

Multinomial Coefficients and the Johnson Scheme

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Abstract. The Johnson scheme is one of the most famous association schemes. The structure is based on binomial coefficients. On the other hand, the generalized Johnson scheme (q -analog of the Johnson scheme) is based on q -binomial coefficients. In this paper, we consider the association scheme which is based on multinomial coefficients and compute intersection numbers (structure constants) of it.

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§1. Introduction

Let V be a set such that $|V| = v$ and let k be a positive integer ($k \leq \frac{v}{2}$). Let X be the set of all k -element subsets of V . Define the i -th relation R_i by $(x, y) \in R_i$, ($x, y \in X$) if and only if $|x \cap y| = k - i$. We get a symmetric association scheme of class k ; $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq k})$. \mathfrak{X} is called the Johnson scheme $J(v, k)$ [2]. This scheme comes from the symmetric group S_v of degree v and the subgroup $S_{v-k} \times S_k$ of S_v . The structure is based on binomial coefficients; e.g., the number of vertices $|X| = \binom{v}{k}$, the intersection numbers and the values of characters.

In this paper, we consider the association schemes whose structure is based on multinomial coefficients. The schemes come from symmetric groups S_n and their subgroups of the form $S_{n_1} \times \cdots \times S_{n_l}$ of S_n . Our association scheme is a natural generalization of the Johnson scheme. This structure is studied by some researchers. Dunkl and Scarabotti computed the character table of this scheme for a special parameter [3, 8]. Krieg and Anderson showed this association scheme was non-commutative whenever $l > 2$ [7, 1]. An algorithm was presented for determining the number of relations of this scheme by Jones [5]. We compute the intersection numbers of the association scheme. Our

results should be helpful for investigation the structure of these association schemes. In Section 2, the notation of association schemes is introduced. In Section 3, we consider a generalization of the combinatorial structure of the Johnson schemes. In Section 4, we compute the intersection numbers of the association scheme.

§2. Association Schemes

Let X be a finite set, R a collection of binary relations on X . For $x \in X$ and $r \subset X \times X$, we put $xr := \{y \in X : (x, y) \in r\}$, $rx := \{y \in X : (y, x) \in r\}$. If the following conditions are satisfied, we say that (X, R) is an association scheme [2, 4];

- (i) R is a partition of $X \times X$;
- (ii) $1 := \{(x, x) : x \in X\} \in R$;
- (iii) If $r \in R$, then $r^* := \{(y, x) : (x, y) \in r\} \in R$;
- (iv) For $f, g, h \in R$, there exists $p_{f,g}^h \in \mathbb{Z}_{\geq 0}$ such that $p_{f,g}^h = |xf \cap gy|$ whenever $(x, y) \in h$.

Let G be a finite group and H a subgroup of G . Let $\Omega = H \backslash G = \{Hg : g \in G\}$ be the set of right cosets of H in G . We define the action of G on $\Omega \times \Omega$ by the diagonal action. Let R be the set of orbits of G on $\Omega \times \Omega$. Then $\mathfrak{X} = (H \backslash G, R)$ is an association scheme. In particular, the correspondence (Hg_1, Hg_2) to $Hg_2g_1^{-1}H$ gives a bijection between R and the set of double cosets of H in G . So $\mathfrak{X} = (H \backslash G, \{Ha_iH\}_{0 \leq i \leq d})$ is an association scheme.

§3. Combinatorial Structures

Let G be a symmetric group $S_{\{1,2,\dots,n\}}$. For a partition of n $((n_1, n_2, \dots, n_l) \vdash n, n_1 + n_2 + \dots + n_l = n, n_1 \geq n_2 \geq \dots \geq n_l > 0, l \geq 2)$, let \mathcal{P} be the set whose members are arrays. Each array is a partition $[Q_1, Q_2, \dots, Q_l]$ of a permutation of a multiset $\underbrace{\{1, \dots, 1\}}_{n_1}, \underbrace{\{2, \dots, 2\}}_{n_2}, \dots, \underbrace{\{l-1, \dots, l-1\}}_{n_{l-1}}, \underbrace{\{l, \dots, l\}}_{n_l}$ (a multiset differs from a set in that each member has a multiplicity) such that $|Q_i| = n_i$ for each $i = 1, 2, \dots, l$.

Example 3.1. Let $\lambda = (2, 2, 1) \vdash 5$. Then we have $\mathcal{P} = \{ [\{1, 1\}, \{2, 2\}, \{3\}], [\{1, 2\}, \{2, 3\}, \{1\}], [\{3, 1\}, \{1, 2\}, \{2\}], [\{2, 2\}, \{3, 1\}, \{1\}], [\{2, 3\}, \{1, 1\}, \{2\}], [\{1, 2\}, \{1, 2\}, \{3\}], [\{2, 3\}, \{2, 1\}, \{1\}], [\{2, 1\}, \{3, 1\}, \{2\}], [\{1, 3\}, \{2, 2\}, \{1\}], [\{3, 2\}, \{1, 1\}, \{2\}], [\{2, 1\}, \{2, 3\}, \{1\}], [\{1, 3\}, \{1, 2\}, \{2\}], [\{1, 2\}, \{3, 2\}, \{1\}] \}$

$[\{3, 1\}, \{2, 1\}, \{2\}], [\{2, 2\}, \{1, 3\}, \{1\}], [\{1, 2\}, \{2, 1\}, \{3\}], [\{2, 3\}, \{1, 2\}, \{1\}],$
 $[\{2, 1\}, \{1, 3\}, \{2\}], [\{2, 1\}, \{3, 2\}, \{1\}], [\{1, 3\}, \{2, 1\}, \{2\}], [\{1, 1\}, \{3, 2\}, \{2\}],$
 $[\{1, 2\}, \{3, 1\}, \{2\}], [\{3, 2\}, \{2, 1\}, \{1\}], [\{2, 1\}, \{1, 2\}, \{3\}], [\{1, 1\}, \{2, 3\}, \{2\}],$
 $[\{1, 2\}, \{1, 3\}, \{2\}], [\{3, 2\}, \{1, 2\}, \{1\}], [\{2, 1\}, \{2, 1\}, \{3\}], [\{3, 1\}, \{2, 2\}, \{1\}],$
 $[\{2, 2\}, \{1, 1\}, \{3\}] \}.$

Notation 1. Let $\lambda = (n_1, n_2, \dots, n_l) \vdash n$. Then we have

$$\begin{aligned}
 |\mathcal{P}| &= \binom{n}{\substack{l \\ n_k}} := \binom{n}{n_1, n_2, \dots, n_l} \\
 &= \binom{n}{n_1} \binom{n - n_1}{n_2} \dots \binom{n - n_1 - n_2 - \dots - n_{l-1}}{n_l} \\
 &= \frac{n!}{n_1! \dots n_l!}.
 \end{aligned}$$

Let P be the following array;

$$P := [\underbrace{\{1, \dots, 1\}}_{n_1}, \underbrace{\{2, \dots, 2\}}_{n_2}, \dots, \underbrace{\{l-1, \dots, l-1\}}_{n_{l-1}}, \underbrace{\{l, \dots, l\}}_{n_l}] \in \mathcal{P}.$$

For any $T \in \mathcal{P}$, denote the i -th entry of T as $T(i)$. So $P(1) = P(2) = \dots = P(n_1) = 1$, $P(n_1 + 1) = P(n_1 + 2) = \dots = P(n_1 + n_2) = 2$, \dots , $P(\sum_{k=1}^{l-1} n_k + 1) = P(\sum_{k=1}^{l-1} n_k + 2) = \dots = P(n) = l$. We define the action of G to P . For any $\pi \in G$,

$$\begin{aligned}
 P^\pi &:= [\underbrace{\{P(\pi(1)), \dots, P(\pi(n_1))\}}_{n_1}, \underbrace{\{P(\pi(n_1 + 1)), \dots, P(\pi(n_1 + n_2))\}}_{n_2}, \\
 &\quad \dots, \underbrace{\{P(\pi(\sum_{k=1}^{l-1} n_k + 1)), \dots, P(\pi(n))\}}_{n_l}].
 \end{aligned}$$

G is a transitive permutation group on \mathcal{P} . $H := S_{\{1, 2, \dots, n_1\}} \times S_{\{n_1 + 1, \dots, n_1 + n_2\}} \times \dots \times S_{\{n_1 + \dots + n_{l-1} + 1, \dots, n_1 + \dots + n_l\}}$ is the stabilizer of $P \in \mathcal{P}$. Let $H \backslash G := \{H = Hg_0, Hg_1, \dots\}$ be the right cosets of G by H . For any $T \in \mathcal{P}$, define $R_T := \{\pi \in G : P^\pi = T\}$ and $L_T := \{\pi \in G : T^\pi = P\}$, then the following consists.

Lemma 1. For any $g \in G$, let $T := P^g$ be an element of \mathcal{P} . Then

$$Hg = R_T.$$

Similarly, $g^{-1}H = L_T$.

By Lemma 1, there is a bijective correspondence between $\{R_T : T \in \mathcal{P}\}$ and right cosets $H \backslash G$. For each $T \in \mathcal{P}$, there is a unique coset $Hg \in H \backslash G$ such that

$$(3.1) \quad P^\pi = T \quad \text{for all } \pi \text{ in } Hg.$$

So there is a bijective correspondence between \mathcal{P} and $H \backslash G$. Identify $H \backslash G$ with \mathcal{P} by the correspondence (3.1).

For any $T_1, T_2 \in \mathcal{P}$, we define the action of G on $\mathcal{P} \times \mathcal{P}$ by $(T_1, T_2)^\pi := (T_1^\pi, T_2^\pi)$, $\pi \in G$. Let $\Lambda_0, \Lambda_1, \dots, \Lambda_d$ be the orbits of G on $\mathcal{P} \times \mathcal{P}$,

$$\mathcal{P} \times \mathcal{P} = \Lambda_0 \cup \Lambda_1 \cup \dots \cup \Lambda_d,$$

where $\Lambda_0 = \{(T, T) : T \in \mathcal{P}\}$.

Lemma 2. *Let $\Lambda_i(P) = \{T \in \mathcal{P} \mid (P, T) \in \Lambda_i\}$. Then $\Lambda_0(P) = \{P\}$, $\Lambda_1(P)$, \dots , $\Lambda_d(P)$ are H -orbits on \mathcal{P} .*

Let $H = Ha_0H, Ha_1H, \dots, Ha_dH$ be the double cosets of G by H . For each $HaH \in H \backslash G/H$, there is a unique orbit Λ_i such that $P^{HaH} \in \Lambda_i(P)$. So there is a one-to-one correspondence between the double cosets of G by H and the orbits of G on $\mathcal{P} \times \mathcal{P}$. Identifying $H \backslash G/H$ with $\{\Lambda_i\}$ by the correspondence. Since $\Lambda_0(P) = \{P\}$, $a_0 = 1_G$. Since $\mathfrak{X} = (H \backslash G, \{Ha_iH\}_{0 \leq i \leq d})$ is an association scheme, $(\mathcal{P}, \{\Lambda_i\}_{0 \leq i \leq d})$ is an association scheme, too.

We think about the number of relations of this association scheme. By Lemma 2, $T_1, T_2 \in \Lambda_i(P)$ if and only if there exists $h \in H$ such that $T_1 = T_2^h$. This means becoming the same array as T_1 when the order in each component is disregarded in T_2 . Let us reword this condition. Let T^* denote the array obtained from T ($\in \mathcal{P}$) by permuting the integers in each component of T into weak increasing order. T^* is called a permissible array[5]. We note that $(T^h)^* = T^*$ for any $h \in H$ and $T \in \mathcal{P}$ since h acts only in each component of T . Let $\mathcal{P}^* := \{T^* : T \in \mathcal{P}\}$. For each $T^* \in \mathcal{P}^*$, there is a unique orbit Λ such that

$$(3.2) \quad T^* \in \Lambda(P).$$

So there is a one-to-one correspondence between \mathcal{P}^* and $\{\Lambda_i\}_{0 \leq i \leq d}$. Identifying $\{\Lambda_i\}$ with \mathcal{P}^* by correspondence (3.2). Then $d + 1 = |\mathcal{P}^*|$.

Example 3.2. Let $\lambda = (2, 2, 1) \vdash 5$. Then we have $\mathcal{P}^* = \{ [\{2, 3\}, \{1, 2\}, \{1\}], [\{1, 3\}, \{2, 2\}, \{1\}], [\{2, 2\}, \{1, 3\}, \{1\}], [\{1, 2\}, \{2, 3\}, \{1\}], [\{2, 3\}, \{1, 1\}, \{2\}], [\{1, 3\}, \{1, 2\}, \{2\}], [\{1, 2\}, \{1, 3\}, \{2\}], [\{1, 1\}, \{2, 3\}, \{2\}], [\{2, 2\}, \{1, 1\}, \{3\}], [\{1, 2\}, \{1, 2\}, \{3\}], [\{1, 1\}, \{2, 2\}, \{3\}] \}$. So $|\mathcal{P}^*| = 11$.

Actually, we want to compute the number of relations. The algorithm has already been known in [5]. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i) \vdash m$ and let $\mu = (\mu_1, \mu_2, \dots, \mu_j)$ be a sequence of natural numbers the sum of which is equal to m . Then define $p(\lambda, \mu, r)$ to be the number of arrays with shape λ and content μ such that the symbols weakly increase along each row, under the additional restriction that the last element in row i is less than or equal to r . Then, using this notation, we have the number of permissible arrays is equal to $p(\lambda, \mu, l)$. The number of permissible arrays is computed by recursively reducing an array by one entry at a time until a trivial situation is achieved.

Algorithm 3 (Jones [5]). *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i) \vdash m \leq n$ and let $\mu = (\mu_1, \mu_2, \dots, \mu_j)$ be a sequence of natural numbers, the sum of which is equal to m .*

(i) *If $\lambda_i = 1$ then $p(\lambda, \mu, r) = \sum_{s \leq r, \mu_s > 0} p(\bar{\lambda}, \bar{\mu}, l)$,*
where $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_{i-1})$, $\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_{s-1}, \mu_s - 1, \mu_{s+1}, \dots, \mu_j)$
and l is the length of μ .

(ii) *If $\lambda_i > 1$ then $p(\lambda, \mu, r) = \sum_{s \leq r, \mu_s > 0} p(\bar{\lambda}, \bar{\mu}, s)$,*
where $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_i - 1)$, $\bar{\mu} = (\mu_1, \mu_2, \dots, \mu_{s-1}, \mu_s - 1, \mu_{s+1}, \dots, \mu_j)$.

The following results are convenient to compute $p(\lambda, \mu, l)$.

- If $\mu = (0, \dots, 0, m, 0, \dots, 0)$, then

$$p(\lambda, \mu, r) = \begin{cases} 1, & s \leq r; \\ 0, & s > r; \end{cases}$$

- If $\mu_1 = \mu_2 = \dots = \mu_r = 0$, then $p(\lambda, \mu, r) = 0$;
- If $\lambda = (m)$, then $p(\lambda, \mu, r) = 1$.

Let $H := S_{\{1, \dots, \lambda_1\}} \times \dots \times S_{\{\lambda_1 + \dots + \lambda_{i-1} + 1, \dots, \lambda_1 + \dots + \lambda_i\}}$ and $K := S_{\{1, \dots, \mu_1\}} \times \dots \times S_{\{\mu_1 + \dots + \mu_{j-1} + 1, \dots, \mu_1 + \dots + \mu_j\}}$ be Young subgroups of the symmetric group $S_{\{1, \dots, n\}}$. The above recurrence formulae equal to the number of (H, K) -double cosets of $S_{\{1, \dots, n\}}$. This means

$$d + 1 = |\mathcal{P}^*| = p(\lambda, \lambda, l),$$

where l is the length of λ .

Example 3.3. Let $\lambda = (2, 2, 1)$. Then we have

$$\begin{aligned}
p(\lambda, \lambda, 3) &= p((2, 2), (1, 2, 1), 3) + p((2, 2), (2, 1, 1), 3) + p((2, 2), (2, 2, 0), 3) \\
&= p((2, 1), (0, 2, 1), 1) + p((2, 1), (1, 1, 1), 2) + p((2, 1), (1, 2, 0), 3) \\
&\quad + p((2, 1), (1, 1, 1), 1) + p((2, 1), (2, 0, 1), 2) + p((2, 1), (2, 1, 0), 3) \\
&\quad + p((2, 1), (1, 2, 0), 1) + p((2, 1), (2, 1, 0), 2) \\
&= p((2), (0, 1, 1), 3) + p((2), (1, 0, 1), 3) \\
&\quad + p((2), (0, 2, 0), 3) + p((2), (1, 1, 0), 3) \\
&\quad + p((2), (0, 1, 1), 3) + p((2), (1, 0, 1), 3) \\
&\quad + p((2), (1, 1, 0), 3) + p((2), (2), 3) \\
&\quad + p((2), (0, 2, 0), 3) + p((2), (1, 1, 0), 3) + p((2), (2), 3) \\
&= 11.
\end{aligned}$$

§4. Intersection Numbers

For a partition λ of n , \mathcal{P} is defined as well as the last section as a set of partition arrays. For any $S, U \in \mathcal{P}$, (S, U) is defined as follows.

$$(S, U) := [Q_1(S, U), Q_2(S, U), \dots, Q_l(S, U)] \in \mathcal{P}.$$

where $Q_i(S, U)$ is a multiset such that $\{U(h) : S(h) = i \ (1 \leq h \leq n)\}$.

Let $(S, U)^*$ denote the array obtained from (S, U) by permuting the integers in each component of (S, U) into weak increasing order. We note that $\mathcal{P}^* = \{(S, U)^* : \forall S, U \in \mathcal{P}\}$ and $(P^{\pi_1}, P^{\pi_2})^* = (P^{\pi_2 \pi_1^{-1}})^* \in \mathcal{P}^*$. For any $(P^{\pi_1}, P^{\pi_2})^* \in \mathcal{P}^*$, there exist a unique orbit Λ such that $(P^{\pi_1}, P^{\pi_2})^* \in \Lambda$. So there is a one-to-one correspondence between the orbits of G on $\mathcal{P} \times \mathcal{P}$ and \mathcal{P}^* . Identifying $\{\Lambda_i\}$ with \mathcal{P}^* by the correspondence. By the last section, since $(\mathcal{P}, \{\Lambda_i\}_{0 \leq i \leq d})$ is an association scheme, the following consists.

Theorem 4. $(\mathcal{P}, \mathcal{P}^*)$ is an association scheme.

For any partition $\lambda \vdash n$, let $J(\lambda)$ be this association scheme. We want to compute the intersection numbers so that we need to consider combinatorial structures. In the remainder of this paper, for any permissible array $T \in \mathcal{P}^*$, we treat the set $\{(S, U) : (S, U)^* = T\}$ as T . The following consist.

(i)

$$\mathcal{P} \times \mathcal{P} = \bigcup_{T \in \mathcal{P}^*} T,$$

where $T_1 \cap T_2 = \emptyset$ ($T_1, T_2 \in \mathcal{P}^*$, $T_1 \neq T_2$);

- (ii) $\{(T, T)^* : T \in \mathcal{P}\} \in \mathcal{P}^*$;
- (iii) If $T \in \mathcal{P}^*$, then $T' \in \mathcal{P}^*$, where $T' = \{(U, S) : (S, U) \in T\}$;
- (iv) For $\alpha, \beta, \gamma \in \mathcal{P}^*$, there exists a non-negative integer $p_{\beta, \gamma}^\alpha$ such that $p_{\beta, \gamma}^\alpha = |\{T \in \mathcal{P} : (S, T) \in \beta, (T, U) \in \gamma\}|$ whenever $(S, U) \in \alpha$.

Notation 2. For any $T \in \mathcal{P}$,

$$Q_i(T, T) = \{T(h) : T(h) = i(1 \leq h \leq n)\} = \{i, \dots, i\}.$$

Hence,

$$(T, T)^* = P = [\{1, \dots, 1\}, \{2, \dots, 2\}, \dots, \{l, \dots, l\}] \in \mathcal{P}^*.$$

Notation 3. The association scheme is non-commutative whenever $l > 2$ [7, 1].

For any $S, U \in \mathcal{P}$, we define $l \times l$ matrix $\epsilon^{S, U}$;

$$(\epsilon^{S, U})_{i, j} := |\{h : S(h) = i \text{ and } U(h) = j \ (1 \leq h \leq n)\}| = |\{x : x \in Q_i(S, U), x = j\}|.$$

Their row sums and their column sums of the matrix are n_1, n_2, \dots, n_l .

On the other hand, let M_{n_1, n_2, \dots, n_l} be the set of $l \times l$ matrices over $\mathbb{N} = \{0, 1, 2, \dots\}$ with row sums (n_1, n_2, \dots, n_l) and column sums (n_1, n_2, \dots, n_l) . For any matrix $A \in M_{n_1, n_2, \dots, n_l}$, we get a unique permissible array;

$$[\underbrace{\{1, \dots, 1\}}_{A_{1,1}}, \underbrace{\{2, \dots, 2\}}_{A_{1,2}}, \dots, \underbrace{\{l, \dots, l\}}_{A_{1,l}}, \dots, \underbrace{\{1, \dots, 1\}}_{A_{l,1}}, \underbrace{\{2, \dots, 2\}}_{A_{l,2}}, \dots, \underbrace{\{l, \dots, l\}}_{A_{l,l}}],$$

where $A_{i,j}$ is (i, j) -entry of A . There is a bijective correspondence between \mathcal{P}^* and M_{n_1, n_2, \dots, n_l} [6]. Identify \mathcal{P}^* with M_{n_1, n_2, \dots, n_l} . Hence each relation is presented by a matrix of M_{n_1, n_2, \dots, n_l} .

Proposition 5. $\epsilon^{S, U} = {}^t \epsilon^{U, S}$. In particular, $(S, U)^* = (U, S)^*$ if and only if $\epsilon^{S, U}$ is a symmetric matrix.

Notation 4. The valency v_α of $\alpha \in \mathcal{P}^*$ was computed in [5];

$$\begin{aligned} v_\alpha &= \binom{n_1}{\alpha_{1,1}, \dots, \alpha_{1,l}} \binom{n_2}{\alpha_{2,1}, \dots, \alpha_{2,l}} \cdots \binom{n_l}{\alpha_{l,1}, \dots, \alpha_{l,l}} \\ &= \frac{n_1! \cdots n_l!}{\alpha_{1,1}! \alpha_{1,2}! \cdots \alpha_{l,l}!}, \end{aligned}$$

where $0! = 1$.

Now we calculate the intersection numbers of $J(\lambda)$. For $S, U \in \mathcal{P}$ such that $\epsilon^{S,U} = \alpha$. This means that

$$\alpha_{i,j} = |\{h : S(h) = i \text{ and } U(h) = j \ (1 \leq h \leq n)\}|.$$

Let us count the number of $T \in \mathcal{P}$ such that $\epsilon^{S,T} = \beta$, $\epsilon^{T,U} = \gamma$. That is

$$\begin{aligned} \beta_{i,j} &= |\{h : S(h) = i \text{ and } T(h) = j \ (1 \leq h \leq n)\}|; \\ \gamma_{i,j} &= |\{h : T(h) = i \text{ and } U(h) = j \ (1 \leq h \leq n)\}|. \end{aligned}$$

And we define $t_k^{i,j}$ ($1 \leq i, j, k \leq l-1$);

$$t_k^{i,j} := |\{h : S(h) = i \text{ and } U(h) = j \text{ and } T(h) = k \ (1 \leq h \leq n)\}|.$$

For fixed $t_k^{i,j}$ ($1 \leq i, j, k \leq l-1$), we compute $t_l^{i,j}$ except $1 \leq i, j, k \leq l-1$ by conditions of α, β, γ . For any i, j ($1 \leq i, j \leq l-1$), $t_l^{i,j}$ is the number of h ($1 \leq h \leq n$) such that $S(h) = i$, $U(h) = j$ and $T(h) = l$. Since $\alpha_{i,j} = \sum_{k=1}^{l-1} t_k^{i,j}$,

$$t_l^{i,j} = \alpha_{i,j} - \sum_{k=1}^{l-1} t_k^{i,j}, \quad 1 \leq i, j \leq l-1.$$

Similarly, the following equations consist.

$$\begin{aligned} t_k^{i,l} &= \beta_{i,k} - \sum_{j=1}^{l-1} t_k^{i,j}, \quad 1 \leq i, k \leq l-1; \\ t_l^{i,l} &= \beta_{i,l} - \sum_{j=1}^{l-1} t_l^{i,j} = \beta_{i,l} - \sum_{j=1}^{l-1} (\alpha_{i,j} - \sum_{k=1}^{l-1} t_k^{i,j}), \quad 1 \leq i \leq l-1; \\ t_k^{l,j} &= \gamma_{k,j} - \sum_{i=1}^{l-1} t_k^{i,j} = \gamma_{k,j} - \sum_{i=1}^{l-1} t_k^{i,j}, \quad 1 \leq j, k \leq l-1; \\ t_l^{l,j} &= \gamma_{l,j} - \sum_{i=1}^{l-1} t_l^{i,j} = \gamma_{l,j} - \sum_{i=1}^{l-1} (\alpha_{i,j} - \sum_{k=1}^{l-1} t_k^{i,j}), \quad 1 \leq j \leq l-1; \\ t_k^{l,l} &= \gamma_{k,l} - \sum_{i=1}^{l-1} t_k^{i,l} = \gamma_{k,l} - \sum_{i=1}^{l-1} (\beta_{i,k} - \sum_{j=1}^{l-1} t_k^{i,j}), \quad 1 \leq k \leq l-1; \\ t_l^{l,l} &= \gamma_{l,l} - \sum_{i=1}^{l-1} t_l^{i,l} = \gamma_{l,l} - \sum_{i=1}^{l-1} (\beta_{i,l} - \sum_{j=1}^{l-1} (\alpha_{i,j} - \sum_{k=1}^{l-1} t_k^{i,j})). \end{aligned}$$

For fixed $t_1^{1,1}$, there are $\binom{\alpha_{1,1}}{t_1^{1,1}}$ choices of h ($1 \leq h \leq n$) such that $T(h) = 1$ on the condition $S(h) = 1$ and $U(h) = 1$. Repeating the same arguments, for fixed $\{t_k^{i,j}\} (1 \leq i, j, k \leq l-1)$, there are

$$\prod_{i,j=1}^l \binom{\alpha_{i,j}}{\sum_{k=1}^l t_k^{i,j}} = \binom{\alpha_{1,1}}{\sum_{k=1}^l t_k^{1,1}} \binom{\alpha_{1,2}}{\sum_{k=1}^l t_k^{1,2}} \cdots \binom{\alpha_{l,l}}{\sum_{k=1}^l t_k^{l,l}}$$

choices of T . We should decide the ranges of $\{t_k^{i,j}\}(1 \leq i, j, k \leq l-1)$. But to avoid becoming complex, we think them in rough ranges; $0 \leq t_k^{i,j} \leq \alpha_{i,j}$. We obtain the formula for the intersection numbers;

Theorem 6. For $\alpha, \beta, \gamma \in \mathcal{P}^*$,

$$p_{\beta, \gamma}^{\alpha} = \sum_{\{0 \leq t_k^{i,j} \leq \alpha_{i,j}\}_{1 \leq i, j, k \leq l-1}} \prod_{i, j=1}^l \binom{\alpha_{i,j}}{t_k^{i,j}}.$$

Example 4.1. Let $l = 2$ and let $n_1 := a \geq n_2 := b$. This is the Johnson scheme $J(a+b, b)$. Relations are the set of 2×2 matrices over $\mathbb{N} = \{0, 1, 2, \dots\}$ with row sums (a, b) and column sums (a, b) . Each of the matrices is decided when $(2, 2)$ -entry of the matrix is decided. The relations of the Johnson scheme actually make these one parameters correspond to the indices of the relations. For $0 \leq e, f, g \leq b$,

$$\alpha = \begin{pmatrix} a-e & e \\ e & b-e \end{pmatrix}, \quad \beta = \begin{pmatrix} a-f & f \\ f & b-f \end{pmatrix}, \quad \gamma = \begin{pmatrix} a-g & g \\ g & b-g \end{pmatrix}.$$

On the intersection numbers, the variable is only $t := t_1^{1,1}$ and let $T_{i,j} := (t_1^{i,j}, t_2^{i,j})$, then we have

$$\begin{aligned} T_{1,1} &= (t, a-e-t); \\ T_{1,2} &= (a-f-t, e-(a-f-t)); \\ T_{2,1} &= (a-g-t, e-(a-g-t)); \\ T_{2,2} &= (f+g+t-a, b-e-(f+g+t-a)). \end{aligned}$$

We compute the intersection numbers;

$$\begin{aligned} p_{\beta, \gamma}^{\alpha} &= \sum_{0 \leq t \leq a-e} \binom{a-e}{t; a-e-t} \binom{e}{a-f-t; e-(a-f-t)} \\ &\quad \times \binom{e}{a-g-t; e-(a-g-t)} \binom{b-e}{f+g+t-a; b-e-(f+g+t-a)} \\ &= \sum_{t=0}^{a-e} \binom{a-e}{t} \binom{e}{a-f-t} \binom{e}{a-g-t} \binom{b-e}{f+g+t-a}. \end{aligned}$$

Example 4.2. Let $l = 3$ and let $\lambda = (n_1, n_2, n_3) \vdash n$ ($n_1 \geq n_2 \geq n_3 > 0$). Relations are the set of 3×3 matrices over $\mathbb{N} = \{0, 1, 2, \dots\}$ with row sums (n_1, n_2, n_3) and column sums (n_1, n_2, n_3) . The number of relations is $p(\lambda, \lambda, 3)$. On the intersection numbers, the variables are $t_k^{i,j}$, $1 \leq i, j, k \leq 2$ and let

$T_{i,j} := (t_1^{i,j}, t_2^{i,j}, t_3^{i,j})$ for 3 relations α, β, γ . Then we have

$$\begin{aligned}
T_{1,1} &= (t_1^{1,1}, t_2^{1,1}, \alpha_{1,1} - t_1^{1,1} - t_2^{1,1}); \\
T_{1,2} &= (t_1^{1,2}, t_2^{1,2}, \alpha_{1,2} - t_1^{1,2} - t_2^{1,2}); \\
T_{1,3} &= (\beta_{1,1} - t_1^{1,1} - t_1^{1,2}, \beta_{1,2} - t_2^{1,1} - t_2^{1,2}, \beta_{1,3} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2})); \\
T_{2,1} &= (t_1^{2,1}, t_2^{2,1}, \alpha_{2,1} - t_1^{2,1} - t_2^{2,1}); \\
T_{2,2} &= (t_1^{2,2}, t_2^{2,2}, \alpha_{2,2} - t_1^{2,2} - t_2^{2,2}); \\
T_{2,3} &= (\beta_{2,1} - t_1^{2,1} - t_1^{2,2}, \beta_{2,2} - t_2^{2,1} - t_2^{2,2}, \beta_{2,3} - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1}) - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2})); \\
T_{3,1} &= (\gamma_{1,1} - t_1^{1,1} - t_1^{2,1}, \gamma_{2,1} - t_2^{1,1} - t_2^{2,1}, \gamma_{3,1} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1})); \\
T_{3,2} &= (\gamma_{1,2} - t_1^{1,2} - t_1^{2,2}, \gamma_{2,2} - t_2^{1,2} - t_2^{2,2}, \gamma_{3,2} - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2}) - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2})); \\
T_{3,3} &= (\gamma_{1,3} - (\beta_{1,1} - t_1^{1,1} - t_1^{1,2}) - (\beta_{2,1} - t_1^{2,1} - t_1^{2,2}), \gamma_{2,3} - (\beta_{1,2} - t_2^{1,1} - t_2^{1,2}) \\
&\quad - (\beta_{2,2} - t_2^{2,1} - t_2^{2,2}), \gamma_{3,3} - (\beta_{1,3} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2})) \\
&\quad - (\beta_{2,3} - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1}) - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2}))).
\end{aligned}$$

We compute the intersection numbers;

$$\begin{aligned}
p_{\beta, \gamma}^{\alpha} &= \sum_{\{0 \leq i, j \leq \alpha_{i,j}\}_{1 \leq i, j, k \leq 2}} \binom{\alpha_{1,1}}{t_1^{1,1}, t_2^{1,1}; \alpha_{1,1} - t_1^{1,1} - t_2^{1,1}} \binom{\alpha_{1,2}}{t_1^{1,2}, t_2^{1,2}; \alpha_{1,2} - t_1^{1,2} - t_2^{1,2}} \\
&\quad \binom{\alpha_{1,3}}{\beta_{1,1} - t_1^{1,1} - t_1^{1,2}; \beta_{1,2} - t_2^{1,1} - t_2^{1,2}; \beta_{1,3} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2})} \\
&\quad \binom{\alpha_{2,1}}{t_1^{2,1}, t_2^{2,1}; \alpha_{2,1} - t_1^{2,1} - t_2^{2,1}} \binom{\alpha_{2,2}}{t_1^{2,2}, t_2^{2,2}; \alpha_{2,2} - t_1^{2,2} - t_2^{2,2}} \\
&\quad \binom{\alpha_{2,3}}{\beta_{2,1} - t_1^{2,1} - t_1^{2,2}; \beta_{2,2} - t_2^{2,1} - t_2^{2,2}; \beta_{2,3} - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1}) - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2})} \\
&\quad \binom{\alpha_{3,1}}{\gamma_{1,1} - t_1^{1,1} - t_1^{2,1}; \gamma_{2,1} - t_2^{1,1} - t_2^{2,1}; \gamma_{3,1} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1})} \\
&\quad \binom{\alpha_{3,2}}{\gamma_{1,2} - t_1^{1,2} - t_1^{2,2}; \gamma_{2,2} - t_2^{1,2} - t_2^{2,2}; \gamma_{3,2} - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2}) - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2})} \\
&\quad \binom{\alpha_{3,3}}{\gamma_{1,3} - (\beta_{1,1} - t_1^{1,1} - t_1^{1,2}) - (\beta_{2,1} - t_1^{2,1} - t_1^{2,2}); \gamma_{2,3} - (\beta_{1,2} - t_2^{1,1} - t_2^{1,2}) - (\beta_{2,2} - t_2^{2,1} - t_2^{2,2}); \\
&\quad \gamma_{3,3} - (\beta_{1,3} - (\alpha_{1,1} - t_1^{1,1} - t_2^{1,1}) - (\alpha_{1,2} - t_1^{1,2} - t_2^{1,2})) - (\beta_{2,3} - (\alpha_{2,1} - t_1^{2,1} - t_2^{2,1}) \\
&\quad - (\alpha_{2,2} - t_1^{2,2} - t_2^{2,2}))}.
\end{aligned}$$

Some generalizations of the Johnson scheme are known. Especially this scheme is important since we want to consider the following problem in the representation theory [9, 10].

Problem 1. *Consider an analogue of Nakayama's conjecture for this scheme.*

Nakayama's conjecture is a theorem of representation theory for symmetric groups. The correspondence of a modular character and a p -core of Young diagrams are stated. In [9], the correspondence of a modular character of Bose-Mesner algebra of the Johnson scheme and a p -core of Young diagrams were given. Our results should be helpful for solving this problem.

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