

On certain rough maximal operators and singular integrals

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Abstract. We study the L^p mapping properties of a class of singular integral operators $T_{\mathcal{P},\Omega,h}$ related to polynomial mappings. We prove that this class of singular operators and some of its related maximal operators are bounded on L^p when the kernel function Ω in $L(\log L)^\alpha(\mathbf{S}^{n-1})$ for some $\alpha > 0$ and the radial function $h(|x|)$ satisfies a mild integrability condition.

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§1. Introduction

Throughout this paper, let \mathbf{R}^n , $n \geq 2$, be the n -dimensional Euclidean space and \mathbf{S}^{n-1} be the unit sphere in \mathbf{R}^n equipped with the normalized Lebesgue measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \setminus \{0\}$ and p' denote the exponent conjugate to p , that is $1/p + 1/p' = 1$.

Let $L(\log L)^\alpha(\mathbf{S}^{n-1})$ (for $\alpha > 0$) denote the space of all those measurable functions Ω on \mathbf{S}^{n-1} which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(y)| \log^\alpha(2 + |\Omega(y)|) d\sigma(y) < \infty.$$

The function spaces $l^\infty(L^\gamma)(\mathbf{R}_+)$ are defined as follows. If $1 \leq \gamma < \infty$,

$$l^\infty(L^\gamma)(\mathbf{R}_+) = \left\{ h : \|h\|_{l^\infty(L^\gamma)(\mathbf{R}_+)} = \sup_{j \in \mathbf{Z}} \left(\int_{2^{j-1}}^{2^j} |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} < C \right\}.$$

If $\gamma = \infty$, $l^\infty(L^\infty)(\mathbf{R}_+) = L^\infty(\mathbf{R}_+)$. Also, for $\gamma \geq 1$ define $\mathcal{H}_\gamma(\mathbf{R}_+)$ to be the set of all measurable functions h on \mathbf{R}_+ satisfying the condition $\|h\|_{L^\gamma(\mathbf{R}_+, dr/r)} = \left(\int_{\mathbf{R}_+} |h(r)|^\gamma dr/r \right)^{1/\gamma} \leq 1$ and define $\mathcal{H}_\infty(\mathbf{R}_+) = L^\infty(\mathbf{R}_+, dt/t)$.

It is easy to verify that the following inclusions hold and are proper:

$$l^\infty(L^\infty)(\mathbf{R}_+) \subset l^\infty(L^\gamma)(\mathbf{R}_+) \subset l^\infty(L^q)(\mathbf{R}_+) \subset l^\infty(L^1)(\mathbf{R}_+)$$

for $1 < q < \gamma < \infty$, and

$$\mathcal{H}_\infty(\mathbf{R}_+) = l^\infty(L^\infty)(\mathbf{R}_+), \text{ and } \mathcal{H}_\gamma(\mathbf{R}_+) \subset l^\infty(L^\gamma)(\mathbf{R}_+)$$

for $1 < \gamma < \infty$.

Let $\mathcal{P} = (P_1, \dots, P_m)$ be a mapping from \mathbf{R}^n into \mathbf{R}^m with P_j being polynomials on \mathbf{R}^n for $1 \leq j \leq m$. To \mathcal{P} we associate a singular integral operator $T_{\mathcal{P}, \Omega, h}$ and its related maximal operators $T_{\mathcal{P}, \Omega, h}^*$, $\mathcal{M}_{\mathcal{P}, \Omega, h}$ and $\mathfrak{S}_{\mathcal{P}, \Omega}^{(\gamma)}$ defined initially for C_0^∞ functions on \mathbf{R}^m by

$$(1.1) \quad T_{\mathcal{P}, \Omega, h} f(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - \mathcal{P}(u)) K_{\Omega, h}(u) du,$$

$$(1.2) \quad T_{\mathcal{P}, \Omega, h}^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|u| > \varepsilon} f(x - \mathcal{P}(u)) K_{\Omega, h}(u) du \right|,$$

$$(1.3) \quad \mathcal{M}_{\mathcal{P}, \Omega, h} f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|y| \leq r} |f(x - \mathcal{P}(u))| |\Omega(u')| |h(|u|)| du,$$

$$(1.4) \quad \mathfrak{S}_{\mathcal{P}, \Omega}^{(\gamma)} f(x) = \sup_{h \in \mathcal{H}_\gamma(\mathbf{R}_+)} |T_{\mathcal{P}, \Omega, h} f(x)|,$$

where h is a measurable function on \mathbf{R}_+ , $K_{\Omega, h}(\cdot)$ is a singular kernel of Calderón-Zygmund type given by $K_{\Omega, h}(y) = \Omega(y') |y|^{-n} h(|y|)$, and $\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies

$$(1.5) \quad \int_{\mathbf{S}^{n-1}} \Omega(u) d\sigma(u) = 0.$$

When $m = n$ and $\mathcal{P}(y) \equiv y$, we shall denote $T_{\mathcal{P}, \Omega, h}$ by $T_{\Omega, h}$, $T_{\mathcal{P}, \Omega, h}^*$ by $T_{\Omega, h}^*$ and $\mathfrak{S}_{\mathcal{P}, \Omega}^{(\gamma)}$ by $\mathfrak{S}_{\Omega}^{(\gamma)}$. Also, if $h \equiv 1$, denote $T_{\Omega, h}$ by T_Ω and $T_{\Omega, h}^*$ by T_Ω^* .

The operators $T_{\mathcal{P}, \Omega, h}$ by $T_{\mathcal{P}, \Omega, h}^*$ defined in (1.1)–(1.2) have their roots in the classical Calderón-Zygmund operators T_Ω and T_Ω^* . In their pioneering work on the theory of singular integrals ([9]), Calderón and Zygmund proved that the operators T_Ω and T_Ω^* are bounded on L^p for $1 < p < \infty$ if $\Omega \in L \log L(\mathbf{S}^{n-1})$. It turns out that their result is the best possible in the sense that the space $L \log L(\mathbf{S}^{n-1})$ cannot be replaced by any other Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with

a ϕ which is increasing and satisfies $\lim_{t \rightarrow \infty} \frac{\phi(t)}{t \log t} = 0$ (e.g., $\phi(t) = t(\log t)^{1-\varepsilon}$, $0 < \varepsilon \leq 1$).

The study of the L^p boundedness of the generalized Calderón-Zygmund operators $T_{h,\Omega}$ and $T_{h,\Omega}^*$ was began by R. Fefferman ([18]) and subsequently by many others under various conditions on Ω and h (see for example, [10], [21], [14], [16], [5], [6]).

Our point of departure is the following L^p boundedness result from [16].

Theorem A. *Suppose that $\Omega \in H^1(\mathbf{S}^{n-1})$ (the 1-Hardy space on \mathbf{S}^{n-1} (see [12])) and $h \in l^\infty(L^\gamma)(\mathbf{R}_+)$ for some $\gamma > 1$. Then $T_{\mathcal{P},\Omega,h}$ is bounded on $L^p(\mathbf{R}^m)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ with bounds on $\|T_{\mathcal{P},\Omega,h}\|_{p,p}$ may depend on $n, m, h(\cdot)$ and $\deg(P_j)$, but they are independent of the coefficients of $\{P_j\}$.*

We point out that the range for p in Theorem A is the full range $(1, \infty)$ whenever $\gamma \geq 2$ and it becomes a tiny open interval around 2 as γ approaches 1. To improve the range of p in Theorem A, Fan and Pan in [16] showed that, if Ω satisfies the stronger condition $\Omega \in L^q(\mathbf{S}^{n-1})$ and \mathcal{P} is an odd polynomial mapping, the L^p boundedness of $T_{\mathcal{P},\Omega,h}$ can be preserved for the full range $1 < p < \infty$, regardless how close γ is to 1. More precisely, they proved the following.

Theorem B ([16]). *Suppose that $\mathcal{P}(-x) = -\mathcal{P}(x)$, $\Omega \in L^q(\mathbf{S}^{n-1})$ and $h \in l^\infty(L^\gamma)(\mathbf{R}_+)$ for some $q > 1$ and $\gamma > 1$. Then $T_{\mathcal{P},\Omega,h}$ and $T_{\mathcal{P},\Omega,h}^*$ are bounded on $L^p(\mathbf{R}^m)$ for $1 < p < \infty$ with bounds on $\|T_{\mathcal{P},\Omega,h}\|_{p,p}$ and $\|T_{\mathcal{P},\Omega,h}^*\|_{p,p}$ independent of the coefficients of $\{P_j\}$.*

In [6], Al-Salman and Pan were able to show that the result in Theorem B continues to hold if the condition $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$ is replaced by the weaker condition $\Omega \in L \log L(\mathbf{S}^{n-1})$ as described in the following theorem.

Theorem C. *Suppose that $\mathcal{P}(-x) = -\mathcal{P}(x)$ and $h \in l^\infty(L^\gamma)(\mathbf{R}_+)$ for some $\gamma > 1$. If $\Omega \in L \log L(\mathbf{S}^{n-1})$, the operators $T_{\mathcal{P},\Omega,h}$ and $T_{\mathcal{P},\Omega,h}^*$ are bounded on $L^p(\mathbf{R}^m)$ for $1 < p < \infty$ with bounds on $\|T_{\mathcal{P},\Omega,h}\|_{p,p}$ and $\|T_{\mathcal{P},\Omega,h}^*\|_{p,p}$ independent of the coefficients of $\{P_j\}$.*

In a recent paper [3], H. Al-Qassem investigated the L^p boundedness of the special class of operators $T_{\Omega,h}$ if h satisfies the stronger condition $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$ and showed that this class of operators behaves completely different from the class of Calderón-Zygmund operators $T_\Omega = T_{1,\Omega}$. In fact, Al-Qassem proved the following:

Theorem D ([3]). *Suppose that $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$ and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. Then $T_{h,\Omega}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

Note that the singular integral operator $T_{\Omega,h}$ is bounded on L^p if $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ and $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$, while the classical Calderón-Zygmund singular integral operator $T_\Omega = T_{\Omega,1}$ is bounded on L^p if $\Omega \in L(\log L)(\mathbf{S}^{n-1})$. It is also worth mentioning that a proof of Theorem D cannot be obtained by a simple application of existing arguments on singular integrals. Even though, there is a more restricted condition on h , if we try to apply previously known arguments then we can prove Theorem D only for p satisfying $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$. To get around this difficulty, Al-Qassem in [3] employed an argument where one of its key ideas is based on the maximal operator $\mathfrak{S}_\Omega^{(\gamma)}$ (see also [19]). Historically, the study of the L^p boundedness of the related maximal operator $\mathfrak{S}_\Omega^{(\gamma)}$ began by L. K. Chen and H. Lin in [11] and subsequently by many other authors [1], [3], [13] and [19].

L. K. Chen and H. Lin in [11] proved the following:

Theorem E. *Assume $n \geq 2$, $1 \leq \gamma \leq 2$ and $\Omega \in C(\mathbf{S}^{n-1})$. Then $\mathfrak{S}_\Omega^{(\gamma)}$ is bounded on $L^p(\mathbf{R}^n)$ for $(\gamma n)' < p < \infty$. Moreover, the range of p is the best possible.*

In [1], Al-Qassem improved the result in Theorem E as described in the following:

Theorem F. *Let $n \geq 2$ and $1 \leq \gamma \leq 2$. Then*

- (a) *If $\Omega \in L(\log L)^{1/\gamma'}(S^{n-1})$ and satisfies (1.5), then $\mathfrak{S}_\Omega^{(\gamma)}$ is bounded on $L^p(R^n)$ for $\gamma' \leq p < \infty$;*
- (b) *There exists an Ω which lies in $L(\log L)^{1/2-\varepsilon}(S^{n-1})$ for all $\varepsilon > 0$ and satisfies (1.5) such that $\mathfrak{S}_\Omega^{(2)}$ is not bounded on $L^2(R^n)$.*

One of the main purposes of this paper is to investigate the L^p boundedness of the operators $T_{\mathcal{P},\Omega,h}$ and $T_{\mathcal{P},\Omega,h}^*$ if $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$ and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. Also, we seek a solution to the following problem which was left unresolved in [3]: Whether there are some results concerning the L^p boundedness of the operators $T_{\Omega,h}$ and $T_{\Omega,h}^*$ if $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for $\gamma = 1$? We shall obtain a positive answer to this problem. The actual statements of our results will be given in the next section.

§2. Main theorems

We shall start with the following result concerning the maximal operator $\mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)}$, which gives the L^p boundedness of $\mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)}$ whenever Ω is allowed to be very rough on the unit sphere.

Theorem 2.1. *Suppose $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ for $1 \leq \gamma \leq 2$. Then $\mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)}$ is bounded on $L^p(\mathbf{R}^m)$ for $\gamma' \leq p < \infty$ and $1 < \gamma \leq 2$; and $\mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)}$ is bounded on $L^\infty(\mathbf{R}^m)$ for $\gamma = 1$. The bounds on $\left\| \mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)} \right\|_{p,p}$ may depend on n, m, γ and $\deg(P_j)$, but they are independent of the coefficients of $\{P_j\}$.*

Here and in the sequel, we mean by the condition $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ for $\gamma = 1$ is that $\Omega \in L^1(\mathbf{S}^{n-1})$.

Theorem 2.2. *Suppose that $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 \leq \gamma \leq \infty$ and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. Then*

- (a) $T_{\mathcal{P},\Omega,h}$ is bounded on $L^p(\mathbf{R}^m)$ for $1 < p < \infty$ if $1 < \gamma \leq \infty$; and
- (b) $T_{\mathcal{P},\Omega,h}$ is bounded on $L^p(\mathbf{R}^m)$ for $1 \leq p \leq \infty$ if $\gamma = 1$.

The bounds on $\|T_{\mathcal{P},\Omega,h}\|_{p,p}$ may depend on n, m, γ and $\deg(P_j)$, but it is independent of the coefficients of $\{P_j\}$.

Theorem 2.3. *Suppose that $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$ and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. Then $T_{\mathcal{P},\Omega,h}^*$ and $\mathcal{M}_{\mathcal{P},\Omega,h}$ are bounded on $L^p(\mathbf{R}^m)$ for $\gamma' < p < \infty$. The bound of the operator norms $\left\| T_{\mathcal{P},\Omega,h}^* \right\|_{p,p}$ and $\|\mathcal{M}_{\mathcal{P},\Omega,h}\|_{p,p}$ may depend on the degrees of the polynomials P_1, \dots, P_m , but it is independent of their coefficients.*

Theorem 2.4. *Suppose that $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$ and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$. If $\mathcal{P}(-x) = -\mathcal{P}(x)$, then $T_{\mathcal{P},\Omega,h}^*$ and $\mathcal{M}_{\mathcal{P},\Omega,h}$ are bounded on $L^p(\mathbf{R}^m)$ for any $p \in (1, \infty)$. The bounds of the operator norms $\left\| T_{\mathcal{P},\Omega,h}^* \right\|_{p,p}$ and $\|\mathcal{M}_{\mathcal{P},\Omega,h}\|_{p,p}$ may depend on the degrees of the polynomials P_1, \dots, P_m , but they are independent of their coefficients.*

Remark 1. Note that

$$(2.1) \quad L^q(\mathbf{S}^{n-1})(q > 1) \subset L(\log L)(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \subset L^1(\mathbf{S}^{n-1}),$$

$$(2.2) \quad L(\log L)^\beta(\mathbf{S}^{n-1}) \subset L(\log L)^\alpha(\mathbf{S}^{n-1}) \text{ if } 0 < \alpha < \beta,$$

$$(2.3) \quad L(\log L)^\alpha(\mathbf{S}^{n-1}) \subset H^1(\mathbf{S}^{n-1}) \text{ for all } \alpha \geq 1, \text{ while}$$

$$(2.4) \quad L(\log L)^\alpha(\mathbf{S}^{n-1}) \not\subset H^1(\mathbf{S}^{n-1}) \not\subset L(\log L)^\alpha(\mathbf{S}^{n-1}) \text{ for all } 0 < \alpha < 1,$$

and all inclusions are proper. Thus, we notice the following: (i) Theorem 2.1 represents an improvement and extension over the result in Theorem E and it is an extension over Theorem F, (ii) Theorem 2.2 represents an improvement in the range of p over Theorem A in the case $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$

and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$, (iii) since $L \log L(\mathbf{S}^{n-1}) \subset L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ for any $\gamma > 1$, Theorem 2.4 represents an improvement over Theorem B in the case $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma < \infty$.

Remark 2. For the case $h \in L^\infty(\mathbf{R}_+)$, the authors in [7] showed that there is a function $f \in L^p$ such that the maximal operator acting on f (i.e. $\mathfrak{S}_\Omega^{(\infty)}(f)$) yields an identically infinite function. It is still an open question whether the L^p boundedness of $\mathfrak{S}_\Omega^{(\gamma)}$ holds for $2 < \gamma < \infty$. We notice that the singular integral operator $T_{\mathcal{P},\Omega,h}$ is bounded on $L^p(\mathbf{R}^m)$ for all $1 \leq \gamma \leq \infty$.

Remark 3. As we mentioned previously that the class of operators $T_{\Omega,h}$ when $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ behaves completely different from the classical class of Calderón-Zygmund operators T_Ω . Also, it may be interesting to point out that Theorem 2.2 implies that the operators $T_{\Omega,h}$ when $h \in \mathcal{H}_1(\mathbf{R}_+)$ are bounded on $L^1(\mathbf{R}^m)$ and $L^\infty(\mathbf{R}^m)$, while the classical Calderón-Zygmund operators T_Ω are not. Furthermore, we notice that the operators $T_{\Omega,h}$ when $h \in \mathcal{H}_1(\mathbf{R}_+)$ are bounded on L^p if $\Omega \in L^1(\mathbf{S}^{n-1})$, while it is well-known that the classical Calderón-Zygmund operator T_Ω is not bounded on L^p for any p if $\Omega \in L^1(\mathbf{S}^{n-1})$ unless Ω is an odd function on \mathbf{S}^{n-1} , i.e., $\Omega(x) = -\Omega(x)$ for $x \in \mathbf{S}^{n-1}$.

Remark 4. The proof of our results will mainly be a consequence of two general lemmas stated in Section 4. The main tools used in this paper come from [1], [3], [4], [19], [14] and [16], among others.

Throughout the rest of the paper the letter C denotes a positive whose value may be different at appearance.

§3. Some definitions and lemmas

We start this section by introducing some notation. For $\omega \in \mathbf{N} \cup \{0\}$ and $k \in \mathbf{Z}$, let $\rho_\omega = 2^{(\omega+1)}$. For a positive integer d , we let $L(\mathbf{R}^n, \mathbf{R}^d)$ denote the space of linear transformations from \mathbf{R}^n into \mathbf{R}^d , V_d denote the space of real-valued homogeneous polynomials of degree d on \mathbf{R}^n with $\theta_d = \dim(V_d)$ and \mathcal{A}_n be the class of polynomials of n variables with real coefficients. For $\mathcal{P} = (P_1, \dots, P_d) \in (\mathcal{A}_n)^d$, we shall use $\deg(\mathcal{P})$ to denote $\max_{1 \leq k \leq d} \deg(P_k)$ and for $P(y) = \sum_{|\alpha|=d} a_\alpha y^\alpha \in V_d$, we set $\|P\| = \sum_{|\alpha|=d} |a_\alpha|$. If d is an even, positive integer, then we have $|x|^d = (x_1^2 + x_2^2 + \dots + x_n^2)^{d/2} \in V_d$. We now choose a basis $\{\eta_1, \dots, \eta_{\theta_d}\}$ for the space V_d such that $\eta_1(x) = |x|^d$ for $x \in \mathbf{R}^n$. It is clear that there are constants C_1 and C_2 such that $C_1 \left(\sum_{j=1}^{\theta_d} |c_j| \right) \leq \|P\| \leq C_2 \left(\sum_{j=1}^{\theta_d} |c_j| \right)$ for every $P = \sum_{j=1}^{\theta_d} c_j \eta_j \in V_d$. We define the linear

transformation $Y_d : V_d \rightarrow V_d$ by $Y_d(P) = \sum_{j=2}^{\theta_d} c_j \eta_j$ for $P = \sum_{j=1}^{\theta_d} c_j \eta_j$. Also, define the linear transformation $Z_d^n : V_d \rightarrow V_d$ by

$$Z_d^n = \begin{cases} id_{\theta_d} & \text{if } d \text{ is odd} \\ Y_d & \text{if } d \text{ is even.} \end{cases}$$

The following result follows from Lemmas 3.3–3.4, 3.7 and Remark 3.6 in [16].

Lemma 3.1. *Let $d \in \mathbf{N}$. Then there exists a positive constant $A_{d,\varepsilon}$ such that*

$$(3.1) \quad \sup_{\lambda \in \mathbf{R}} \int_{\mathbf{S}^{n-1}} |P(y) - \lambda|^{-\varepsilon} d\sigma(y) \leq A_{d,\varepsilon} \|Z_d^n(P)\|^{-\varepsilon}$$

for every $P \in V_d$, and $\varepsilon \in [0, \varepsilon(d))$, where $\varepsilon(d) = \frac{2}{[3+(-1)^{d+1}]d}$. If U is a subspace of V_d satisfying $|x|^d \notin U$, then there exists a constant $A'_{d,\varepsilon}$ such that

$$(3.2) \quad \sup_{\lambda \in \mathbf{R}} \int_{\mathbf{S}^{n-1}} |P(y) - \lambda|^{-\varepsilon} d\sigma(y) \leq A'_{d,\varepsilon} \|P\|^{-\varepsilon}$$

holds for $\varepsilon \in [0, \varepsilon(d))$ and all $P \in U$. The constant $A'_{d,\varepsilon}$ may depend on the subspace U if d is even, but it is independent of U if d is odd.

Lemma 3.2. *Let $\omega \in \mathbf{N} \cup \{0\}$ and $\Omega_\omega(\cdot)$ be a function on \mathbf{S}^{n-1} satisfying the following conditions: (i) $\|\Omega_\omega\|_{L^2(\mathbf{S}^{n-1})} \leq \rho_\omega^2$, and (ii) $\|\Omega_\omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Suppose that $F : \mathbf{R}^n \rightarrow \mathbf{R}$ is a function given by*

$$(3.3) \quad F(x) = \sum_{j=0}^l P_j(x) + W(|x|),$$

where $P_j(\cdot)$ is a homogeneous polynomial of degree j , $0 \leq j \leq l$ and $W(\cdot)$ is an arbitrary function. Then there exist a positive constant C independent of k , and ω such that

$$(3.4) \quad \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega+1)^{1/2} \left(\rho_\omega^k \|Z_l(P_l)\| \right)^{-\frac{1}{8l(\omega+1)}}.$$

If U is a subspace of V_l satisfying $|x|^l \notin U$, then there exists a constant C' such that

$$(3.5) \quad \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C'(\omega+1)^{1/2} \left(\rho_\omega^k \|P_l\| \right)^{-\frac{1}{8l(\omega+1)}}.$$

holds for $k \in \mathbf{Z}$ and $F \in U$ given by (3.3) with $P_l \in U$. The constant C' may depend on the subspace U if l is even, but it is independent of U if l is odd.

Proof. By a change of variable, we have

$$\begin{aligned} & \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ & \leq \left(\int_1^{\rho_\omega} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(\rho_\omega^k tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

By writing

$$\begin{aligned} & \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(\rho_\omega^k tx)} d\sigma(x) \right|^2 \\ & = \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}} \Omega_\omega(x) \overline{\Omega_\omega(y)} e^{i(F(\rho_\omega^k tx) - F(\rho_\omega^k ty))} d\sigma(x) d\sigma(y) \end{aligned}$$

and using Van der Corput's lemma we get

$$\begin{aligned} \left| \int_1^{\rho_\omega} e^{i(F(\rho_\omega^k tx) - F(\rho_\omega^k ty))} \frac{dt}{t} \right| & \leq C \min \left\{ (\omega + 1), \left| \rho_\omega^{kl} (P_l(x) - P_l(y)) \right|^{-\frac{1}{l}} \right\} \\ & \leq C(\omega + 1) \left| \rho_\omega^{kl} (P_l(x) - P_l(y)) \right|^{-\frac{1}{8l}}. \end{aligned}$$

Therefore, by Hölder's inequality and (3.1) we get

$$\begin{aligned} & \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \\ & \leq C(\omega + 1)^{1/2} \|\Omega_\omega\|_{L^2(\mathbf{S}^{n-1})} \left(\rho_\omega^{kl} \|Z_l(P_l)\| \right)^{-\frac{1}{8l}}. \end{aligned}$$

By condition (i) on Ω_ω , we get

$$\left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2} \rho_\omega^2 \left(\rho_\omega^{kl} \|Z_l(P_l)\| \right)^{-\frac{1}{8l}}.$$

By interpolating between the preceding estimate and the trivial estimate

$$\left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2}$$

we obtain (3.4). The proof of the inequality (3.5) follows by the same argument as proving (3.4) except we need to apply (3.2) instead of (3.1). This completes the proof of the lemma. \square

Lemma 3.3. *Let $\omega \in \mathbf{N} \cup \{0\}$, $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq \infty$ and $\Omega_\omega(\cdot)$ be a function on \mathbf{S}^{n-1} satisfying the following conditions: (i) $\|\Omega_\omega\|_{L^2(\mathbf{S}^{n-1})} \leq \rho_\omega^2$, and (ii) $\|\Omega_\omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Let F be given as (3.3). Then there exist a positive constant C independent of k , and ω such that*

$$(3.6) \quad \left| \int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right| h(t) \frac{dt}{t} \right| \leq C(\omega+1)^{1/\gamma'} \left(\rho_\omega^k \|Z_l(P_l)\| \right)^{-\frac{1}{8l\gamma'(\omega+1)}}.$$

If U is a subspace of V_l satisfying $|x|^l \notin U$, then there exists a constant C' such that

$$(3.7) \quad \left| \int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right| h(t) \frac{dt}{t} \right| \leq C'(\omega+1)^{1/\gamma'} \left(\rho_\omega^k \|P_l\| \right)^{-\frac{1}{8l(\omega+1)}}.$$

holds for $k \in \mathbf{Z}$ and $F \in U$ given by (3.3) with $P_l \in U$. The constant C' may depend on the subspace U if l is even, but it is independent of U if l is odd.

Proof. Let us first prove (3.6). By Hölder's inequality we have

$$\begin{aligned} & \left| \int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right| h(t) \frac{dt}{t} \right| \\ & \leq \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \\ & \leq \left(\int_0^\infty |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \\ & \leq \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'}. \end{aligned}$$

Now, we need to consider two cases:

CASE 1. $\gamma \in (1, 2]$. Since $\left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right| \leq 1$ we get immediately

$$\begin{aligned} & \left| \int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right| \frac{|h(t)|}{t} dt \right| \\ & \leq \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/\gamma'} \end{aligned}$$

which in turn leads to (3.6) by invoking Lemma 3.2.

CASE 2. $\gamma \in (2, \infty]$. By Hölder's inequality we get

$$\begin{aligned} & \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \\ & \leq C(\omega + 1)^{(1/\gamma' - 1/2)} \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega_\omega(x) e^{-iF(tx)} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{1/2} \end{aligned}$$

which easily implies (3.6) by applying Lemma 3.2 and thus the proof of (3.6) is complete. The proof of (3.7) is similar. \square

Definition 3.4. For suitable mappings $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^m$ and $\Omega_\omega : \mathbf{S}^{n-1} \rightarrow \mathbf{R}$, we define the measures $\{\lambda_{\Phi,\omega,t} : t \in \mathbf{R}_+\}$ on \mathbf{R}^m by

$$\int_{\mathbf{R}^m} f d\lambda_{\Phi,\omega,t} = \int_{\mathbf{S}^{n-1}} f(\Phi(yt)) \Omega_\omega(y) d\sigma(y).$$

Also, we define the measures $\{\sigma_{\Phi,\Omega,k,\omega} : k \in \mathbf{Z}\}$ and the maximal operator $\sigma_{\Phi,\Omega,\omega}^*$ on \mathbf{R}^m by

$$\int_{\mathbf{R}^m} f d\sigma_{\Phi,k,\omega} = \int_{\rho_\omega^k \leq |u| < \rho_\omega^{k+1}} f(\Phi(u)) K_{\Omega_\omega,h}(u) du,$$

and

$$\sigma_{\Phi,\omega}^*(f) = \sup_{k \in \mathbf{Z}} |\sigma_{\Phi,k,\omega} * f|,$$

where $|\sigma_{\Phi,k,\omega}|$ is defined in the same way as $\sigma_{\Phi,k,\omega}$, but with Ωh replaced by $|\Omega h|$.

Let $\mathcal{Q}(t) = (\mathcal{Q}_1(t), \dots, \mathcal{Q}_m(t))$ be a mapping defined on \mathbf{R} with $\mathcal{Q}_j \in \mathcal{A}_1$ for $1 \leq j \leq m$. Let

$$\mathcal{M}_{\mathcal{Q}}f(x) = \sup_{R>0} \frac{1}{R} \int_{|t|<R} |f(x - \mathcal{Q}(t))| dt.$$

We shall need the following L^p boundedness result due to Stein and Wainger in [26].

Lemma 3.5. *For every $1 < p \leq \infty$, there exists a positive constant C_p such that*

$$(3.8) \quad \|\mathcal{M}_{\mathcal{Q}}f\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^m)$. The constant C_p may depend on the degrees of the polynomials $\{\mathcal{Q}_j\}$, but it is independent of the coefficients of $\{\mathcal{Q}_j\}$.

Lemma 3.6. *Let $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$ and Let $\mathcal{P} = (P_1, \dots, P_m)$ be a polynomial mapping from \mathbf{R}^n into \mathbf{R}^m . Let Ω_ω be a function on \mathbf{S}^{n-1} satisfying $\|\Omega_\omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Then for $\gamma' < p \leq \infty$ and $f \in L^p(\mathbf{R}^m)$, there exists a positive constant C_p which is independent of ω such that*

$$(3.9) \quad \|\sigma_{\mathcal{P},\omega}^*(f)\|_p \leq C_p(\omega + 1)^{1/\gamma'} \|f\|_p.$$

Proof. By Hölder's inequality we have

$$\sigma_{\mathcal{P},\omega}^*(f) \leq \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} \left(M_\omega^*(|f|^{\gamma'}) \right)^{1/\gamma'} \leq C \left(M_\omega^*(|f|^{\gamma'}) \right)^{1/\gamma'},$$

where

$$M_\omega^*(f) = \sup_{k \in \mathbf{Z}} \left| \int_{\rho_\omega^k \leq |y| < \rho_\omega^{k+1}} f(x - \mathcal{P}(y)) \Omega_\omega(y') |y|^{-n} dy \right|.$$

We notice the proof of this lemma is completed if we can show that

$$(3.10) \quad \|M_\omega^*(f)\|_{L^p(\mathbf{R}^m)} \leq C_p(\omega + 1) \|f\|_{L^p(\mathbf{R}^m)}$$

for $1 < p \leq \infty$ and for some constant $C_p > 0$ independent of ω , and the coefficients of P_1, \dots, P_m . However, (3.10) follows as a simple consequence of (3.8). \square

Lemma 3.7. *Let $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$ and Let $\mathcal{P} = (P_1, \dots, P_m)$ be a polynomial mapping from \mathbf{R}^n into \mathbf{R}^m . Let Ω_ω be a function on \mathbf{S}^{n-1} satisfying $\|\Omega_\omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Then for $\gamma' < p < \infty$, there exists a positive constant C_p which is independent of ω such that*

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_{\mathcal{P},k,\omega} * g_k|^2 \right)^{1/2} \right\|_p \leq C_p(\omega + 1)^{1/\gamma'} \left\| \left(\sum_{k \in \mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_p$$

holds for arbitrary measurable functions $\{g_k\}$ on \mathbf{R}^m .

The proof of this lemma follows the same argument as in [3] (see also [17]). We omit the details.

§4. General results

We shall need the following lemma which has its roots in [14], [16] and [5]. A proof of this lemma can be obtained by the same proof (with only minor modifications) as that of Lemma 3.2 in [5]. We omit the details.

Lemma 4.1. *Let $N \in \mathbf{N}$ and $\{\sigma_k^{(l)} : k \in \mathbf{Z}, 0 \leq l \leq N\}$ be a family of Borel measures on \mathbf{R}^n with $\sigma_k^{(0)} = 0$ for every $k \in \mathbf{Z}$. Let $\{a_l : 1 \leq l \leq N\} \subseteq \mathbf{R}_+/(0, 2)$, $\{m_l : 1 \leq l \leq N\} \subseteq \mathbf{N}$, $\{\alpha_l : 1 \leq l \leq N\} \subseteq \mathbf{R}_+$, and let $L_l \in L(\mathbf{R}^n, \mathbf{R}^{m_l})$ for $1 \leq l \leq N$. Suppose that for all $k \in \mathbf{Z}$, $1 \leq l \leq N$, for all $\xi \in \mathbf{R}^n$ and for some $C > 0$, $B > 1$, $\lambda > 0$, $C > 0$ and for some $B > 1$ we have the following:*

$$(i) \quad \left\| \sigma_k^{(l)} \right\| \leq CB^\lambda;$$

$$(ii) \quad \left| \hat{\sigma}_k^{(l)}(\xi) \right| \leq CB^\lambda |a_l^{kB} L_l(\xi)|^{-\frac{\alpha_l}{B}};$$

$$(iii) \quad \left| \hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi) \right| \leq CB^\lambda |a_l^{kB} L_l(\xi)|^{\frac{\alpha_l}{B}};$$

(iv) *For some $p_0 \in (2, \infty)$,*

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_k^{(l)} * g_k|^2 \right)^{1/2} \right\|_{p_0} \leq CB^\lambda \left\| \left(\sum_{k \in \mathbf{Z}} |g_k|^2 \right)^{1/2} \right\|_{p_0}$$

holds for all functions $\{g_k\}$ on \mathbf{R}^n .

Then for $p'_0 < p < p_0$ there exists a positive constant C_p such that

$$\left\| \sum_{k \in \mathbf{Z}} \sigma_k^{(N)} * f \right\|_{L^p(\mathbf{R}^n)} \leq C_p B^\lambda \|f\|_{L^p(\mathbf{R}^n)}$$

and

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\sigma_k^{(N)} * f|^2 \right)^{1/2} \right\|_{L^p(\mathbf{R}^n)} \leq C_p B^\lambda \|f\|_{L^p(\mathbf{R}^n)}$$

hold for all f in $L^p(\mathbf{R}^n)$. The constant C_p is independent of the linear transformations $\{L_l\}_{l=1}^N$.

The proof of Theorem 1.2 (b) will rely heavily on the following lemma. Before stating this lemma, we introduce some notation. For $1 \leq p, q < \infty$, let $L^p(L^q(\mathbf{R}_+, dt/t), \mathbf{R}^n)$ be the space of all measurable functions $f_t(x)$ defined on $\mathbf{R}^n \times \mathbf{R}_+$ with mixed norm $\|f\|_{L^p(L^q(\mathbf{R}_+, dt/t), \mathbf{R}^n)}$, where

$$\begin{aligned} \|f\|_{L^p(L^q(\mathbf{R}_+, dt/t), \mathbf{R}^n)} &= \left\| \left\| f(\cdot)(\cdot) \right\|_{L^q(\mathbf{R}_+, dt/t)} \right\|_{L^p(\mathbf{R}^n)} \\ &= \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}_+} |f_t(x)|^q dt/t \right)^{p/q} dx \right)^{1/p}. \end{aligned}$$

If $p = \infty$ or $q = \infty$, we can define $L^p(L^q(\mathbf{R}_+, dt/t), \mathbf{R}^n)$ by the usual modification.

Lemma 4.2. *Let $n, m \in \mathbf{N}$, $L \in L(\mathbf{R}^n, \mathbf{R}^m)$, $a \geq 2$, $C > 0$ and $q_0 > 1$. Let $\{\sigma_t : t \in \mathbf{R}_+\}$ be a family of finite Borel measures on \mathbf{R}^n . Suppose that there are constants $\alpha \in (0, 1]$, $C > 0$ and $B > 0$ such that the following hold for $k \in \mathbf{Z}$, $t \in \mathbf{R}_+$ and $\xi \in \mathbf{R}^n$:*

$$(4.1) \quad \|\sigma_t\| \leq 1;$$

$$(4.2) \quad \int_{a^{kB}}^{a^{(k+1)B}} |\hat{\sigma}_t(\xi)|^2 \frac{dt}{t} \leq CB(a^{kB} |L(\xi)|)^{\pm \frac{\alpha}{B}};$$

$$(4.3) \quad \left\| \sup_{k \in \mathbf{Z}} \int_{a^{kB}}^{a^{(k+1)B}} \|\sigma_t\| * |f| \frac{dt}{t} \right\|_{L^p(\mathbf{R}^n)} \leq CB \|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in L^p(\mathbf{R}^n)$ and $1 < p \leq \infty$. Then for $q \leq p < \infty$ and $2 \leq q < \infty$, there exists a positive constant C_p such that

$$(4.4) \quad \left\| \|\sigma_t * f\|_{L^q(\mathbf{R}_+, dt/t)} \right\|_{L^p(\mathbf{R}^n)} = \left\| \left(\int_0^\infty |\sigma_t * f|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} \leq C_p B^{1/q} \|f\|_{L^p(\mathbf{R}^n)}$$

for all $f \in L^p(\mathbf{R}^n)$. The constant C_p is independent of B and L .

Proof. By an argument in [16], we may assume that $m \leq n$ and $L(\xi) = (\xi_1, \dots, \xi_m)$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$. We first prove (4.4) for the case $q = 2$. The proof of this case follow a similar argument employed in the proof of Theorem 2.1 in [4] except for minor modifications. For the reader's convenience, we shall only present a sketch of the proof of this case and omit some details. Let $\{\psi_j\}_{-\infty}^\infty$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $[a^{-(j+1)B}, a^{-(j-1)B}]$. More precisely, we require the following: $\psi_j \in C^\infty$, $0 \leq \psi_j \leq 1$, $\sum_j \psi_j(t) = 1$; $\text{supp} \psi_j \subseteq [a^{-(j+1)B}, a^{-(j-1)B}]$; $\left| \frac{d^s \psi_j(t)}{dt^s} \right| \leq \frac{C}{t^s}$, where C can be chosen to be independent of B . Let $\widehat{\Psi}_j(\xi) = \psi_j(|L(\xi)|)$. Decompose

$$f * \sigma_t(x) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} (\Psi_{k+j} * \sigma_t * f)(x) \chi_{[a^{kB}, a^{(k+1)B})}(t) := \sum_{j \in \mathbf{Z}} F_j(x, t)$$

and hence

$$\left(\int_0^\infty |\sigma_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \leq \sum_{j \in \mathbf{Z}} \left(\int_0^\infty |F_j(x, t)|^2 \frac{dt}{t} \right)^{1/2} := \mathcal{M}_j f(x)$$

holds for $f \in \mathcal{S}(\mathbf{R}^n)$. To this end, we first compute the L^2 -norm of $\mathcal{M}_j(f)$. By the same argument as in [4] we get

$$(4.5) \quad \|\mathcal{M}_j(f)\|_2 \leq CB^{1/2}a^{-\beta/2|j|} \|f\|_2.$$

On the other hand, we compute the L^p -norm of $\mathcal{M}_j(f)$. For $p \geq 2$, there exists a nonnegative function g in $L^{(p/2)'}$ with $\|g\|_{(p/2)'} \leq 1$ such that

$$\begin{aligned} \|\mathcal{M}_j(f)\|_p^2 &= \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}^n} \int_{a^{kB}}^{a^{(k+1)B}} |\Psi_{k+j} * \sigma_t * f(x)|^2 \frac{dt}{t} g(x) dx \\ &\leq \sum_{k \in \mathbf{Z}} \int_{\mathbf{R}^n} \int_{a^{kB}}^{a^{(k+1)B}} |\sigma_t| * |\Psi_{k+j} * f(x)|^2 \frac{dt}{t} g(x) dx \\ &\leq C \left\| \sum_{k \in \mathbf{Z}} |\Psi_{k+j} * f|^2 \right\|_{p/2} \left\| \sup_{k \in \mathbf{Z}} \int_{a^{kB}}^{a^{(k+1)B}} |\sigma_t| * (\tilde{g}) \frac{dt}{t} \right\|_{(p/2)'} \\ &\leq CB \left\| \sum_{k \in \mathbf{Z}} |\Psi_{k+j} * f|^2 \right\|_{p/2}, \end{aligned}$$

where $\tilde{g}(x) = g(-x)$. By using (iii), the Littlewood-Paley theory, we have

$$(4.6) \quad \|\mathcal{M}_j(f)\|_p \leq CB^{1/2} \|f\|_p \text{ for } 2 \leq p < \infty.$$

By interpolation between (4.5) and (4.6) we get

$$(4.7) \quad \|\mathcal{M}_j(f)\|_p \leq CB^{1/2}a^{-\beta|j|} \|f\|_p \text{ for } 2 \leq p < \infty$$

and hence we have

$$(4.8) \quad \left\| \left(\int_0^\infty |\sigma_t * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leq \sum_{j \in \mathbf{Z}} \|\mathcal{M}_j(f)\|_p \leq C_p B^{1/2} \|f\|_p$$

for $2 \leq p < \infty$. Also, by condition (i) we have

$$(4.9) \quad |\sigma_t * f(x)| \leq \|f\|_\infty \text{ for } f \in L^\infty(\mathbf{R}^n) \text{ and for almost every } x \in \mathbf{R}^n.$$

Now, we define a linear operator T on any function f on \mathbf{R}^n by $T(f)(x) = \sigma_t * f(x)$. We use now an idea appearing in [19] (see also [3]). From the inequalities (4.8) and (4.9), we interpret that $\|T(f)\|_{L^p(L^2(\mathbf{R}_+, dt/t), \mathbf{R}^n)} \leq CB^{1/2} \|f\|_{L^p(\mathbf{R}^n)}$ for $2 \leq p < \infty$ and $\|T(f)\|_{L^\infty(L^\infty(\mathbf{R}_+, dt/t), \mathbf{R}^n)} \leq C \|f\|_{L^\infty(\mathbf{R}^n)}$. Applying the real interpolation theorem for Lebesgue mixed normed spaces to the above results (see [8]), we conclude that $\|T(f)\|_{L^p(L^q(\mathbf{R}_+, dt/t), \mathbf{R}^n)} \leq CB^{1/q} \|f\|_{L^p(\mathbf{R}^n)}$ for $q \leq p < \infty$ and $2 \leq q < \infty$ which in turn implies (4.4). Thus Lemma 4.2 is proved. \square

Lemma 4.3. *Let $N \in \mathbf{N}$ and $\{\sigma_t^{(l)} : t \in \mathbf{R}_+, 0 \leq l \leq N\}$ be a family of Borel measures on \mathbf{R}^n with $\sigma_t^{(0)} = 0$ for all $t \in \mathbf{R}_+$. Let $\{a_l : 1 \leq l \leq N\} \subseteq \mathbf{R}_+/(0, 2)$, $\{m_l : 1 \leq l \leq N\} \subseteq \mathbf{N}$, $\{\alpha_l : 1 \leq l \leq N\} \subseteq \mathbf{R}_+$, and let $L_l \in L(\mathbf{R}^n, \mathbf{R}^{m_l})$ for $1 \leq l \leq N$. Suppose that for all $t \in \mathbf{R}_+$, $1 \leq l \leq N$, for all $\xi \in \mathbf{R}^n$ and for some $C > 0$ and for some $B > 1$ we have we have the following:*

$$\begin{aligned}
& \text{(i)} \quad \left\| \sigma_t^{(l)} \right\| \leq C; \\
& \text{(ii)} \quad \int_{a_l^{kB}}^{a_l^{(k+1)B}} \left| \hat{\sigma}_t^{(l)}(\xi) \right|^2 \frac{dt}{t} \leq CB \left| a_l^{kB} L_l(\xi) \right|^{-\frac{\alpha_l}{B}}; \\
& \text{(iii)} \quad \int_{a_l^{kB}}^{a_l^{(k+1)B}} \left| \hat{\sigma}_t^{(l)}(\xi) - \hat{\sigma}_t^{(l-1)}(\xi) \right|^2 \frac{dt}{t} \leq CB \left| a_l^{kB} L_l(\xi) \right|^{\frac{\alpha_l}{B}}; \\
& \text{(iv)} \quad \text{For } f \in L^p(\mathbf{R}^n) \text{ and } 1 < p \leq \infty, \\
(4.10) \quad & \left\| \sup_{k \in \mathbf{Z}} \int_{a^{kB}}^{a^{(k+1)B}} \left\| \sigma_t^{(l)} * f \right\| \frac{dt}{t} \right\|_p \leq CB \|f\|_p
\end{aligned}$$

Then for $2 \leq p < \infty$ and $2 \leq q < \infty$, there exists a positive constant C_p such that

$$\begin{aligned}
(4.11) \quad & \left\| \left\| \sigma_t^{(N)} * f \right\|_{L^q(\mathbf{R}_+, dt/t)} \right\|_{L^p(\mathbf{R}^n)} = \left\| \left(\int_0^\infty \left| \sigma_t^{(N)} * f \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)} \\
& \leq C_p B^{1/q} \|f\|_{L^p(\mathbf{R}^n)}
\end{aligned}$$

for all $f \in L^p(\mathbf{R}^n)$. The constant C_p is independent of B and the linear transformations $\{L_l\}_{l=1}^N$.

. The idea of the proof will be very much similar to the one appearing in the proof of Theorem 7.6 in [16]. Without loss of generality, we may assume that $0 < \alpha_l \leq 1$, $m_l \leq n$ and $L_l(\xi) = (\xi_1, \dots, \xi_{m_l})$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ and $1 \leq l \leq N$. Define the family of measures $\{\mu_t^{(l)} : 1 \leq l \leq N, t \in \mathbf{R}_+\}$ as follows: choose and fix a function $\theta \in \mathbf{C}_0^\infty(\mathbf{R})$ such that $\theta(s) = 1$ for $|t| \leq \frac{1}{2}$ and $\theta(s) = 0$ for $|t| \geq 1$. Let $\psi(t) = \theta(t^2)$ and for $t \in \mathbf{R}_+$, let

$$\begin{aligned}
(4.12) \quad & \hat{\mu}_t^{(l)}(\xi) \\
& = \hat{\sigma}_t^{(l)}(\xi) \prod_{l < j \leq N} \psi(a_j^{kB} |L_l(\xi)|) - \hat{\sigma}_t^{(l-1)}(\xi) \prod_{l-1 < j \leq N} \psi(a_j^{kB} |L_l(\xi)|)
\end{aligned}$$

when $1 \leq l \leq N - 1$ and

$$(4.13) \quad \hat{\mu}_t^{(N)}(\xi) = \hat{\sigma}_t^{(N)}(\xi) - \hat{\sigma}_t^{(N-1)}(\xi) \psi(a_N^{kB} |L_l(\xi)|).$$

By straightforward calculations, conditions (i)–(iii) and (4.12)–(4.13) we get

$$(4.14) \quad \left\| \mu_t^{(l)} \right\| \leq C,$$

$$(4.15) \quad \int_{a_l^{kB}}^{a_l^{(k+1)B}} \left| \hat{\mu}_t^{(l)}(\xi) \right|^2 \frac{dt}{t} \leq CB(a_l^{kB} |L_l(\xi)|)^{\pm \frac{\alpha_l}{B}} \text{ for all } 1 \leq l \leq N.$$

By condition (iv), it is easy to see that

$$(4.16) \quad \left\| \sup_{k \in \mathbf{Z}} \int_{a_l^{kB}}^{a_l^{(k+1)B}} \left| \mu_t^{(l)} \right| * f \left| \frac{dt}{t} \right\|_p \leq CB \|f\|_p$$

for $1 < p < \infty$, $f \in L^p(\mathbf{R}^n)$ and $1 \leq l \leq N$. By (4.14)–(4.16) and invoking Lemma 4.2, for $1 \leq l \leq N$, $q \leq p < \infty$ and $2 \leq q < \infty$, there exists a positive constant C_p such that

$$(4.17) \quad \left\| \left(\int_0^\infty \left| \mu_t^{(l)} * f \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p \leq C_p B^{1/q} \|f\|_p$$

holds for all f in $L^p(\mathbf{R}^n)$. Since $\sigma_t^{(0)} = 0$, we find that

$$(4.18) \quad \sigma_k^{(N)} = \sum_{l=1}^N \mu_k^{(l)}$$

and hence by (4.17) we get (4.11). The proof of Lemma 4.3 is complete. \square

§5. Proof of the main results

Proof of Theorem 2.1. Assume that Ω belongs to $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ for $1 \leq \gamma \leq 2$ and satisfies (1.5). We decompose Ω as in [3] (see also [6]). Let $E_0 = \{x \in \mathbf{S}^{n-1} : |\Omega(x)| < 2\}$ and for $\omega \in \mathbf{N}$, let $E_\omega = \{x \in \mathbf{S}^{n-1} : 2^\omega \leq |\Omega(x)| < 2^{\omega+1}\}$. For $\omega \in \mathbf{N} \cup \{0\}$, set $D = \left\{ \omega \in \mathbf{N} : \left\| \Omega \chi_{E_\omega} \right\|_1 \geq 2^{-4\omega} \right\}$ and define the sequence of functions $\{\Omega_\omega\}_{\omega \in D \cup \{0\}}$ by

$$\Omega_0(x) = \sum_{\omega \in \{0\} \cup (\mathbf{N}-D)} \Omega(x) \chi_{E_\omega}(x) - \sum_{\omega \in \{0\} \cup (\mathbf{N}-D)} \left(\int_{\mathbf{S}^{n-1}} \Omega(x) \chi_{E_\omega}(x) d\sigma(x) \right)$$

and for $\omega \in D$,

$$\Omega_\omega(x) = \left(\left\| \Omega \chi_{E_\omega} \right\|_1 \right)^{-1} \left(\Omega(x) \chi_{E_\omega}(x) - \int_{\mathbf{S}^{n-1}} \Omega(x) \chi_{E_\omega}(x) d\sigma(x) \right).$$

Then one can easily verify that the following hold for all $\omega \in D \cup \{0\}$ and for some positive constant C :

$$(5.1) \quad \|\Omega_\omega\|_2 \leq C\rho_\omega^2, \quad \|\Omega_\omega\|_1 \leq C;$$

$$(5.2) \quad \sum_{\omega \in D \cup \{0\}} (\omega + 1)^{1/\gamma'} \left\| \Omega \chi_{E_\omega} \right\|_1 \leq C \|\Omega\|_{L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})},$$

$$(5.3) \quad \int_{\mathbf{S}^{n-1}} \Omega_\omega(u) d\sigma(u) = 0, \quad \Omega = \sum_{\omega \in D \cup \{0\}} \left\| \Omega \chi_{E_\omega} \right\|_1 \Omega_\omega.$$

By (5.3), we have

$$(5.4) \quad \mathfrak{S}_{\mathcal{P}, \Omega}^{(\gamma)} f(x) \leq \sum_{\omega \in D \cup \{0\}} \left\| \Omega \chi_{E_\omega} \right\|_1 \mathfrak{S}_{\mathcal{P}, \Omega_\omega}^{(\gamma)} f(x).$$

We notice that the proof of Theorem 2.1 is completed if we can show that the inequality

$$(5.5) \quad \left\| S_{\mathcal{P}, \Omega_\omega}^{(\gamma)} f \right\|_p \leq C_p (\omega + 1)^{1/\gamma'} \|f\|_p$$

holds for $\gamma' \leq p < \infty$ if $1 < \gamma \leq 2$ and for $p = \infty$ if $\gamma = 1$. Let $0 < n_1 < n_2 < \dots < n_{\tilde{N}} = \deg(\mathcal{P})$ be non-negative integers, and polynomials

$\{P_\nu^l : 1 \leq \nu \leq N, 1 \leq l \leq \tilde{N}\}$ such that for $x \in \mathbf{R}^n$, $\mathcal{P}(x) = \sum_{l=1}^{\tilde{N}} \mathcal{P}^l(x) + A(|x|)$,

where $\mathcal{P}^l(x) = (P_1^l(x), \dots, P_N^l(x)) \in (\mathcal{H}_{n, n_l})^N$, $A(t) = (A_1(t), \dots, A_N(t))$ with $t \in \mathbf{R}$, $Z_{n_l}^n(P_\nu^l) = P_\nu^l$, and $A_\nu \in \mathcal{A}_1$ for $1 \leq \nu \leq N$ and $1 \leq l \leq \tilde{N}$.

For $1 \leq l \leq \tilde{N}$, let δ_l denote the number of elements of $\{\beta \in (\mathbf{N} \cup \{0\})^n : |\beta| = n_l\}$ and write $\{\beta \in (\mathbf{N} \cup \{0\})^n : |\beta| = n_l\} = \{\beta(1), \dots, \beta(\delta_l)\}$. Write

$P_j^l(x) = \sum_{k=1}^{\delta_l} \eta_{k,j} x^{\beta(k)}$ and define the linear mappings $L_l : \mathbf{R}^N \rightarrow \mathbf{R}^{\delta_l}$ by

$L_l(\xi) = \left(\sum_{j=1}^m \eta_{1,j}^l \xi_j, \dots, \sum_{j=1}^m \eta_{\delta_l,j}^l \xi_j \right)$ for $1 \leq j \leq N$, $1 \leq l \leq \tilde{N}$. Let $\Phi_l(x) =$

$\sum_{j=1}^l \mathcal{P}^j(x) + \mathcal{W}(|x|)$ for $1 \leq l \leq \tilde{N}$ and $\Phi_0(x) = \mathcal{W}(|x|)$. For simplicity, let

$\sigma_{k,\omega}^{(l)} = \sigma_{\Phi_l, k, \omega}$, $\lambda_{\omega, t}^{(l)} = \lambda_{\Phi_l, \omega, t}$ and $\sigma_w^{*(l)}(f)(x) = \sup_{k \in \mathbf{Z}} \left| \sigma_{k,\omega}^{(l)} * f(x) \right|$ for $1 \leq l \leq \tilde{N}$.

Now, by definition of $\lambda_{\omega, t}^{(l)}$, it is easy to verify that

$$(5.6) \quad \left\| \lambda_{\omega, t}^{(l)} \right\| \leq C,$$

$$(5.7) \quad \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \hat{\lambda}_{\omega, t}^{(l)}(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2} \text{ for } 1 \leq l \leq \tilde{N}.$$

By invoking Lemma 3.1 we have

$$(5.8) \quad \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \hat{\lambda}_{\omega,t}^{(l)}(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2} \left| \rho_\omega^{n_l k} L_l(\xi) \right|^{-\frac{\alpha_l}{\omega+1}}.$$

We also, by a change of variable we have

$$\begin{aligned} & \left| \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \hat{\lambda}_{\omega,t}^{(l)}(\xi) - \hat{\lambda}_{\omega,t}^{(l-1)}(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \right| \\ & \leq \left(\int_1^{\rho_\omega} \left(\int_{\mathbf{S}^{n-1}} \left| e^{-i\xi \cdot [\Phi_l(\rho_\omega^k ty) - \Phi_{l-1}(\rho_\omega^k ty)]} - 1 \right| |\Omega_\omega(y)| d\sigma(y) \right)^2 \frac{dt}{t} \right)^{1/2} \\ & \leq C(\omega + 1)^{1/2} \left| \rho_\omega^{n_l k} L_l(\xi) \right|. \end{aligned}$$

By combining the last estimate with the trivial estimate

$$\left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \hat{\lambda}_{\omega,t}^{(l)}(\xi) - \hat{\lambda}_{\omega,t}^{(l-1)}(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2}$$

we obtain

$$(5.9) \quad \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \hat{\lambda}_{\omega,t}^{(l)}(\xi) - \hat{\lambda}_{\omega,t}^{(l-1)}(\xi) \right|^2 \frac{dt}{t} \right)^{1/2} \leq C(\omega + 1)^{1/2} \left| \rho_\omega^{n_l k} L_l(\xi) \right|^{\frac{\alpha_l}{\omega+1}}.$$

Also, by Lemma 3.5 and the definition of $\lambda_{\omega,t}^{(l)}$ we get

$$(5.10) \quad \left\| \sup_{k \in \mathbf{Z}} \left(\int_{\rho_\omega^k}^{\rho_\omega^{k+1}} \left| \lambda_{\omega,t}^{(l)} * f \right| \frac{dt}{t} \right) \right\|_p \leq C_p(\omega + 1) \|f\|_p$$

for $1 \leq l \leq \tilde{N}$ and $1 < p \leq \infty$.

Assume $1 < \gamma \leq 2$. By duality, we have

$$\begin{aligned} \mathfrak{S}_{\mathcal{P}, \Omega_\omega}^{(\gamma)} f(x) &= \sup_{h \in \mathcal{H}_\gamma(\mathbf{R}_+)} \left| \int_0^\infty \int_{\mathbf{S}^{n-1}} h(t) f(x - \mathcal{P}(tu)) \Omega_\omega(u) d\sigma(u) \frac{dt}{t} \right| \\ &= \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} f(x - \mathcal{P}(tu)) \Omega_\omega(u) d\sigma(u) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'} \\ (5.11) \quad &= \left(\int_0^\infty \left| \lambda_{\omega,t}^{(\tilde{N})} * f(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'}. \end{aligned}$$

Now by (5.6)–(5.11) and invoking Lemma 4.3 we get (5.5) for $\gamma' \leq p < \infty$ if $1 < \gamma \leq 2$. For $\gamma = 1$, (5.5) follows easily by (5.6). The proof of Theorem 2.1 is complete. \square

Proof of Theorem 2.2. Assume that $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 \leq \gamma \leq \infty$ and Ω belongs to $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ for $1 \leq \gamma \leq 2$ and satisfies (1.5).

Proof of part (a). Notice that $(T_{\mathcal{P},\Omega,h}f)(x) = \lim_{\varepsilon \rightarrow 0} T_{\mathcal{P},\Omega,h}^{(\varepsilon)}f(x)$, where $T_{\mathcal{P},\Omega,h}^{(\varepsilon)}$ is the truncated singular integral operator given by

$$(5.12) \quad T_{\mathcal{P},\Omega,h}^{(\varepsilon)}f(x) = \int_{|y|>\varepsilon} f(x - \mathcal{P}(y))K_{\Omega,h}(y) dy.$$

Let us first consider the case $1 < \gamma \leq 2$. We follow a similar argument as in [19]. Without loss of generality, we may assume that $\|h\|_{L^\gamma(\mathbf{R}_+, dr/r)} = 1$. By (5.3) we deal with $T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}$ instead of $T_{\mathcal{P},\Omega,h}^{(\varepsilon)}$. Notice that, by Hölder's inequality we have

$$\begin{aligned} \left| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f(x) \right| &\leq \int_\varepsilon^\infty |h(t)| \left| \lambda_{\omega,t}^{(\tilde{N})} * f(x) \right| \frac{dt}{t} \\ &\leq \left(\int_0^\infty \left| \lambda_{\omega,t}^{(\tilde{N})} * f(x) \right|^{\gamma'} \frac{dt}{t} \right)^{1/\gamma'}. \end{aligned}$$

Therefore, $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f \right\|_p \leq \left\| \mathfrak{S}_{\mathcal{P},\Omega}^{(\gamma)}f \right\|_p \leq C(1 + \omega)^{1/\gamma'} \|f\|_p$ for $\gamma' \leq p < \infty$ and $1 < \gamma \leq 2$ and C is independent of ε . By a standard duality argument, $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f \right\|_p \leq C(1 + \omega)^{1/\gamma'} \|f\|_p$ for $1 < p \leq \gamma$ and $1 < \gamma \leq 2$. By a passage to the limit (as $\varepsilon \rightarrow 0$) and applying Fatou's lemma we get $\left\| T_{\mathcal{P},\Omega_\omega,h}f \right\|_p \leq C(1 + \omega)^{1/\gamma'} \|f\|_p$ for $\gamma' \leq p < \infty$ and for $1 < p \leq \gamma$. If $\gamma = 2$, then we are done; otherwise an application of the real interpolation theorem gives $\left\| T_{\mathcal{P},\Omega_\omega,h}f \right\|_p \leq C(1 + \omega)^{1/\gamma'} \|f\|_p$ for the remaining range of $p : \gamma < p < \gamma'$.

Now we consider the case $2 < \gamma \leq \infty$. By the above argument and by (5.3), Theorem 2.2 is proved for the case $2 < \gamma \leq \infty$ if we show that $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f \right\|_p \leq C_p(1 + \omega)^{1/\gamma'} \|f\|_p$ for $1 < p < \infty$. To this end, decompose $T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f = \sum_{k \in \mathbf{Z}} \sigma_{\mathcal{P},k,\omega} * f$ and then invoking Lemmas 3.3, 3.6, 3.7 and 4.1 along with following a similar argument as in the proof of Theorem 2.1 to get $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f \right\|_p \leq C_p(1 + \omega)^{1/\gamma'} \|f\|_p$, where C_p is independent of ε . In particular, $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)}f \right\|_p \leq C_p(1 + \omega)^{1/\gamma'} \|f\|_p$ for $2 \leq p < \infty$. By the routine duality

argument, $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)} f \right\|_p \leq C_p(1+\omega)^{1/\gamma'} \|f\|_p$ for $1 < p \leq 2$. This completes the proof of Theorem 2.2 (a).

Proof of Theorem 2.2 (b). Assume $\gamma = 1$. Again we deal with the truncated operator $T_{\mathcal{P},\Omega,h}^{(\varepsilon)}$ instead of $T_{\mathcal{P},\Omega,h}$. Without loss of generality, we may assume that $\|h\|_{L^\gamma(\mathbf{R}_+,dr/r)} = 1$. It is easy to see that $\left| T_{\mathcal{P},\Omega,h}^{(\varepsilon)} f(x) \right| \leq \|f\|_{L^\infty(\mathbf{R}^m)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})}$ for all $f \in L^\infty(\mathbf{R}^m)$ and for almost every $x \in \mathbf{R}^m$. In particular we have $\left\| T_{\mathcal{P},\Omega_\omega,h}^{(\varepsilon)} f \right\|_{L^\infty(\mathbf{R}^m)} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^\infty(\mathbf{R}^m)}$ for all $f \in S(\mathbf{R}^m)$. By the routine duality argument, we have $\left\| T_{\mathcal{P},\Omega,h}^{(\varepsilon)} f \right\|_{L^1(\mathbf{R}^m)} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^1(\mathbf{R}^m)}$ for all $f \in S(\mathbf{R}^m)$. Thus by interpolation between the last two estimates we get $\left\| T_{\mathcal{P},\Omega,h}^{(\varepsilon)} f \right\|_{L^p(\mathbf{R}^m)} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^m)}$ for $1 < p < \infty$ and all $f \in S(\mathbf{R}^m)$. Finally, using density argument we get $\left\| T_{\mathcal{P},\Omega,h}^{(\varepsilon)} f \right\|_{L^p(\mathbf{R}^m)} \leq \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^m)}$ for $1 \leq p < \infty$ and for all $f \in L^p(\mathbf{R}^m)$. \square

Proof of Theorem 2.3. By (5.3), we have

$$\mathcal{M}_{\mathcal{P},\Omega,h} f(x) \leq \sum_{\omega \in D \cup \{0\}} \left\| \Omega \chi_{\mathbf{E}_\omega} \right\|_{L^1(\mathbf{S}^{n-1})} \mathcal{M}_{\mathcal{P},|\Omega_\mu|,h}.$$

By Lemma 3.6 and noticing that

$$\left\| \mathcal{M}_{\mathcal{P},|\Omega_\mu|,h} f \right\|_p \leq C \left\| \sigma_{\mathcal{P},\omega}^*(|f|) \right\|_p \leq C_p(1+\omega)^{1/\gamma'} \|f\|_p,$$

we get

$$\begin{aligned} \left\| \mathcal{M}_{\mathcal{P},\Omega,h} f \right\|_p &\leq C_p \sum_{\omega \in D \cup \{0\}} (1+\omega)^{1/\gamma'} \left\| \Omega \chi_{\mathbf{E}_\omega} \right\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p \\ &\leq C_p \|\Omega\|_{L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})} \|f\|_p. \end{aligned}$$

The proof of the L^p boundedness of $T_{\mathcal{P},\Omega,h}^*$ follows by the above estimates and following the same argument as in [5] (see also [17]). We omit the details. \square

Proof of Theorem 2.4. The proof of this theorem follows by the above estimates and the arguments in [5]. Again we omit the details. \square

§6. A further result

We shall end the paper by presenting an additional result.

Theorem 6.1. *Let $h \in \mathcal{H}_\gamma(\mathbf{R}_+)$ for some $1 \leq \gamma \leq \infty$. Let $P(x)$ be a real-valued polynomial on \mathbf{R}^n and $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ and satisfies (1.5). Define the operator H on \mathbf{R}^n by*

$$Hf(x) = p.v. \int_{\mathbf{R}^n} e^{iP(x-y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy$$

Then H is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ if $1 < \gamma \leq \infty$ and for $1 \leq p < \infty$ if $\gamma = 1$. The bound for of the L^p norm of H may depend on the degree of the polynomial P , but it is independent of the coefficients of P .

By a well-known method, Theorem 6.1 follows from Theorem 2.2. For more information, see the proof of Theorem 9.1 in [16].

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