k}, \ldots)$ be, respectively, two elements of $\mathcal{R}^r_{n,.}$ and $\mathcal{R}^d_{n,.}$. Then, the direct product of $P^r_{n.}$ by $Q^d_{n.}$, denoted by $\mathcal{P}^{r+d}_{n,.}$ is given by $\mathcal{P}^{r+d}_{n,.} = P^r_{n,.} Q^d_{n,.}$, that is

$$\mathcal{P}^{r+d}_{n,.} = (\mathcal{P}^{r+d}_{n,1} = P^r_{n,1} Q^d_{n,1}, \mathcal{P}^{r+d}_{n,2} = P^r_{n,2} Q^d_{n,2}, \ldots, \mathcal{P}^{r+d}_{n,k} = P^r_{n,k} Q^d_{n,k}, \ldots).$$

The projection $h^{r+d}_{n,k+1} : \mathcal{R}^{r+d}_{n,k+1} \to \mathcal{R}^{r+d}_{n,k}$ is defined by setting zero any term of the form $\partial_{[\mu]_{k+1}} \mathbf{a}_{i \le n} = 0$, using Lemma 15. □

2.2.3. Ideals and projectors

Consider the sets of dofs $\mathcal{D}_{n,k} \subset \mathcal{D}$. The dofs taken in $\mathcal{D}_{n,k}$ are the indeterminates of the polynomial graded ring

$$\mathcal{R}_{n,k} \equiv \mathbb{C}\left[\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a}_n, \partial_{[\mu]_k} \mathbf{a_1}, \partial_{[\mu]_k} \mathbf{a_2}, \ldots, \partial_{[\mu]_k} \mathbf{a}_n\right].$$

Theorem 1 states the inverse limit $\mathcal{R}_{n,.} = \bigoplus_{r \ge 0} \lim_{\overleftarrow{k}} \mathcal{R}^r_{n,k}$ in the infinite number of indeterminate elements of $\mathcal{D}$. We fix in this section the number of indeterminates to be n which is no longer displayed. $\mathcal{R}_{n,.}$ is henceforth denoted by $\mathcal{R}$ for the sake of simplicity. The dofs are taken in the general form $\mathbf{a}$ or $\mathbf{b}$. Let us also mention that the following developments are valid in general algebra [14].

Proposition 17. *Any principal ideal of* $\mathcal{R}$*, generated by a fixed* $\mathbf{a}_0 \in \mathcal{D}$*, can be written as* $< \mathbf{a}_0 >= \mathbf{a}_0 \mathcal{R}$*.*

The form of $< \mathbf{a}_0 >$ is due to the fact that the ring $\mathcal{R}$ is commutative. Given a fixed indeterminate $\mathbf{a}_0 \in \mathcal{D}$, the projection $i_{\mathbf{a}_0}$ onto the ideal $< \mathbf{a}_0 >$ can be defined as

$$i_{\mathbf{a}_0}(P) = P, \text{ if } P \in < \mathbf{a}_0 >,$$

$$i_{\mathbf{a}_0}(P) = 0, \text{ if } < P > \cap < \mathbf{a}_0 >= \emptyset, \tag{2.27}$$

$< P >$ being the ideal generated by P, such that $i_{\mathbf{a}_0}$ has the natural following properties:

(i) $i^2_{\mathbf{a}_0} = i_{\mathbf{a}_0}$;

(ii) $i_{\mathbf{a}_0}$ is linear on $\mathcal{R}$;

(iii) $\mathcal{R} = Im\, i_{\mathbf{a}_0} \oplus Ker\, i_{\mathbf{a}_0}$, where $Im\, i_{\mathbf{a}_0}$ denotes the range of $i_{\mathbf{a}_0}$ and $Ker\, i_{\mathbf{a}_0}$ its kernel.

Now, we pay attention to more pratical and basic relations useful for computational techniques in NCFT. The following properties are of particular significance in NCFT.

Proposition 18. *Let $\mathbb{I}$ be the unity of $\mathcal{R}$, and $\mathbf{a}_0$ be a fixed dof in this space. Then, the following relations hold, $\forall \mathbf{a}$ a non constant dof,*

$$\forall \lambda \in \mathbb{C} \setminus \{0\}, \qquad i_{\mathbf{a}_0}(\lambda \mathbb{I}) = 0, \; i_{\mathbf{a}_0}(0) = 0, \tag{2.28}$$

$$\mathbf{a} \neq \lambda \mathbf{a}_0, \;\; \forall\, \lambda \in \mathbb{C}, \qquad i_{\mathbf{a}_0}(\mathbf{a}) = 0, \tag{2.29}$$

$$\textit{if}\, \exists\, \lambda \in \mathbb{C}, \;\; \mathbf{a} = \lambda \mathbf{a}_0, \qquad i_{\mathbf{a}_0}(\mathbf{a}) = \mathbf{a}. \tag{2.30}$$

Proof. The proposition is immediate if one considers that $i_{\mathbf{a}_0}$ is a projection and any dof is an independent and irreducible element of $\mathcal{R}$. □

Proposition 19. *Given $\mathbf{a}_0$ a dof, the binary relation $=_{\mathbf{a}_0}$ in $\mathcal{R}$ hereafter called "to be equal to (modulo $\mathbf{a}_0$)" and defined as follows*

$$\forall \mathcal{F}, \mathcal{G} \in \mathcal{R}, \quad \mathcal{F} =_{\mathbf{a}_0} \mathcal{G} \Leftrightarrow i_{\mathbf{a}_0}(\mathcal{F}) = i_{\mathbf{a}_0}(\mathcal{G}), \tag{2.31}$$

is an equivalence relation in $\mathcal{R}$.

Remark 10. The equivalence relation (2.31) has nothing to see with the usual equivalence relation 'to be equal to (modulo the ideal $< \mathbf{a}_0 >$)'. Indeed, if $F \equiv G$ modulo $< \mathbf{a}_0 >$, this means that $F - G \in \mathbf{a}_0 \mathcal{R}$, in other terms, $F = G + \mathbf{a}_0 \cdot B$, for some $B \in \mathcal{R}$. This does not imply that F and G have the same image under the projector onto $< \mathbf{a}_0 >$. Besides, in the ring of series, the derivation with respect to a given indeterminate keeps its ordinary sense satisfying the Leibniz rule. In symbol, we write

$$\frac{\partial}{\partial \mathbf{a}_0}(f \cdot g) = \frac{\partial}{\partial \mathbf{a}_0}(f) \cdot g + f \cdot \frac{\partial}{\partial \mathbf{a}_0}(g).$$

It is noteworthy that taking the derivative of series with respect to a given $\mathbf{a}_0$, one must implicitly use the projection on the ideal spanned by $\mathbf{a}_0$, i.e.

$$\frac{\partial}{\partial \mathbf{a}_0}(f) = \frac{\partial}{\partial \mathbf{a}_0}(i_{\mathbf{a}_0} f) \quad \Leftrightarrow \quad \frac{\partial}{\partial \mathbf{a}_0}(f - i_{\mathbf{a}_0} f) = 0.$$

Hence, the derivation targets all terms included in f involving the quantity $\mathbf{a}_0$, i.e. the sum of terms being equal to f modulo $\mathbf{a}_0$. The remaining terms cancel under the projector $i_{\mathbf{a}_0}$.

2.3. $\star$-algebra revisited

In this subsection, we provide the definition of the $\star$-product in terms of projective limits of space functions. Some relevant calculations in NCFT are also given.

2.3.1. $\mathbf{a} \star \mathbf{b}$ as a projective limit

Let $\mathcal{R}_{n,.}$ be the ring of series defined by inverse limits of sequences of the ring of polynomials $\mathcal{R}_{n,k}$ in the indeterminates of $\mathcal{D}_{n,k}$ over $\mathbb{C}$. From the relation (2.2) and Theorem 1, we aim at constructing a star product $\mathbf{a} \star \mathbf{b}$ in the form of an inverse limit.

Any partial sum of

$$\mathbf{a} \star \mathbf{b} = \mathbf{ab} + \frac{\sqrt{-1}}{2}\theta^{\mu\nu}\partial_\mu \mathbf{a} \partial_\nu \mathbf{b} + \cdots$$

is a homogeneous polynomial of degree $r = 2$ and is built with two indeterminates which belong to $D_2^0 = \{\mathbf{a}, \mathbf{b}\}$. Then, we set the following sequence

$$P_{2,.}^2 = \left(P_{2,0}^2, P_{2,1}^2, P_{2,2}^2, P_{2,3}^2, \dots, P_{2,k}^2, \dots\right),$$

$$\forall k \in \mathbb{N}, \quad P_{2,k}^2 = \sum_{p=0}^{k} \frac{1}{p!}\left(\frac{\sqrt{-1}}{2}\right)^p \theta^{[\mu\nu]_p}\partial_{[\mu]_p}\mathbf{a}\partial_{[\nu]_p}\mathbf{b}. \tag{2.32}$$

$\forall k \in \mathbb{N}$, the $(2, k+1)$ h-projection, defined by vanishing all derivatives of the form $\partial_{[\mu]_{k+1}}\mathbf{a}$ as well as that of the form $\partial_{[\mu]_{k+1}}\mathbf{b}$, has the following action on the general term $P_{2,k+1}^2$

$$h_{2,k+1}^2(P_{2,k+1}^2) = P_{2,k}^2 \tag{2.33}$$

as expected. This result is summarized in the following

Theorem 3. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two dofs of the field space. Then,* $\mathbf{a} \star \mathbf{b} \in \mathcal{R}_{2,.}$ *and we have explicitly,* $\mathbf{a} \star \mathbf{b} = P_{2,.}^2 = \lim_{\overleftarrow{k}} P_{2,k}^2$.

It readily follows the statement

Corollary 2. *For any dof* $\mathbf{a}$ *and* $\mathbf{b}$:

$$\theta^{[\mu\nu]_p}\partial_{[\mu]_p}(\mathbf{a} \star \mathbf{b}) \in \mathcal{R}_{2,.}^2, \qquad \forall p \in \mathbb{N}, \quad \mathbf{a}^{\star p} \in \mathcal{R}_{1,.}^p. \tag{2.34}$$

For all $n \in \mathbb{N}$, consider a multi-index $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}^n$ such that $p \neq q$, $i_p \neq i_q$ and defining a family of dofs $\{\mathbf{a}_{i_k}\}_{i_k}$, and let $J = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$ be another multi-index. Then

$$\mathbf{a}_{i_1}^{j_1} \star \mathbf{a}_{i_2}^{j_2} \cdots \star \mathbf{a}_{i_n}^{j_n} \in \mathcal{R}^{j_1+j_2+\cdots+j_n}_{max(i_1,i_2,\dots,i_n)}. \tag{2.35}$$

Proof. For a fixed μ and by the Leibniz rule, we write

$$\partial_\mu(\mathbf{a} \star \mathbf{b}) = \partial_\mu \mathbf{a} \star \mathbf{b} + \mathbf{a} \star \partial_\mu \mathbf{b}. \tag{2.36}$$

It is then immediate that (2.36), being a sum of $\star$-products of two $\star$-factors, belongs to $\mathcal{R}^2_{2,\cdot}$. Indeed, we give now a prescribed sequence which realizes the l.h.s. term of (2.36) as an inverse limit (the r.h.s. is treated in the same manner). The sequence which generates $\partial_\mu \mathbf{a} \star \mathbf{b}$ by an inverse limit is given by:

$$P^2_{2,0} = 0, \;\; P^2_{2,1} = \partial_\mu \mathbf{a}\mathbf{b}, \;\; P^2_{2,2} = \partial_\mu \mathbf{a}\mathbf{b} + \frac{\sqrt{-1}}{2}\theta^{\rho\sigma}\partial_{\rho\mu}\mathbf{a}\partial_\sigma \mathbf{b},$$
$$P^2_{2,k} = \sum_{d=0}^{k} \left(\frac{\sqrt{-1}}{2}\right)^d \frac{1}{d!}\,\theta^{[\rho\sigma]_d}\partial_{[\rho]_d}(\partial_\mu \mathbf{a})\,\partial_{[\sigma]_d}\mathbf{b}.$$

One can see that the projection (2.21) is well defined. The inverse limit in k is of degree 2 and defines $\partial_\mu \mathbf{a} \star \mathbf{b}$. This proves (2.34) for a single index μ. Proceeding by induction, for p a nonnegative integer, we have

$$\theta^{[\mu\nu]_p}\partial_{[\mu]_p}(\mathbf{a} \star \mathbf{b}) = \theta^{[\mu\nu]_p}\sum_{l=0}^{p} C^l_p\,\partial_{[\mu]_l}\mathbf{a} \star \partial_{[\mu]_{p-l}}\mathbf{b}, \tag{2.37}$$

$$C^l_p = p!/(l!(p-l)!). \tag{2.38}$$

This sum of $\star$-products belongs to $\mathcal{R}^2_{2,\cdot}$. This ends the proof of the first relation of (2.34).

The proof of (2.35) proceeds from induction on the number n of indeterminates. The case $n = 2$ corresponds to the definition of the Moyal product. Assume that the relation is true up to the order n. Let us set

$$T^{j_1,j_2,\dots,j_n}_{i_1,i_2,\dots,i_n} = \mathbf{a}_{i_1}^{j_1} \star \mathbf{a}_{i_2}^{j_2} \cdots \star \mathbf{a}_{i_n}^{j_n} \in \mathcal{R}^{j_1+j_2+\cdots+j_n}_{max(i_1,i_2,\dots,i_n)}. \tag{2.39}$$

The identity (2.34) proves, by induction on the number of indeterminate n, that the derivative $\partial_{[\mu]_k}T^{j_1,j_2,\dots,j_n}_{i_1,i_2,\dots,i_n} \in \mathcal{R}^{j_1+j_2+\cdots+j_n}_{max(i_1,i_2,\dots,i_n)}$.

Now let $\mathbf{a}_{i_{n+1}}$ be another indeterminate and j_{n+1} another integer.

$$\begin{aligned} &T^{j_1,j_2,\dots,j_n}_{i_1,i_2,\dots,i_n} \star \mathbf{a}^{j_{n+1}}_{i_{n+1}} \\ &\quad = T^{j_1,j_2,\dots,j_n}_{i_1,i_2,\dots,i_n}\mathbf{a}^{j_{n+1}}_{i_{n+1}} + \sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{\sqrt{-1}}{2}\right)^k \theta^{[\mu\nu]_k}\partial_{[\mu]_k}T^{j_1,j_2,\dots,j_n}_{i_1,i_2,\dots,i_n}\partial_{[\nu]_k}\mathbf{a}^{j_{n+1}}_{i_{n+1}} \end{aligned} \tag{2.40}$$

which is a sum of quantities belonging to the direct product of $\mathcal{R}^{j_1+j_2+\cdots+j_n}_{max(i_1,i_2,\ldots,i_n)}$ and $\mathcal{R}^{j_{n+1}}_{i_{n+1}}$ by Lemma 16 and Theorem 2. The remaining equality is obtained by induction on the number n of indeterminates. □

Remark 11. This definition of the $\star$-product as inverse limit allows to highlight generalized n-ary laws [13] that one could further investigate. Consider the generalized product combining n elements defined as a series element of $\mathcal{R}_{n,\cdot}$ such that

$$(a_1, a_2, \ldots, a_n)_\star = \sum_{p=0}^{k} \sum_{p_1+p_2+\cdots+p_n=p} \Theta^{[\mu_1\mu_2\ldots\mu_n]_p} \partial_{[\mu_1]_{p_1}} \mathbf{a}_1 \partial_{[\mu_2]_{p_2}} \mathbf{a}_2 \ldots \partial_{[\mu_n]_{p_n}} \mathbf{a}_n, \tag{2.41}$$

where $\Theta^{[\mu_1\mu_2\ldots\mu_n]_p}$ is some tensor which has to be chosen such that appropriate properties of the generalized law, namely n-associativity or n-commutativity is satisfied. The Moyal $\star$-product is a binary law and clearly defines some restriction of this kind of product.

2.3.2. Some relevant computations

We are interested in such properties of the $\star$-product of polynomials, whatever the powers, like those which are relevant in NCFT. Concrete examples have been worked out in Section 3. The $\star$-product of any two dofs $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \star \mathbf{b} = \mathbf{ab} + \left(\frac{\sqrt{-1}}{2}\right) \theta^{\mu\nu} \partial_\mu \mathbf{a} \partial_\nu \mathbf{b} + \cdots + \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!} \theta^{[\mu\nu]_n} \partial_{[\mu]_n} \mathbf{a} \partial_{[\nu]_n} \mathbf{b} + \cdots$$

Thus, each term of this series belongs to different ideals since each partial derivative is considered as an irreducible generator of the total ring

$$\mathbf{a} \star \mathbf{b} \in < \mathbf{ab} > \oplus < \partial_\mu \mathbf{a} \partial_\nu \mathbf{b} > \oplus \cdots \oplus < \partial_{[\mu]_n} \mathbf{a} \partial_{[\nu]_n} \mathbf{b} > \oplus \cdots$$

Hence, the projection $i_{(.)}$ onto a fixed ideal generated by either $\mathbf{a}$ or $\partial \mathbf{a}$ is well defined. The following proposition is straightforward.

Proposition 20. *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \partial\mathbf{a}, \partial\mathbf{b}, \partial\mathbf{c}$ *be independent irreducible indeterminates chosen among the generators of the commutative ring of series* $\mathcal{R}$. *Then,*

(i) *the following basic relations are valid:*

$$\mathbf{a} \star \mathbf{b} =_{\mathbf{a}} \mathbf{ab}, \qquad \mathbf{b} \star \mathbf{a} =_{\mathbf{a}} \mathbf{ab}, \tag{2.42}$$

$$\forall P(\check{\mathbf{a}}, \mathbf{b}, \mathbf{c}, \ldots) \in \mathcal{R}, \qquad \mathbf{a} \star P(\check{\mathbf{a}}, \mathbf{b}, \mathbf{c}, \ldots) =_{\mathbf{a}} \mathbf{a} P(\check{\mathbf{a}}, \mathbf{b}, \mathbf{c}, \ldots), \tag{2.43}$$

where $P(\breve{\mathbf{a}}, \mathbf{b}, \mathbf{c}, \ldots)$ *is some series which does not contain the dofs* $\mathbf{a}$. *Furthermore,*

$$\mathbf{a} \star \mathbf{b} =_{\partial_{[\mu]_n}\mathbf{b}} \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{n!}\,\theta^{[\rho\mu]_n}\partial_{[\rho]_n}\mathbf{a}\partial_{[\mu]_n}\mathbf{b}. \tag{2.44}$$

(ii) *Finally, we have the following relation with polynomial functions: For all* $F, G \in \mathcal{R}$, *if* $i_{\mathbf{b}}F = 0$ *then* $F \star G =_{\mathbf{b}} F \star i_{\mathbf{b}}G$.

Proof. These relations are inferred from the expansion of the $\star$-product while the last property can be deduced as follows

$$\begin{array}{rcl} G & = & i_{\mathbf{b}}(G) + G_0 \ \text{ with } \ i_{\mathbf{b}}G_0 = 0 \\ F \star G & = & F \star i_{\mathbf{b}}(G) + F \star G_0 \\ F \star G & =_{\mathbf{b}} & i_{\mathbf{b}}(F \star i_{\mathbf{b}}(G) + F \star G_0) =_{\mathbf{b}} (F \star i_{\mathbf{b}}(G)) \end{array}$$

which is the expected relation. □

Remark 12. When $\theta \to 0$, then $\star$-product becomes the usual multiplication and the projection $i_{(.)}$ becomes a projection onto the dof ideals, and any projection onto ideal generated by derivatives $\partial\mathbf{a}$ vanishes as expected.

§3. Applications

In this section, we give some relevant applications of the previous study in NCFT. First, we need to define the classical extension of Lagrange formulation of field theory for higher order dynamical systems [8, 7]. Such higher order systems actually lie within the framework of NCFT, dealing with infinite order of derivative. In particular, we address the issue of defining Euler-Lagrange equations for NC and nonlocal field systems. Second, we discuss some features of the Ostrogradski formulation of Hamiltonian dynamics in this context, by computing the infinite discrete sequence of conjugate momenta recovering, by the way and in a new manner, that the field phase space in NCFT is infinite dimensional [5].

A higher order dynamics can be understood, in some simple situation, by a given classical field system characterized by a dynamical scalar field ϕ and a Lagrange function $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi, \ldots, \partial_{[\mu]_k}\phi)$, k being the fixed maximal order of derivation as occurring in the Lagrangian.

The general formula for the Euler-Lagrange equation of motion is obtained by taking successively integration by part from the action variation. One gets

$$\sum_{n=0}^{k}(-1)^n\,\partial_{[\mu]_n}\,\frac{\partial\mathcal{L}}{\partial\partial_{[\mu]_n}\phi} = 0, \tag{3.1}$$

where it is assumed that the sum runs over all different multi-indices $[\mu]_n \neq [\mu']_n$ making hence $\partial_{[\mu]_n}\phi \neq \partial_{[\mu']_n}\phi$.

In NCFT, the maximal order of differentiation is infinite. The sum (3.2) now becomes infinite which is understood as an algebraic series of Section 2, namely

$$\sum_{n=0}^{\infty}(-1)^n\, \partial_{[\mu]_n}\, \frac{\partial \mathcal{L}}{\partial \partial_{[\mu]_n}\phi} = 0, \tag{3.2}$$

the same assumption on $[\mu]_n$ as the above mentioned holds. Let us precise that, here, the algebra of series is built over a singleton set $D_1^0 = \{\phi\}$.

In a more general statement, we introduce the following notation

$$\begin{aligned}[\lambda]_k &= (\lambda_1, \lambda_2, \dots, \lambda_k) \\ &\equiv (\underbrace{\chi_1, \chi_1, \dots, \chi_1}_{q_1-times}, \underbrace{\chi_2, \chi_2, \dots, \chi_2}_{q_2-times}, \dots, \underbrace{\chi_p, \chi_p, \dots, \chi_p}_{q_p-times})\end{aligned} \tag{3.3}$$

where the equivalence is valid under some permutation of the indices of $[\lambda]_k$. Note that $p \leq k$ and $\sum_{i=1}^{p} q_i = k$. The number of nontrivial permutations of $[\lambda]_k$ is well-known to be the hypergeometric number

$$H([\lambda]_k) := \frac{k!}{\prod_{i=1}^{p}(q_i)!}, \qquad H([\lambda]_0) := 1.$$

Claim 1. *Let $\mathcal{L}_\star = \mathcal{L}_\star(\phi, \partial_\mu\phi, \dots, \partial_{[\mu]_k}\phi, \dots)$ be a NC Lagrange function in a NC scalar field theory. The NC analogue of Euler-Lagrange equation of motion of ϕ is expressed by*

$$\sum_{k=0}^{\infty}(-1)^k\, H([\lambda]_k)^{-1}\, \partial_{[\lambda]_k}\, \frac{\partial \mathcal{L}}{\partial \partial_{[\lambda]_k}\phi} = 0. \tag{3.4}$$

Let us test the validity of such a statement in the particular case of the free NC scalar field theory described by the NC Lagrangian in a D dimensional Minkowski spacetime

$$\mathcal{L}_\star = \frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi - \frac{m^2}{2}\phi \star \phi. \tag{3.5}$$

The following statement holds

Proposition 21. *The Euler-Lagrange equation of motion of the system described by the Lagrangian* (3.5) *is the ordinary Klein-Gordon equation*

$$\sum_{k=0}^{\infty}(-1)^k\, H([\lambda]_k)^{-1}\, \partial_{[\lambda]_k}\, \frac{\partial \mathcal{L}}{\partial \partial_{[\lambda]_k}\phi} = 0 = (\partial_\mu\partial^\mu + m^2)\phi. \tag{3.6}$$

Proof. Given a multi-index $[\lambda]_k$, we have the equalities

$$(3.7)\quad \frac{\partial(\phi\star\phi)}{\partial\partial_{[\lambda]_k}\phi} = \left(\tfrac{\sqrt{-1}}{2}\right)^k \frac{(1+(-1)^k)}{\prod_{i=1}^p (q_i)!}\,\theta^{[\lambda\sigma]_k}\,\partial_{[\sigma]_k}\phi,$$

$$(3.8)\quad \frac{\partial(\partial_\mu\phi\star\partial^\mu\phi)}{\partial\partial_{[\lambda]_k}\phi} = \left(\tfrac{\sqrt{-1}}{2}\right)^{k-1}(1+(-1)^{k-1})\sum_{i=1}^p \frac{\theta^{[\lambda\sigma]^i_{k-1}}}{\prod_{j=1}^p (q_j-\delta_{ji})!}\;\partial^{\chi_i}{}_{[\sigma]_{k-1}}\phi,$$

$$\theta^{[\lambda\sigma]^i_{k-1}} := \underbrace{\theta^{\chi_1\sigma_1}\theta^{\chi_1\sigma_2}\dots\theta^{\chi_1\sigma_{q_1}}}_{q_1-times}\underbrace{\theta^{\chi_2\sigma_{q_1+1}}\theta^{\chi_2\sigma_{q_1+2}}\dots\theta^{\chi_2\sigma_{q_1+q_2}}}_{q_2-times}\dots$$
$$\underbrace{\theta^{\chi_i\sigma_{q_1+q_2+\dots+q_{i-1}+1}}\theta^{\chi_i\sigma_\cdot}\dots\theta^{\chi_i\sigma_\cdot}}_{(q_i-1)-times}\dots\underbrace{\theta^{\chi_p\sigma_\cdot}\theta^{\chi_p\sigma_\cdot}\dots\theta^{\chi_p\sigma_\cdot}}_{q_p-times}.$$

In order to prove equations (3.7) and (3.8), we consider (2.44)

$$(3.9)\qquad \phi\star\phi =_{\partial_{[\lambda]_k}\phi} \kappa_k\,\theta^{[\rho\sigma]_k}\,\partial_{[\rho]_k}\phi\,\partial_{[\sigma]_k}\phi,\quad \kappa_k := \left(\frac{\sqrt{-1}}{2}\right)^k\frac{1}{k!}$$

$$(3.10)\qquad \partial_\mu\phi\star\partial^\mu\phi =_{\partial_{[\lambda]_k}\phi} \kappa_{k-1}\,\theta^{[\rho\sigma]_{k-1}}\,\partial_{[\rho]_{k-1}\mu}\phi\,\partial^\mu{}_{[\sigma]_{k-1}}\phi.$$

We can now deduce

$$(3.11)\qquad \frac{\partial(\phi\star\phi)}{\partial\partial_{[\lambda]_k}\phi} = \kappa_k\,\theta^{[\rho\sigma]_k}\left(\frac{\partial\partial_{[\rho]_k}\phi}{\partial\partial_{[\lambda]_k}\phi}\,\partial_{[\sigma]_k}\phi + \partial_{[\rho]_k}\phi\frac{\partial\partial_{[\sigma]_k}\phi}{\partial\partial_{[\lambda]_k}\phi}\right),$$

from which, up to the combinatorial factor $k!/(\prod_{i=1}^p(q_i)!) = H([\lambda]_k)$ defining the number of permutations of $[\lambda]_k$ which can be reproduced in the sum $\theta^{[\rho\sigma]_k}\partial_{[\rho]_k}\phi$ and $\theta^{[\rho\sigma]_k}\partial_{[\sigma]_k}\phi$ (keeping in mind that $[\lambda]_k$ may have repeated indices), the relation (3.7) is obtained. In a similar manner, (3.8) can be obtained from (3.10) noting however that one first has to choose a $\lambda = \chi_i = \mu$ in the sum $\theta^{[\rho\sigma]_{k-1}}\partial_{[\rho]_k\mu}\phi$ before computing the overall combinatorial factor.

We show now the identity (3.6). Let us set

$$\sum_{k=0}^\infty(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}}{\partial\partial_{[\lambda]_k}\phi} = \frac{1}{2}\sum_{k=1}^\infty(-1)^k H([\lambda]_k)^{-1}$$
$$\times\partial_{[\lambda]_k}\left[\left(\frac{\sqrt{-1}}{2}\right)^{k-1}(1+(-1)^{k-1})\sum_{i=1}^p\frac{\theta^{[\lambda\sigma]^i_{k-1}}}{\prod_{j=1}^p(q_j-\delta_{ji})!}\;\partial^{\chi_i}{}_{[\sigma]_{k-1}}\phi\right]$$
$$-\frac{m^2}{2}\sum_{k=0}^\infty(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\left[\left(\frac{\sqrt{-1}}{2}\right)^k\frac{(1+(-1)^k)}{\prod_{i=1}^p(q_i)!}\,\theta^{[\lambda\sigma]_k}\,\partial_{[\sigma]_k}\phi\right].$$

The antisymmetry of the deformation tensor implies $\partial_{[\lambda]_k}\theta^{[\lambda\sigma]_k}\partial_{[\sigma]_k} = 0$, if $k\neq 0$. Furthermore, if $k=0$, the quantities q_i cancel. Besides, $\partial_{[\lambda]_k} = \partial_{\chi_i[\lambda]_{k-1}}$ so that $\partial_{\chi_i[\lambda]_{k-1}}\theta^{[\lambda\sigma]_{k-1}}\partial^{\chi_i}{}_{[\sigma]_{k-1}} = 0$, if $k\neq 1$. Finally, we get

$$(3.12)\qquad \sum_{k=0}^\infty(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}}{\partial\partial_{[\lambda]_k}\phi} = -m^2\phi + (-1)^{k=1}\,\partial_{\chi_i}\,\partial^{\chi_i}\phi$$

which achieves the proof. □

Remark 13. The final Klein-Gordon equation was of course expected since we well know that the free NCFT perfectly coincides with its commutative counterpart from the action formulation. However here, and for the first time to our best knowledge of the literature, we can use only the Lagrange formulation to describe the dynamics of the free NC system. Lagrange formulation is indeed more difficult to handle but this actually can be considered as the first steps towards a well-defined generalized Hamiltonian formulation according to Ostrogradski [21].

Now let us consider a new system with an interaction leading to a nontrivial NCFT, in the sense that the dynamics of this system cannot be reduced to its commutative analogue.

Consider a NC $U(1)$ induced gauge theory described by the following Lagrangian density in NC Minkowski spacetime in $1+1$ dimensions, i.e. $\mathbb{R}^{1,1}$:

$$\mathcal{L}_D = \sqrt{-1}\bar{\psi} \star \left(\gamma^\mu \partial_\mu + \sqrt{-1}e\gamma^\mu A_\mu \star\right) \psi. \tag{3.13}$$

where the real valued gauge potential is A_μ, ψ and $\bar{\psi}$ are the two spinors describing fermionic particles, e is the gauge constant coupling which is nothing but the absolute value of the charge of the fermionic particle (electron or positron), γ^μ are the Dirac matrices generated by the usual Pauli matrices $\sigma_{i=1,2,3}$, obeying the Clifford anticommuting algebra:

$$\{\,\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{3.14}$$

$$\gamma^0 = \sigma^1,\;\; \gamma^1 = \sqrt{-1}\sigma^2,\;\; \gamma^0\gamma^1 = \gamma_5 = -\sigma^3,\;\; \bar{\psi} = \psi^\dagger\gamma^0. \tag{3.15}$$

Units such that $\hbar = 1 = c$ are considered. We read off the dofs from these actions: A_μ, ψ and $\bar{\psi}$. Thenceforth, the algebra of series is built over the set $\mathcal{D}_3^0 = \{A_\mu, \psi, \bar{\psi}\}$.

Explicitly, the Lagrangian (3.13) is written as:

$$\mathcal{L}_D = \sqrt{-1}\psi^\dagger \star \left(\partial_0 + \sqrt{-1}eA_0\star\right)\psi + \sqrt{-1}\psi^\dagger\gamma_5 \star \left(\partial_1 + \sqrt{-1}eA_1\star\right)\psi.$$

From the variation calculus of the actions $S = \int d^2x\, \mathcal{L}_D$, we can deduce the following statement.

Proposition 22. *The equations of motion of ψ and $\bar{\psi}$ with respect to the NC Lagrangian $\mathcal{L}_D$ defined by* (3.13) *are*

$$\frac{\delta S}{\delta \psi} = 0 \quad \Leftrightarrow \quad \partial_\mu\bar{\psi}\gamma^\mu - \sqrt{-1}e\bar{\psi} \star A_\mu\gamma^\mu = 0, \tag{3.16}$$

$$\frac{\delta S}{\delta \bar{\psi}} = 0 \quad \Leftrightarrow \quad \gamma^\mu\partial_\mu\psi + \sqrt{-1}e\gamma^\mu A_\mu \star \psi = 0. \tag{3.17}$$

The equation of motion of A_μ is the Lagrangian constraint

$$\frac{\delta S}{\delta A_\mu} = 0 \quad \Leftrightarrow \quad -e\,\gamma^\mu \psi \star \bar{\psi} = 0. \tag{3.18}$$

Now, let us prove that according to our previous formulation, we can define rigorous analogues of Euler-Lagrange equations for spin fields from the Lagrangian.

Proposition 23. *The NC Euler-Lagrange equations of ψ, $\bar{\psi}$ and A_μ are*

$$\sum_{k=0}^{\infty} (-1)^k\, H([\lambda]_k)^{-1}\, \partial_{[\lambda]_k} \frac{\partial \mathcal{L}_D}{\partial \partial_{[\lambda]_k} \psi} = 0 \;\Leftrightarrow\; \partial_\mu \bar{\psi} \gamma^\mu - \sqrt{-1} e \bar{\psi} \star A_\mu \gamma^\mu = 0, \tag{3.19}$$

$$\sum_{k=0}^{\infty} (-1)^k\, H([\lambda]_k)^{-1}\, \partial_{[\lambda]_k} \frac{\partial \mathcal{L}_D}{\partial \partial_{[\lambda]_k} \bar{\psi}} = 0 \;\Leftrightarrow\; \gamma^\mu \partial_\mu \psi + \sqrt{-1} e \gamma^\mu A_\mu \star \psi = 0, \tag{3.20}$$

$$\sum_{k=0}^{\infty} (-1)^k\, H([\lambda]_k)^{-1}\, \partial_{[\lambda]_k} \frac{\partial \mathcal{L}_D}{\partial \partial_{[\lambda]_k} A_\mu} = 0 \;\Leftrightarrow\; -e\,\gamma^\mu \psi \star \bar{\psi} = 0. \tag{3.21}$$

The proof of Proposition 23 requires a preliminar lemma applying the projective limit formulation in the current situation.

Lemma 24. *For any fields Φ and Υ with the Grassmann parity $\epsilon(\Upsilon) = \epsilon = 0, 1$, for any integer $k = 0, 1, \ldots,$ we obtain*

$$\frac{\partial (\Phi \star \Upsilon)}{\partial \partial_{[\lambda]_k} \Upsilon} = (-1)^\epsilon\, \kappa_k\, H([\lambda]_k)\, \theta^{[\nu\lambda]_k} \partial_{[\nu]_k} \Phi, \qquad \kappa_k = \left(\frac{\sqrt{-1}}{2} \right)^k \frac{1}{k!} \tag{3.22}$$

$$\frac{\partial (\Upsilon \star \Phi)}{\partial \partial_{[\lambda]_k} \Upsilon} = \kappa_k\, H([\lambda]_k)\, \theta^{[\lambda\nu]_k} \partial_{[\nu]_k} \Phi. \tag{3.23}$$

Given a field A

$$\frac{\partial (\Phi \star A \star \Upsilon)}{\partial \partial_{[\lambda]_k} A} = H([\lambda]_k)\, \theta^{[\rho\lambda]_k} \sum_{m=0}^{k} \kappa_m \frac{(-1)^{k-m}}{(k-m)!} \partial_{[\rho]_m} (\Phi \star \partial_{[\bar{\rho}]_{k-m}} \Upsilon) \tag{3.24}$$

where $[\bar{\rho}]_{k-m}$ denotes the complementary index of $[\rho]_m$ in $[\rho]_k$, namely $[\bar{\rho}]_{k-m} = (\rho_{m+1}, \rho_{m+2}, \ldots, \rho_k)$.

Proof. From (2.44),

$$\Phi \star \Upsilon =_{\partial_{[\mu]_k} \Upsilon} \;\kappa_k \theta^{[\rho\mu]_k} \partial_{[\rho]_k} \Phi\, \partial_{[\mu]_k} \Upsilon. \tag{3.25}$$

The equations (3.22) and (3.23) are obtained from (3.25), keeping in mind that derivatives are left derivatives with respect to the Grassmann parity of fields. For instance,

$$\frac{\partial}{\partial \partial_\mu \Upsilon} (\Phi \star \Upsilon) = (-1)^\epsilon \frac{\sqrt{-1}}{2} \theta^{\rho\mu} \partial_\rho \Phi.$$

The remaining factor is obtained by similar consideration as previously discussed.

The proof of (3.24) may start with the momentum Fourier space. We have, with normalized measures

$$
\begin{aligned}
(f\star g)(x) &= \int d^Nk\, d^Np\, \tilde f(k)\, \tilde g(p)\, e^{-\sqrt{-1}k\theta p} e^{\sqrt{-1}px}\star e^{\sqrt{-1}kx}\\
&= \int d^Nk\, d^Np\, \tilde f(k)\, \tilde g(p) \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!}\, \theta^{[\mu\nu]_n} \partial_{[\nu]_n} e^{\sqrt{-1}px}\star \partial_{[\mu]_n} e^{\sqrt{-1}kx}\\
&= \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!}\, \theta^{[\mu\nu]_n} \partial_{[\nu]_n} g(x)\star \partial_{[\mu]_n} f(x). \qquad (3.26)
\end{aligned}
$$

Then, one readily obtains

$$
\begin{aligned}
\Phi\star A\star\Upsilon &= \Phi\star \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!}\, \theta^{[\mu\nu]_n} \partial_{[\nu]_n}\Upsilon\star\partial_{[\mu]_n} A\\
&= \sum_{m=0}^{\infty} \kappa_m\, \theta^{[\rho\sigma]_m} \sum_{n=0}^{\infty} \frac{(\sqrt{-1})^n}{n!}\, \theta^{[\mu\nu]_n}\, \partial_{[\rho]_m}(\Phi\star\partial_{[\nu]_n}\Upsilon)\partial_{[\sigma]_m\,[\mu]_n} A. \qquad (3.27)
\end{aligned}
$$

The projection onto the term of k^{th} order derivative of A can be written

$$
\begin{aligned}
&\Phi\star A\star\Upsilon =_{\partial_{[\lambda]_k}A} \qquad (3.28)\\
&\sum_{m=0}^{k} \kappa_m\, \theta^{[\rho\sigma]_m} \frac{(\sqrt{-1})^{k-m}}{(k-m)!} \theta^{[\mu\nu]_{k-m}}\, \partial_{[\rho]_m}(\Phi\star\partial_{[\nu]_{k-m}}\Upsilon)\partial_{[\sigma]_m\,[\mu]_{k-m}} A.
\end{aligned}
$$

Renaming the independent variables such that $\mu_1,\dots,\mu_{k-m}$ as $\sigma_{m+1},\dots,\sigma_k$ and $\nu_1,\dots,\nu_{k-m}$ as $\rho_{m+1},\dots,\rho_k$, and using the antisymmetry of $\theta^{\rho\sigma}$, one proves that

$$
\begin{aligned}
&\theta^{[\rho\sigma]_m}\theta^{[\mu\nu]_{k-m}}\partial_{[\rho]_m}(\Phi\star\partial_{[\nu]_{k-m}}\Upsilon)\partial_{[\sigma]_m\,[\mu]_{k-m}} A\\
&\quad = (-1)^{k-m}\theta^{[\rho\sigma]_k}\partial_{[\rho]_m}(\Phi\star\partial_{[\bar\rho]_{k-m}}\Upsilon)\partial_{[\sigma]_k} A.
\end{aligned}
$$

Combining the last relation and (3.28), it can be deduced the identity (3.24) after taking the derivative in $\partial_{[\lambda]_k}A$ onto (3.28). This ends the proof of the lemma. □

Proof of Propostion 23. The proof is immediate from the addition of different quantities already computed in the Lemma 24. Using the relation (3.23) and considering $\Upsilon=\bar\psi$ and Φ to be $\partial_\mu\psi$ or $\gamma^\mu A_\mu\star\psi$, it comes

$$
\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k\, H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}_D}{\partial\partial_{[\lambda]_k}\bar\psi} \qquad (3.29)\\
&=\sum_{k=0}^{\infty}(-1)^k\,\partial_{[\lambda]_k}\left(\sqrt{-1}\kappa_k\,\theta^{[\lambda\sigma]_k}\gamma^\mu\partial_{[\sigma]_k\mu}\psi - e\kappa_k\,\theta^{[\lambda\sigma]_k}\partial_{[\sigma]_k}(\gamma^\mu A_\mu\star\psi)\right).
\end{aligned}
$$

The antisymmetry of $\theta^{[\lambda\sigma]_k}$ cancels any term such that $k \neq 0$. It clearly remains the usual covariant derivative equation of motion of ψ (3.19). Similar computations lead to the equation of motion of $\bar{\psi}$ (3.20). Let us now derive the constraint (3.21). We have from (3.24)

$$\begin{aligned}(3.30)\quad &\sum_{k=0}^{\infty}(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}_D}{\partial\partial_{[\lambda]_k}A_\mu}\\ &= -e\sum_{k=0}^{\infty}(-1)^k\,\theta^{[\rho\lambda]_k}\sum_{m=0}^{k}\frac{(-\sqrt{-1})^{k-m}\kappa_m}{(k-m)!}\partial_{[\lambda]_k[\rho]_m}(\bar{\psi}\star\gamma^\mu\partial_{[\bar{\rho}]_{k-m}}\psi).\end{aligned}$$

Due still to the antisymmetry of $\theta^{\rho\lambda}$, it turns out that if $m \neq 0$, the sum over m vanishes. Indeed, assume $m \geq 1$, one sees that for any function B,

$$\theta^{[\rho\lambda]_k}\partial_{[\lambda]_k[\rho]_m}B = \theta^{\rho_1\lambda_1}\cdots\theta^{\rho_m\lambda_m}\cdots\theta^{\rho_k\lambda_k}\partial_{\rho_1\lambda_1\ldots\rho_m\lambda_m}B = 0.$$

Therefore, we get from (3.30)

$$\begin{aligned}(3.31)\quad &\sum_{k=0}^{\infty}(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}_D}{\partial\partial_{[\lambda]_k}A_\mu}\\ &= -e\sum_{k=0}^{\infty}\frac{(-1)^{2k}(\sqrt{-1})^k}{k!}\theta^{[\rho\lambda]_k}\partial_{[\lambda]_k}(\bar{\psi}\star\gamma^\mu\partial_{[\rho]_k}\psi)\\ &= \sum_{k=0}^{\infty}\frac{(\sqrt{-1})^k}{k!}\,\theta^{[\rho\lambda]_k}\partial_{[\lambda]_k}(\bar{\psi}\star\gamma^\mu\partial_{[\rho]_k}\psi).\end{aligned}$$

Furthermore,

$$\theta^{[\rho\lambda]_k}\partial_{[\lambda]_k}(\bar{\psi}\star\gamma^\mu\partial_{[\rho]_k}\psi) = \theta^{[\rho\lambda]_k}\sum_{l=0}^{k}C_k^l\,\partial_{[\lambda]_l}\bar{\psi}\star\gamma^\mu\partial_{[\lambda]_{k-l}[\rho]_k}\psi,$$

with $C_k^l = k!/(l!(k-l)!)$, the last sum vanishes unless $l = k$. Finally, we obtain

$$\begin{aligned}(3.32)\quad &\sum_{k=0}^{\infty}(-1)^k H([\lambda]_k)^{-1}\,\partial_{[\lambda]_k}\frac{\partial\mathcal{L}_D}{\partial\partial_{[\lambda]_k}A_\mu}\\ &= -e\sum_{k=0}^{\infty}\frac{(\sqrt{-1})^k}{k!}\theta^{[\rho\lambda]_k}\partial_{[\lambda]_k}\bar{\psi}\star\gamma^\mu\partial_{[\rho]_k}\psi = -e\gamma^\mu\psi\star\bar{\psi},\end{aligned}$$

where the last identity stems from (3.26). □

Remark 14. In NCFTs, most of computations are usually performed under integral, namely within an action formulation, since the main suitable properties are satisfied only under the integral (see Section 2). Actually, one should, in more general framework of a NC module field theory over Moyal algebra seen as a Hilbert space, work with the generalized trace class integral action which involves the same relevant properties [5]. Hence, the integral action formulation seems to be a way of predilection of the study of NCFTs. However, we have now shown that the Lagrangian algebraic formulation could also have a well defined sense with a connection to the action calculus. In particular, the equation of motion extracted from action variational principle could correspond to the NC Euler-Lagrange equations. We point out that a clear Lagrangian and Hamiltonian formulations of the dynamics are still lacking in NCFTs [6]. The above development shows that the nonlocal character of the NC theory, manifested by algebraic formal series, can be usefully exploited to extend the classical Euler-Lagrange equation of motion to the framework of the NC theory.

As particular application of Lemma 24, let us study the formal NC momentums obtained by derivation of inverse limit series. We have to define first the meaning of NC momentums in the sense of nonlocal theories, i.e. NC analogues of Ostrogradski formulas. The following statement can be found in [8].

Proposition 25. *Let* $L = L(x_n, \dot{x}_n, \ddot{x}_n, \dots, x_n^{(m_n)})$ *be a Lagrange function describing a system with* $x_n(t)$ *as dofs,* $n = 1, 2, \dots$ *and* m_n *being the maximal order of all time derivatives of the coordinate* x_n *appearing in* L.

The quantities p_{n,α_n} $(\alpha_n = 0, 1, \dots, m_n - 1)$ *defined by the recurrence relations*

$$p_{n,m_n-1} = \frac{\partial L}{\partial x_n^{(m_n)}}, \tag{3.33}$$

$$p_{n,i_n-1} = \frac{\partial L}{\partial x_n^{(i_n)}} - \frac{d}{dt} p_{n,i_n}, \qquad i_n = 1, 2, \dots, m_n - 1 \tag{3.34}$$

allow to redefine the Euler-Lagrange equations of motion as

$$\sum_{k_n=0}^{m_n} (-1)^{k_n} \left(\frac{d}{dt}\right)^{k_n} \frac{\partial L}{\partial x_n^{(k_n)}} = \frac{\partial L}{\partial x_n} - \frac{d}{dt} p_{n,0} = 0, \ \ n = 1, 2, \dots. \tag{3.35}$$

Proof. See [8]. □

The NC counterpart in field theory of Proposition 25 is now investigated. From the NC Euler-Lagrange equation (3.4)

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n H([\lambda]_n)^{-1} \, \partial_{[\lambda]_n} \frac{\partial \mathcal{L}}{\partial \partial_{[\lambda]_n} \phi} \\ &\quad = \frac{\partial \mathcal{L}}{\partial \phi} + (-1)\partial_{\lambda_1} \sum_{n=1}^{\infty} (-1)^{n-1} H([\lambda]_n)^{-1} \, \partial_{[\bar{\lambda}]_{n-1}} \frac{\partial \mathcal{L}}{\partial \partial_{[\bar{\lambda}]_{n-1}\lambda_1} \phi}, \end{aligned} \tag{3.36}$$

we can define the initial value

$$\pi_{0,\lambda_1} = \frac{\partial \mathcal{L}}{\partial\partial_{\lambda_1}\phi} + \sum_{n=2}^{\infty}(-1)^{n-1} H([\lambda]_n)^{-1}\, \partial_{[\bar\lambda]_{n-1}} \frac{\partial \mathcal{L}}{\partial\partial_{[\bar\lambda]_{n-1}\lambda_1}\phi} \tag{3.37}$$

and the general term of the Ostrogradski formulas, $k \geq 1$

$$\begin{aligned} \pi_{k-1,[\lambda]_k} &= H([\lambda]_k)^{-1} \frac{\partial \mathcal{L}}{\partial\partial_{[\lambda]_k}\phi} \\ &\quad + \sum_{n=k+1}^{\infty} (-1)^{n-k} H([\lambda]_n)^{-1}\, \partial_{[\bar\lambda]_{n-k}} \frac{\partial \mathcal{L}}{\partial\partial_{[\bar\lambda]_{n-k}[\lambda]_k}\phi}. \end{aligned} \tag{3.38}$$

The NC Euler-Lagrange equation can now be written in a simple way

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\lambda \pi_{0,\lambda} = 0. \tag{3.39}$$

The following proposition is henceforth proved.

Proposition 26. *Let $\mathcal{L}_\star = \mathcal{L}_\star(\phi, \partial_\mu\phi, \dots, \partial_{[\mu]_k}\phi, \dots)$ be a NC Lagrange function describing a system with ϕ and $\partial\phi$ as dofs.*

The quantities $\pi_{k-1,[\lambda]_k}$ $(k = 1, 2, \dots)$ defined by the general term

$$\begin{aligned} \pi_{k-1,[\lambda]_k} &= H([\lambda]_k)^{-1} \frac{\partial \mathcal{L}}{\partial\partial_{[\lambda]_k}\phi} \\ &\quad + \sum_{n=k+1}^{\infty} (-1)^{n-k} H([\lambda]_n)^{-1}\, \partial_{[\bar\lambda]_{n-k}} \frac{\partial \mathcal{L}}{\partial\partial_{[\bar\lambda]_{n-k}[\lambda]_k}\phi}. \end{aligned} \tag{3.40}$$

allow to redefine the Euler-Lagrange equations of motion as

$$\sum_{n=0}^{\infty}(-1)^n H([\lambda]_n)^{-1}\, \partial_{[\lambda]_n} \frac{\partial \mathcal{L}}{\partial\partial_{[\lambda]_n}\phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\lambda\, \pi_{0,\lambda} = 0. \tag{3.41}$$

Remark 15. As a first remark, the classical Ostrogradski quantities p_{n,i_n} (3.34) are defined recursively with a certain initial value p_{n,m_n} (3.33) defined by the fixed maximal order of time derivative of the dof x_n. According to the above construction in NCFT, the definition of these quantities actually uses a different route defining at first the series of the fundamental quantity $\pi_{0,\lambda}$ from which the general term can be deduced. Secondly, the quantity $\pi_{0,\lambda}$, as one easily observes, converges to the usual momentum conjugated to the field ϕ (leading term in (3.37)) in the limit $\theta \to 0$ and for first order derivative theories. The classical commutative Euler-Lagrange equation is well recovered by the identity (3.41). In addition, the Ostrogradski formalism is established provided that the derivatives $x_n^{(k_n)}$ of each dof x_n up to the order $m_n - 1$ are considered as independent. Here, the construction meets this idea by seeing the fields and their derivatives as totally free indeterminates. This is a new reason why the Ostrogradski formalism makes sense in the projective limit ring. Finally, the quantities $\pi_{k,[\lambda]_k}$, $k = 0, 1, \ldots$, explicitly prove the existence of an infinite dimensional phase space. Indeed, they could be used to define the canonical conjugate momentums in an extended Ostrogradski Hamiltonian formulation which has to be properly defined in a forthcoming work.

§4. Concluding remarks

In this paper, we have investigated in detail, from the fundamental point of view of the noncommutativity, new properties which are straightforwardly involved in the relevant computational aspects of the NCFT, built on the $\star$-product. In order to avoid cumbersome calculations arising from the nonlocality of the theory, source of difficulties, different approaches have been proposed such as the theories of lowest order terms of the Seiberg Witten Map [23] and "two-time" generalized Ostrogradski formalism [6]. Here we adopt a different approach where the Moyal algebra of functions is developed as an appropriate ring of formal series or ring of projective limits of polynomials in the indeterminate defined by the fields and their derivatives. Thus, a new definition of the $\star$-product has been provided, making functional derivations as algebraic derivations with respect to a given indeterminate field. The $\star$-product is regarded as an inverse limit operator. It appears then possible to set appropriate rules operating over nonlocal quantities in the NC spacetime, preserving all the NC properties of the $\star$-algebra at all levels of computation. Avoiding the action formulation, we have given concrete examples in any dimensions by making explicit directly by Lagrange formalism, the NC Euler-Lagrange equations of the NC free scalar field theory and a $U(1)$ induced gauge theory coupled to NC Dirac Lagrangian density. Applications of this study allow also to consider Ostrogradski formulas for higher order theories even if an infinite dimensional phase space is considered.

In the forthcoming work, we expect to give a new insight to the nonlocal theories induced by infinite series generated by the Moyal product in the framework of the ring of projective limits of vector fields with a thorough study of their convergence domain. A second issue which remains to be understood within the framework of this study concerns the Hamiltonian formulation of the dynamics in NCFTs.

Aknowledgments

The authors are thankful to the referee for his useful comments which allow them to improve the paper. This work is partially supported by the ICTP through the OEA-ICMPA-Prj-15. The ICMPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, Lett. Math. Phys. **1** (1977), 521-530; Ann. Phys. **111** (1978), 61-110 (Part I); Ann. Phys. **111** (1978), 111-151 (Part II).

[2] J. Ben Geloun and M. N. Hounkonnou, *Energy momentums tensors in noncommutative renormalizable scalar field theory*, Phys. Lett. B **653** (2007), 343-345.

[3] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994;

[4] A. Connes and M. Rieffel, *Yang-mills for noncommutative two-tori*, Contemp. Math. Oper. Algebra Math. Phys. **62** (1987), 237.

[5] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2001), 977–1029; [e-print `hep-th/0106048`];

[6] J. Gomis, K. Kamimura and J. LLosa, *Hamiltonian formalism for space-time noncommutative theories*, Phys. Rev. D **63** (2001), 045003; [e-print `hep-th/0006235`].

[7] J. Govaerts, *Hamiltonian quantisation and constrained dynamics*, Leuven Univ. Press, Louvain, 1991.

[8] J. Govaerts and M. S. Rashid, *The Hamiltonian formulation of higher order dynamical systems*, [e-print `hep-th/9403009`], *unpublished.*

[9] H. Grosse and R. Wulkenhaar, *Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base*, Comm. Math. Phys. **256** (2005), 305–374; [e-print `hep-th/0401128`].

[10] M. Hayakawa, *Perturbative analysis on infrared aspect of noncommutative QED on $\mathbb{R}^4$*, Phys Lett. B **478** (2000), 394; [e-print `hep-th/9912094`].

[11] L. Hörmander, *The Weyl calculus of pseudodifferential operators*, Commun. Pure Appl. Math. **32** (1979), 359-443.

[12] M. N. Hounkonnou, F. Massamba and J. Ben Geloun, *2-dimensional noncommutative field theory on the light cone*, JGSP **6** (2006), 38-46.

[13] R. Kerner, *Ternary algebraic structures and their applications in physics*, [e-print `math-ph/0011023`], *unpublished.*

[14] S. Lang, *Algebra*, 2^{nd}*Ed.*, Yale Univ., Addison Wesley Pub. C[ie] inc., New Haven, Connecticut, 1984.

[15] J. Madore, *An introduction to noncommutative geometry and its physical applications*, Cambridge Univ. Press., Cambridge, 1999.

[16] J. M. Maillard, *On the twisted convolution product and the Weyl transform of tempered distributions*, J. Geom. Phys. **3** (1986), 231-261.

[17] I. G. Mcdonald *Symmetric functions and orthogonal polynomials Dean Jacqueline B. Lewis Memorial Lectures*, University Lecture Series , Vol. 12, American Mathematical Society, 1998.

[18] A. Micu and M. Sheikh-Jabbari, *Noncommutative* ϕ^4 *at two loops*, JHEP**01**(2001)025; [e-print `hep-th/0008057`].

[19] J. E. Moyal, *Quantum mechanics as a statistical theory*, Proceedings of the Cambridge Philosophical Society **45** (1949), 99-124.

[20] P. J. Olver, *Applications of Lie groups to differential equations*, 2^{nd}*Ed.*, Springer-Verlag, New-York, 1993.

[21] M. Ostrogradski, Mem. Ac. St. Petersbourg **6** (1850), 385.

[22] V. Rivasseau, *Noncommutative renormalization*, Seminaire Poincaré X, Espace Quantique (2007) 15–95.

[23] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP**09**(1999)032; [e-print `hep-th/9908142`].

[24] H. S. Snyder, *Quantized space-time*, Phys. Rev. **71** (1947), 38.

[25] R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378** (2003), 207–299; [e-print `hep-th/0109162`].

[26] J.-C. Wallet, *Noncommuative induced gauge theories on Moyal spaces*, [e-print `arXiv:0708.2471 [hep-th]`] (2007).

Joseph Ben Geloun
International Chair of Mathematical Physics and Applications
ICMPA-UNESCO Chair
University of Abomey-Calavi
072 B.P.: 50 Cotonou, Rep. of Benin.
E-mail: `jobengeloun@gmail.com`

Mahouton Norbert Hounkonnou
International Chair of Mathematical Physics and Applications
ICMPA-UNESCO Chair
University of Abomey-Calavi
072 B.P.: 50 Cotonou, Rep. of Benin.
E-mail: `norbert_hounkonnou@cipma.net`, `hounkonnou@yahoo.fr`

F. Massamba
Department of Mathematics
University of Botswana
Private Bag 0022 Gaborone, Rep. of Botswana
and
International Chair of Mathematical Physics and Applications
ICMPA-UNESCO Chair
University of Abomey-Calavi
072 B.P.: 50 Cotonou, Rep. of Benin.
E-mail: massfort@yahoo.fr, massambaf@mopipi.ub.bw