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On the Killing vector fields of generalized metrics

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Abstract We consider a manifold endowed with a metric tensor in its tangent bundle pulled back by its own projection. We shall give necessary and sufficient conditions for a vector field to be an infinitesimal isometry of a metric of this type in general and for some special classes. We also examine translations, i.e., the special class of Killing vector fields whose integral curves are geodesics of an associated Finsler manifold. As applications, we determine the Killing vector fields of Funk metrics, and we give a new proof for the fact that perturbing a Riemannian manifold by a one-form metrically equivalent to a Killing field yields a Randers manifold for which the original vector field is a Killing field as well.

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§1. Introduction

By a generalized metric we shall mean a symmetric, non-degenerate $(0, 2)$ tensor in the pull-back bundle $\tau^*\tau$ of the tangent bundle $\tau : TM \to M$ over τ. The study of metrics of this type dates back to the 1950's [13, 25]. A new classification for them has been published recently [10]. These metrics are natural generalizations of Finsler structures, since manifolds endowed with generalized metrics are the most general spaces where 'the metric depends also on the direction'. Some of their characteristic properties in which they differ from Finsler manifolds were already pointed out in [13], e.g., the fact that their autoparallel and extremal curves do not necessarily coincide, even with a natural choice of a covariant derivative. These metrics may be interesting not only from a geometrical, but also from a physical viewpoint, since they furnish a natural geometric description of the so-called bilocal field theories introduced by Yukawa in the 1940's. Yukawa's main goal was to explain mass quantization and to eliminate certain types of divergences in quantum field theory. For bilocal field theories, we may refer to Yukawa's original papers [27, 28], or, for more recent reviews on multi-local theories, see [15, 22]. In this paper, however, we restrict ourselves to the geometric aspects of generalized metrics; we wish to consider physical implications in a later article.

The infinitesimal symmetries of space-time are expressed by so-called Killing vector fields in general relativity. Therefore, it is an important problem to determine the Killing vector fields of different classes of generalized metrics. In a Euclidean space, translations are distinguished from other types of isometries by the property that their orbits are straight lines. This property is used to generalize the notion of translations to more general classes of metrics: translations are Killing vector fields whose integral curves are at the same time geodesics (in some sense). In this paper we also study the translations of a certain type of generalized metrics.

The outline of the paper is the following. Sections 2 − 4 may be regarded as preparatory sections, since they contain no new results; they only make the paper more or less self-contained. Coming to the original results, in section 5 we have collected those which are relevant to all generalized metrics. We discuss the Killing vector fields of special types of metrics in section 6. In section 7 we study the translations of weakly normal and Miron regular metrics. Section 8 contains applications to Randers manifolds and Funk metrics. Finally, in section 9 we discuss some open problems.

§2. Preliminary constructions

We begin by recalling some definitions and basic facts concerning the technical tools that we shall use later. As a general reference, see [8, 21].

We work on an n-dimensional connected smooth manifold M whose topology is of Hausdorff type and has a countable base. The symbol $C^\infty(M)$ stands for the ring of smooth real-valued functions on M, and $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of (smooth) vector fields on M. The symbol $\tau : TM \to M$ is the tangent bundle of M, and the tangent bundle of TM is denoted by τ_{TM}. We shall denote the open submanifold of TM formed by the non-zero tangent vectors by $\overset{\circ}{T}M$, and the restriction of τ to $\overset{\circ}{T}M$ by $\overset{\circ}{\tau}$. If N is another manifold, and $f : M \to N$ is a smooth map, then its tangent map is denoted by $f_* : TM \to TN$. If f is a diffeomorphism, the push-forward of a vector field X on M by f is

$$f_\sharp X := f_* \circ X \circ f^{-1}.$$

A subset W of the product manifold $\mathbb{R} \times M$ is said to be *radial* if, for any $p \in M$, $W \cap (\mathbb{R} \times \{p\}) = I \times \{p\}$, where I is an open interval that contains $0 \in \mathbb{R}$.

Let X be a vector field on a manifold M. The *flow* of X is a map $\varphi : W \to M$ such that $W \subset \mathbb{R} \times M$ is a radial set, and $c_p := \varphi(.,p) : I_p \to M$ is the maximal integral curve of X starting from the point $p \in M$, i.e., $\dot{c}_p = X \circ c_p$, $c_p(0) = p$, and any other curve satisfying these two conditions is a restriction of c_p. If $W = \mathbb{R} \times M$, the vector field X is said to be *complete*.

If f is a smooth function on M, then the function

$$f^c : TM \to \mathbb{R}, \qquad v \in TM \mapsto f^c(v) := vf$$

is a smooth function on TM and is called the *complete lift* of f. It can be shown that any vector field on TM is determined by its action on complete lifts, and if $X \in \mathfrak{X}(M)$, there is a unique vector field X^c on TM such that $X^c f^c = (Xf)^c$ for any smooth function f on M [21]. The vector field X^c is said to be the *complete lift* of X. Let $\varphi : W \to M$ be the flow of X. If we fix the first argument of φ, the map $\varphi_t := \varphi(t,.)$ is a diffeomorphism between two open submanifolds of M, and the map

$$\tilde{\varphi} : (t,v) \mapsto \tilde{\varphi}(t,v) := (\varphi_t)_*(v) \qquad ((t,\tau(v)) \in W)$$

is the flow of X^c.

The pull-back bundles of τ by τ and $\mathring{\tau}$ will play an important role in our presentation, and will be denoted by $\tau^*\tau$ and $\mathring{\tau}^*\tau$, respectively. The shorthand for their sections will be $\mathfrak{X}(\tau)$ and $\mathfrak{X}(\mathring{\tau})$. These sections will also be called vector fields along the projection.

We have the *canonical short exact sequence*

$$0 \to \tau^*TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} \tau^*TM \to 0,$$

where $\mathbf{i}(z,v)$ is the initial velocity of the parametrized straight line $t \mapsto z + tv$ for all $(z,v) \in \tau^*TM$, and $\mathbf{j}$ is defined by $w \in T_zTM \mapsto (z, \tau_*(w))$. The set of vertical vectors is $VTM := \operatorname{Im}\mathbf{i} = \operatorname{Ker}\mathbf{j}$, it is the total space of the *vertical subbundle* of τ_{TM}, denoted by τ_{TM}^v. The module of the vertical vector fields is $\mathfrak{X}^v(TM)$. Note that the Lie bracket of two vertical vector fields is always vertical.

The bundle maps $\mathbf{i}$ and $\mathbf{j}$ give rise to $C^\infty(TM)$-homomorphisms between $\mathfrak{X}(\tau)$ and $\mathfrak{X}(TM)$ denoted by the same symbols. Thus we obtain the exact sequence

$$0 \to \mathfrak{X}(\tau) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \mathfrak{X}(\tau) \to 0$$

of $C^\infty(TM)$-homomorphisms.

If X is a vector field on M, we define

$$\hat{X}(z) := (z, X(\tau(z))) \qquad (z \in TM), \qquad X^v := \mathbf{i}\hat{X}.$$

Obviously, $\hat{X}$ is a vector field along τ, while X^v is a vertical vector field. The vector field $\hat{X}$ is said to be a *basic vector field* along τ, and X^v is called the *vertical lift* of X. Further important canonical objects are given by

$$\delta(z) := (z, z) \quad (z \in TM), \qquad C := \mathbf{i}\delta, \qquad \text{and} \qquad J := \mathbf{i} \circ \mathbf{j},$$

the *canonical section of* $\tau^*\tau$, the *Liouville vector field* on TM and the *vertical endomorphism*, respectively. We associate to J the *vertical differential* d_J on TM. By definition,

$$d_J f := df \circ J, \qquad f \in C^\infty(TM).$$

Then $d_J f$ is a (semibasic) one-form on TM.

If $\alpha \in \mathcal{T}^0_k(M)$ is a symmetric or skew-symmetric k-form on M, then the tensor fields $\hat{\alpha}$ and $\bar{\alpha}$ defined by

$$\hat{\alpha}_v(v_1, \dots, v_k) := \alpha_p(v_1, \dots, v_k), \qquad \bar{\alpha}_v(v_1, \dots, v_{k-1}) := \alpha_p(v, v_1, \dots, v_{k-1})$$
$$(v, v_i \in T_pM, 1 \leqq i \leqq k; p \in M)$$

are symmetric or skew-symmetric k- and $(k-1)$-forms along τ, respectively. In particular, if $f \in C^\infty(M)$, then $f^v := \hat{f} = f \circ \tau \in C^\infty(TM)$ is the *vertical lift* of f.

Let $\tilde{X}$ and $\tilde{Y}$ be two vector fields along τ. Choose a vector field η on TM such that $\mathbf{j}\eta = \tilde{Y}$. We define the *canonical v-covariant derivative* of $\tilde{Y}$ with respect to $\tilde{X}$ by

$$\nabla^v_{\tilde{X}} \tilde{Y} = \nabla^v_{\tilde{X}} \mathbf{j}\eta := \mathbf{j}\left[\mathbf{i}\tilde{X}, \eta\right].$$

It can easily be seen that the definition is independent of the choice of η. The operator $\nabla^v_{\tilde{X}}$ can be extended to any tensor α of type $(0, s)$ along τ, to be a kind of tensor derivation:

$$(\nabla^v_{\tilde{X}}\alpha)\left(\tilde{Y}_1, \dots, \tilde{Y}_s\right) := \left(\mathbf{i}\tilde{X}\right)\alpha\left(\tilde{Y}_1, \dots, \tilde{Y}_s\right) - \sum_{i=1}^{s} \alpha\left(\tilde{Y}_1, \dots, \nabla^v_{\tilde{X}}\tilde{Y}_i, \dots, \tilde{Y}_s\right)$$

$\left(\tilde{Y}_1, \dots, \tilde{Y}_s \in \mathfrak{X}(\tau)\right)$.

If X is a vector field on M, we may define a Lie derivative $\mathcal{L}_X$ in the tensor algebra of $\tau^*\tau$ in the following way:

$$\mathcal{L}_X : \qquad f \in C^\infty(TM) \mapsto X^c f, \qquad \tilde{Y} \in \mathfrak{X}(\tau) \mapsto \mathbf{i}^{-1}\left[X^c, \mathbf{i}\tilde{Y}\right],$$

and extend it to any types of tensors by the usual product rule (for details, see [6, 21]). In particular, $\mathcal{L}_X\delta = 0$, and, for $Y \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_X\hat{Y} = \mathbf{i}^{-1}[X^c, Y^v] = \mathbf{i}^{-1}([X, Y]^v) = \widehat{[X, Y]}.$$

§3. Generalized metrics

In this section we introduce generalized metrics and some of their special classes. Our main source is reference [10].

Definition 3.1. *Let g be a symmetric and non-degenerate tensor of type* $(0, 2)$ *in the bundle $\tau^*\tau$ or in $\overset{\circ}{\tau}{}^*\tau$. Then g is said to be a* generalized metric *or briefly a* metric.

It is crucial that g need not be defined on the zero section, since, if g is homogeneous and is defined in the whole $\tau^*\tau$ (and, of course, is smooth), then it is the lift of a pseudo-Riemannian metric on M.

Using non-degeneracy, the *first Cartan tensor* $\mathcal{C}$ and the *lowered first Cartan tensor* $\mathcal{C}_\flat$ of a generalized metric g are defined by the following formulae:

$$g\left(\mathcal{C}(\tilde{X}, \tilde{Y}), \tilde{Z}\right) := \mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \tilde{Z}\right) := \left(\nabla^v_{\tilde{X}} g\right)\left(\tilde{Y}, \tilde{Z}\right) \qquad \left(\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\tau)\right).$$

The one-form

$$\vartheta_g : \xi \in \mathfrak{X}(TM) \mapsto \vartheta_g(\xi) := g(\mathbf{j}\xi, \delta)$$

on TM is called the *Lagrange one-form* associated to g, and its exterior derivative $\omega_g := d\vartheta_g$ is the *Lagrange two-form* associated to g. The *absolute energy* of g is $E := \frac{1}{2} g(\delta, \delta)$.

Definition 3.2. *A metric g along τ or $\overset{\circ}{\tau}$ is said to be* variational *if the first Cartan tensor $\mathcal{C}$ associated to it is symmetric,* weakly variational *if* $\mathcal{C}_\flat\left(\tilde{X}, \tilde{Y}, \delta\right) = \mathcal{C}_\flat\left(\tilde{Y}, \tilde{X}, \delta\right)$ *for every $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tau)$,* normal *if* $\mathcal{C}\left(\tilde{X}, \delta\right) = 0$ *for every $\tilde{X} \in \mathfrak{X}(\tau)$, and* weakly normal *if* $\mathcal{C}_\flat\left(\tilde{X}, \delta, \delta\right) = 0$ *for every $\tilde{X} \in \mathfrak{X}(\tau)$. The metric is* Miron regular [12] *if the tensor*

$$\tilde{B} : \tilde{X} \in \mathfrak{X}(\tau) \mapsto \tilde{B}\left(\tilde{X}\right) := \tilde{X} + \mathcal{C}\left(\tilde{X}, \delta\right)$$

has maximal rank at every point of TM (or $\overset{\circ}{T}M$).

Now, for the sake of the reader's convenience, we summarize some results of [10] we shall make use of.

(1) A metric g is variational if and only if there is a smooth function L on TM (or on $\overset{\circ}{T}M$) such that $g = \nabla^v \nabla^v L$. In this case, we shall call L a *Lagrangian.*

(2) A metric g is weakly variational if and only if there is a smooth function L on TM (or on $\overset{\circ}{T}M$) such that $\vartheta_g = d_J L$.

(3) If g is weakly normal and Miron regular, then E is positively homogeneous of degree 2, and the symmetric tensor $\nabla^v\nabla^v E$ is non-degenerate. In other words, E is a (possibly indefinite) Finsler energy function. Furthermore, $\vartheta_g = d_J E$.

(4) If g is normal, then there is a (possibly indefinite) Finsler energy function E such that $g = \nabla^v\nabla^v E$.

§4. Ehresmann connections and covariant derivatives

Following the terminology used e.g. in [7], by an *Ehresmann connection* we shall mean a split canonical short exact sequence:

$$0 \rightleftharpoons \tau^*TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftharpoons}} TTM \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftharpoons}} \tau^*TM \rightleftharpoons 0.$$

The requirement that this is a splitting means that $\mathcal{V}\circ\mathbf{i} = \mathbf{j}\circ\mathcal{H} = 1_{\tau^*TM}$, and $\operatorname{Im}\mathcal{H} = \operatorname{Ker}\mathcal{V}$. We allow the possibility that $\mathcal{H}$ and $\mathcal{V}$ are defined only on $\overset{\circ}{T}M$ rather than on the whole TM. The type $(1,1)$ tensor field $\mathbf{h} := \mathcal{H}\circ\mathbf{j}$ on TM is said to be the *horizontal projector* belonging to $\mathcal{H}$, and $\operatorname{Im}\mathbf{h}_v$ is called the *horizontal subspace* of T_vTM if $v \in TM$. The map $\mathbf{v} := 1_{TM} - \mathbf{h}$ is the *vertical projector* belonging to $\mathbf{h}$. As in the case of $\mathbf{i}$ and $\mathbf{j}$, we denote by the same symbols the arising $C^\infty(TM)$-homomorphism between the modules of vector fields as the corresponding bundle maps. If $X \in \mathfrak{X}(M)$ is a vector field on M, then $X^h := \mathcal{H}\hat{X} = \mathbf{h}X^c \in \mathfrak{X}(TM)$ is its *horizontal lift*.

The *torsion* of an Ehresmann connection is the (1,2) tensor T along τ determined by the formula

$$\mathbf{i}T\left(\hat{X},\hat{Y}\right) := \left[X^h, Y^v\right] - \left[Y^h, X^v\right] - [X,Y]^v \qquad (X,Y \in \mathfrak{X}(M)).$$

If a metric and an Ehresmann connection with vanishing torsion are given on TM (or on $\overset{\circ}{T}M$), we can construct a metric covariant derivative D in $\tau^*\tau$ as follows (see [4, 10]). First, we consider *Berwald's covariant derivative* in $\tau^*\tau$ given by

$$\nabla_{\mathbf{i}\tilde{X}}\tilde{Y} := \mathbf{j}\left[\mathbf{i}\tilde{X}, \mathcal{H}\tilde{Y}\right], \qquad \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} := \mathcal{V}\left[\mathcal{H}\tilde{X}, \mathbf{i}\tilde{Y}\right] \qquad \left(\tilde{X},\tilde{Y} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

Observe that its vertical part coincides with the canonical v-covariant derivative. Next, we introduce the *second Cartan tensor* $\mathcal{C}^h$ by means of the relation

$$g\left(\mathcal{C}^h\left(\tilde{X},\tilde{Y}\right),\tilde{Z}\right) := \left(\nabla_{\mathcal{H}\tilde{X}}g\right)\left(\tilde{Y},\tilde{Z}\right) \qquad \left(\tilde{X},\tilde{Y},\tilde{Z} \in \mathfrak{X}(\tau)\right).$$

Third, using the Christoffel trick, we define two other tensors along τ:

$$g\left(\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),\tilde{Z}\right) = g\left(\mathcal{C}(\tilde{X},\tilde{Y}),\tilde{Z}\right) + g\left(\mathcal{C}(\tilde{Y},\tilde{Z}),\tilde{X}\right) - g\left(\mathcal{C}(\tilde{Z},\tilde{X}),\tilde{Y}\right),$$
$$g\left(\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y}),\tilde{Z}\right) = g\left(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}\right) + g\left(\mathcal{C}^h(\tilde{Y},\tilde{Z}),\tilde{X}\right) - g\left(\mathcal{C}^h(\tilde{Z},\tilde{X}),\tilde{Y}\right).$$

With the help of $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ we define D by the rules

$$D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\tilde{X},\tilde{Y}\right), \qquad D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\tilde{X},\tilde{Y}\right)$$
$$\left(\tilde{X},\tilde{Y},\tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})\right).$$

Finally, this covariant derivative operator can also be extended to any type of tensors by the usual product rule. Then it will be metric, i.e., $Dg = 0$. If g arises from a Finsler energy function, and $\mathcal{H}$ is the canonical Ehresmann connection on the Finsler manifold (section 7), then D coincides with the well-known *Cartan's covariant derivative* [20, 21].

§5. Killing vector fields in general

In this section g will be a generalized metric on M. For the sake of definiteness, we shall assume that g is defined only on $\overset{\circ}{T}M$. The same arguments, however, remain valid when its domain is the whole TM.

Definition 5.1. *A diffeomorphism $f : U \to V$ between two open subsets of M is a* local isometry *if its tangent map leaves g invariant, i.e.,*

$$g_{f_*(v)}(f_*(w_1), f_*(w_2)) = g_v(w_1, w_2)$$

for any $p \in U$ and $v, w_1, w_2 \in \overset{\circ}{T}_pM$. A vector field $X \in \mathfrak{X}(M)$ with flow $\varphi : W \subset \mathbb{R} \times M \to M$ is said to be an infinitesimal isometry *if φ_t is a local isometry between two open subsets of M for all $t \in \mathbb{R}$ such that the domain of φ_t is not empty. A vector field $X \in \mathfrak{X}(M)$ is called a* Killing vector field *if $\mathcal{L}_X g = 0$.*

Proposition 5.2. *Let g be a metric and $X \in \mathfrak{X}(M)$ a vector field. Then X is an infinitesimal isometry of M if and only if it is a Killing vector field.*

Proof. We shall repeatedly use the dynamic interpretation of the Lie bracket of two vector fields [24]: if $X, Y \in \mathfrak{X}(M)$, and φ is the flow of X, then

$$[X,Y](p) = \lim_{t\to 0}\frac{1}{t}\{(\varphi_{-t})_*[Y(\varphi_t(p))] - Y(p)\} = \lim_{t\to 0}\frac{1}{t}((\varphi_{-t})_\sharp Y - Y)(p),$$

for all $p \in M$. Now let us begin with proving the necessity, and assume that X is an infinitesimal isometry. For arbitrarily chosen vector fields Y and Z on M, define a function $f \in C^\infty(\overset{\circ}{T}M)$ by $f := g\left(\hat{Y}, \hat{Z}\right)$. If $v \in \overset{\circ}{T}_pM$, $t \in \mathbb{R}$ and $(t,p) \in W$, we have

$$\begin{aligned} f((\varphi_t)_* v) &= g_{(\varphi_t)_*(v)}(Y(\varphi_t(p)), Z(\varphi_t(p))) \\ &= g_{(\varphi_t)_*(v)}\{(\varphi_t)_*[(\varphi_{-t})_\sharp Y](p), (\varphi_t)_*[(\varphi_{-t})_\sharp Z](p)\} \\ &= g_v((\varphi_{-t})_\sharp Y(p), (\varphi_{-t})_\sharp Z(p)), \end{aligned}$$

using, in the last step, that φ_t is a local isometry for every sufficiently small $t \in \mathbb{R}$. Now we use the fact that the curve $c_v : t \mapsto (\varphi_t)_*(v)$ is an integral curve of X^c to obtain

$$\begin{aligned} &X^c(v)f = \lim_{t\to 0} \frac{1}{t}[f((\varphi_t)_*(v)) - f(v)] \\ &= \lim_{t\to 0} \frac{1}{t}[g_v((\varphi_{-t})_\sharp Y(p), (\varphi_{-t})_\sharp Z(p)) - g_v(Y(p), Z(p))] \\ &= \lim_{t\to 0}\left[\frac{g_v((\varphi_{-t})_\sharp Y(p) - Y(p), (\varphi_{-t})_\sharp Z(p))}{t} + \frac{g_v(Y(p), (\varphi_{-t})_\sharp Z(p) - Z(p))}{t}\right] \\ &= g_v\left(\lim_{t\to 0}\frac{1}{t}((\varphi_{-t})_\sharp Y(p) - Y(p)), \lim_{t\to 0}(\varphi_{-t})_\sharp Z(p)\right) \\ &\quad + g_v\left(Y(p), \lim_{t\to 0}\frac{1}{t}((\varphi_{-t})_\sharp Z(p) - Z(p))\right) = g_v([X,Y](p), Z(p)) \\ &\quad + g_v(Y(p), [X,Z](p)) = \left\{g\left(\widehat{[X,Y]}, \hat{Z}\right) + g\left(\hat{Y}, \widehat{[X,Z]}\right)\right\}(v), \\ &\qquad X^c g\left(\hat{Y}, \hat{Z}\right) = X^c f = g\left(\widehat{[X,Y]}, \hat{Z}\right) + g\left(\hat{Y}, \widehat{[X,Z]}\right) \\ &\qquad\quad = g\left(\mathcal{L}_X \hat{Y}, \hat{Z}\right) + g\left(\hat{Y}, \mathcal{L}_X \hat{Z}\right). \end{aligned}$$

Thus we conclude

$$(\mathcal{L}_X g)\left(\hat{Y}, \hat{Z}\right) = X^c g\left(\hat{Y}, \hat{Z}\right) - g\left(\mathcal{L}_X \hat{Y}, \hat{Z}\right) - g\left(\hat{Y}, \mathcal{L}_X \hat{Z}\right) = 0,$$

i.e., X is a Killing vector field.

To prove the converse, assume that X is a Killing vector field, consider the flow $\varphi : W \subset \mathbb{R} \times M \to M$ of X, and let $p \in M$, $v, w_1, w_2 \in \overset{\circ}{T}_pM$ be arbitrary. We shall again denote the maximal integral curve of X^c starting from v by $c_v : I_p \to TM$. (The domain of this curve depends only on p.) We define the function $\ell : I_p \to \mathbb{R}$ in the following way:

$$\ell(t) := g_{(\varphi_t)_*(v)}((\varphi_t)_*(w_1), (\varphi_t)_*(w_2)) = g_{c_v(t)}((\varphi_t)_*(w_1), (\varphi_t)_*(w_2)).$$

It is enough to show that ℓ is constant. To this end, we define two vector fields along c_v:

$$Y(t) := (\varphi_t)_*(w_1), \qquad Z(t) := (\varphi_t)_*(w_2) \qquad (t \in I_p).$$

Then Y and Z can be extended, at least locally, to vector fields $\tilde{Y}$ and $\tilde{Z}$ on an open subset U of TM such that

$$Y(t) = \tilde{Y}(c_v(t)), \qquad Z(t) = \tilde{Z}(c_v(t)) \qquad (t \in I)$$

($I \subset I_p$ is another open interval). Now with the help of the function given by

$$f(q) := g_q(\tilde{Y}(q), \tilde{Z}(q)) \qquad (q \in U),$$

we have $\ell \restriction I = f \circ c_v$. Thus,

$$\begin{aligned} \ell'(t) &= (f \circ c_v)'(t) = \dot{c}_v(t) f = X^c(c_v(t)) f = (X^c f)(c_v(t)) \\ &= \left[g(\mathcal{L}_X \tilde{Y}, \tilde{Z}) + g(\tilde{Y}, \mathcal{L}_X \tilde{Z}) \right] (c_v(t)), \\ \mathbf{i}\left(\mathcal{L}_X \tilde{Y}\right)(q) &= \left[X^c, \mathbf{i}\tilde{Y} \right](q) = \lim_{t \to 0} \frac{1}{t} \left\{ (\varphi_{-t})_*[\mathbf{i}\tilde{Y}(\varphi_t(q))] - \mathbf{i}\tilde{Y}(q) \right\} \\ &= \mathbf{i} \lim_{t \to 0} \frac{1}{t} \left\{ (\varphi_{-t})_*[\tilde{Y}(\varphi_t(q))] - \tilde{Y}(q) \right\} = 0 \qquad (q \in c_v(I)) \end{aligned}$$

due to the construction of $\tilde{Y}$. We obtain, in a similar way, that $\mathcal{L}_X \tilde{Z} = 0$. Hence ℓ is indeed constant. $\square$

If the metric g is positive definite and homogeneous, i.e., the function $g\left(\hat{X}, \hat{Y}\right)$ is positively homogeneous of degree 0 for any $X, Y \in \mathfrak{X}(M)$, then we may define the length of an arc $c : [\alpha, \beta] \to M$ by

$$\ell(c) := \int_\alpha^\beta \sqrt{E \circ \dot{c}} = \int_\alpha^\beta \sqrt{g_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t))} dt.$$

The distance of two points $p, q \in M$ is then given by

$$d(p, q) := \inf\{\ell(c) | c : [0,1] \to M, c(0) = p, c(1) = q\}.$$

We say that g is *reversible* if $g_{-v}(w_1, w_2) = g_v(w_1, w_2)$ for any $v, w_1, w_2 \in T_pM$ and $p \in M$. In this case, d is symmetric, and (M, d) becomes a metric space.

It is known that every Killing field is complete on a complete Riemannian manifold [17]. This result can be easily generalized as follows.

Proposition 5.3. *Let g be a homogeneous, reversible and positive definite metric, and suppose that X is a Killing vector field of g. If M is complete as a metric space, the vector field X is complete as well.*

Proof. Let $c_p : [0, \alpha[\to M$ be an integral curve of X starting from p. We show that c_p can be extended to $[0, \alpha]$. Since $\ddot{c}_p = X^c \circ \dot{c}_p$, and

$$X^c E = \frac{1}{2} X^c g(\delta, \delta) = \frac{1}{2} (\mathcal{L}_X g)(\delta, \delta) = 0,$$

the function $E \circ \dot{c}_p$ is constant. Let $\lambda := \sqrt{E(\dot{c}_p(t))}$ ($t \in [0, \alpha[$ is arbitrary). Thus, if $t, t' \in]0, \alpha[$,

$$d(c_p(t), c_p(t')) \leqq \left| \int_t^{t'} \sqrt{E \circ \dot{c}_p} \right| = \lambda |t - t'|.$$

This implies, by the completeness of M, that the limit $\lim_{t \to \alpha} c_p(t)$ exists. $\square$

Now we suppose that an Ehresmann connection is specified on M whose torsion vanishes. Let D be the covariant derivative operator constructed in section 4.

The following proposition was formulated in [19] for the special case of Finsler manifolds. It generalizes the skew-symmetry of the covariant differential of a Killing field in Riemannian geometry.

Proposition 5.4. *If X is a Killing vector field on M,*

$$g\left(D_{\mathcal{H}\tilde{Y}} \hat{X}, \tilde{Z}\right) + g\left(\tilde{Y}, D_{\mathcal{H}\tilde{Z}} \hat{X}\right) + g\left(\mathcal{C}\left(\mathcal{V}X^c, \tilde{Y}\right), \tilde{Z}\right) = 0$$

for any $\tilde{Y}, \tilde{Z} \in \mathfrak{X}(\overset{\circ}{\tau})$.

Proof. Since the left-hand side is tensorial in $\tilde{Y}$, $\tilde{Z}$, it is enough to verify the formula for basic vector fields $\hat{Y}$, $\hat{Z}$. Using the condition that X is a Killing field, we obtain

$$
\begin{aligned}
0 &= (\mathcal{L}_X g)\left(\hat{Y},\hat{Z}\right) = X^c g\left(\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right) \\
&\overset{Dg=0}{=} g\left(D_{X^c}\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) + g\left(\hat{Y},D_{X^c}\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right) \\
&= g\left(\nabla_{X^h}\hat{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right) - \widehat{[X,Y]},\hat{Z}\right) + (Y \leftrightarrow Z) \\
&= g\left(\mathcal{V}\left(\left[X^h,Y^v\right] - [X,Y]^v\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right) \\
&\quad + (Y \leftrightarrow Z) \\
&\overset{T=0}{=} g\left(\mathcal{V}\left[Y^h,X^v\right] + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{X},\hat{Y}\right) + \frac{1}{2}\overset{\circ}{\mathcal{C}}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right) + (Y \leftrightarrow Z) \\
&= g\left(\nabla_{Y^h}\hat{X} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h\left(\hat{Y},\hat{X}\right),\hat{Z}\right) \\
&\quad + \frac{1}{2}\left\{\mathcal{C}_\flat\left(\mathcal{V}X^c,\hat{Y},\hat{Z}\right) + \mathcal{C}_\flat\left(\hat{Y},\hat{Z},\mathcal{V}X^c\right) - \mathcal{C}_\flat\left(\hat{Z},\mathcal{V}X^c,\hat{Y}\right)\right\} + (Y \leftrightarrow Z) \\
&= g\left(D_{Y^h}\hat{X},\hat{Z}\right) + g\left(\hat{Y},D_{Z^h}\hat{X}\right) + \mathcal{C}_\flat\left(\mathcal{V}X^c,\hat{Y},\hat{Z}\right) \\
&= g\left(D_{Y^h}\hat{X},\hat{Z}\right) + g\left(\hat{Y},D_{Z^h}\hat{X}\right) + g\left(\mathcal{C}\left(\mathcal{V}X^c,\hat{Y}\right),\hat{Z}\right),
\end{aligned}
$$

where the symbol $(Y \leftrightarrow Z)$ means an expression consisting of all preceding terms, with Y and Z interchanged. □

§6. Special classes of generalized metrics

For any metric g, we introduce the (1,1) tensor $\overset{*}{\mathcal{C}}$ along τ by the prescription

$$
\overset{*}{\mathcal{C}} : \tilde{X} \in \mathfrak{X}(\tau) \mapsto \mathcal{C}\left(\tilde{X},\delta\right),
$$

where $\mathcal{C}$ is the first Cartan tensor of the metric.

Proposition 6.1. *Let g be a weakly variatonal and Miron regular metric with $\vartheta_g = d_J L$. A vector field X on M is a Killing vector field for g if and only if the function $X^c L$ is a vertical lift and $\mathcal{L}_X\overset{*}{\mathcal{C}} = 0$.*

Proof.

(1) *Necessity*

Suppose that X is a Killing field. If $Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} Y^v X^c L &= X^c Y^v L - [X^c, Y^v]L = X^c (d_J L)(Y^c) - d_J L[X,Y]^c \\ &= X^c \vartheta_g(Y^c) - \vartheta_g[X,Y]^c = X^c g\left(\hat{Y}, \delta\right) - g\left(\widehat{[X,Y]}, \delta\right) \\ &= (\mathcal{L}_X g)\left(\hat{Y}, \delta\right) = 0, \end{aligned}$$

thus $X^c L$ is a vertical lift. To verify the necessity of the second condition, let Z be another vector field on M. Using our assumption $\mathcal{L}_X g = 0$ repeatedly, we get

$$\begin{aligned} g\left((\mathcal{L}_X \overset{*}{\mathcal{C}})(\hat{Y}), \hat{Z}\right) &= g\left(\mathcal{L}_X(\overset{*}{\mathcal{C}}(\hat{Y})) - \overset{*}{\mathcal{C}}\widehat{[X,Y]}, \hat{Z}\right) \\ &= X^c g\left(\overset{*}{\mathcal{C}}(\hat{Y}), \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}\widehat{[X,Y]}, \hat{Z}\right) - g\left(\overset{*}{\mathcal{C}}(\hat{Y}), \widehat{[X,Z]}\right) \\ &= X^c g\left(\mathcal{C}(\hat{Y}, \delta), \hat{Z}\right) - g\left(\mathcal{C}(\widehat{[X,Y]}, \delta), \hat{Z}\right) - g\left(\mathcal{C}(\hat{Y}, \delta), \widehat{[X,Z]}\right) \\ &= X^c\left(\nabla^v_{\hat{Y}} g\right)\left(\delta, \hat{Z}\right) - \left(\nabla^v_{\widehat{[X,Y]}} g\right)\left(\delta, \hat{Z}\right) - \left(\nabla^v_{\hat{Y}} g\right)\left(\delta, \widehat{[X,Z]}\right) \\ &= X^c Y^v g\left(\delta, \hat{Z}\right) - X^c g\left(\hat{Y}, \hat{Z}\right) - [X^c, Y^v] g\left(\delta, \hat{Z}\right) + g\left(\widehat{[X,Y]}, \hat{Z}\right) \\ &\quad - Y^v g\left(\delta, \widehat{[X,Z]}\right) + g\left(\hat{Y}, \widehat{[X,Z]}\right) \\ &= Y^v X^c g\left(\delta, \hat{Z}\right) - Y^v g\left(\delta, \widehat{[X,Z]}\right) = Y^v(\mathcal{L}_X g)\left(\delta, \hat{Z}\right) = 0, \end{aligned}$$

which implies, by the non-degeneracy of g, that $\mathcal{L}_X \overset{*}{\mathcal{C}} = 0$.

(2) *Sufficiency*

If $X^c L$ is a vertical lift, we obtain

$$(\mathcal{L}_{X^c} \vartheta_g)(Y^c) = X^c Y^v L - [X^c, Y^v]L = Y^v X^c L = 0$$

for any vector field Y on M, which implies $\mathcal{L}_{X^c} \vartheta_g = 0$. Since the Lie derivative and the exterior derivative commute, we also have $\mathcal{L}_{X^c} \omega_g = \mathcal{L}_{X^c} d\vartheta_g = 0$. The second condition implies

$$\mathcal{L}_X \tilde{B} = \mathcal{L}_X\left(1_{\mathfrak{X}(\tau)} + \overset{*}{\mathcal{C}}\right) = 0.$$

As $\mathcal{L}_X g$ is tensorial, and g is Miron regular, it is sufficient to show that $(\mathcal{L}_X g)\left(\tilde{B}(\hat{Y}), \hat{Z}\right) = 0$ for any vector fields Y and Z on M. Using

$\omega_g(J\xi,\eta) = g\left(\tilde{B}(\mathbf{j}\xi),\mathbf{j}\eta\right)$ $(\xi,\eta \in \mathfrak{X}(TM))$, we get

$$\begin{aligned}
&(\mathcal{L}_X g)\left(\tilde{B}(\hat{Y}),\hat{Z}\right)\\
&= X^c g\left(\tilde{B}(\hat{Y}),\hat{Z}\right) - g\left(\mathcal{L}_X\tilde{B}(\hat{Y}),\hat{Z}\right) - g\left(\tilde{B}(\hat{Y}),\widehat{[X,Z]}\right)\\
&= X^c g\left(\tilde{B}(\hat{Y}),\hat{Z}\right) - g\left(\tilde{B}\widehat{[X,Y]},\hat{Z}\right) - g\left(\tilde{B}(\hat{Y}),\widehat{[X,Z]}\right)\\
&= X^c\omega_g(Y^v,Z^c) - \omega_g([X^c,Y^v],Z^c) - \omega_g(Y^v,[X^c,Z^c])\\
&= (\mathcal{L}_{X^c}\omega_g)(Y^v,Z^c) = 0,
\end{aligned}$$

thus concluding the proof. □

The metric g does not determine L uniquely, since a vertical lift can be added to L without changing $d_J L$. Moreover, we have

Corollary 6.2. *With conditions similar to those in* 6.1*, if g is defined on the whole TM, and X is a Killing vector field, L can be chosen such that $X^c L = 0$.*

Proof. By 6.1, there is a smooth function $\widetilde{L}$ on TM such that $X^c\widetilde{L}$ is a vertical lift. Let us define L by

$$L(v) := \widetilde{L}(v) - \widetilde{L}(0_{\tau(v)}),$$

then L differs from $\widetilde{L}$ only by a vertical lift, and $X^c L = 0$. □

Now we introduce two canonical inclusions. The first one will be

$$i_1 : M \to TM, \qquad p \in M \mapsto i_1(p) := 0_p.$$

In other words, i_1 is an embedding of M into TM that assigns to each point p the zero vector at p. The second inclusion is given by the prescription

$$\begin{gathered}
i_2 : TM \to TTM, \qquad v \in TM \mapsto i_2(v) := \dot{c}_v(0),\\
\text{where} \qquad c_v : t \in \mathbb{R} \mapsto 0_{\tau(v)} + tv.
\end{gathered}$$

We shall also use the shorthand $\bar{\tau} := i_1 \circ \tau$.

Proposition 6.3. *Let g be a variational metric defined on the whole TM. A vector field X on M is a Killing vector field if and only if there is a Lagrangian L for g such that $X^c L = 0$.*

Proof.

(1) *Necessity*

Suppose that X is a Killing vector field, and $\widetilde{L}$ is an arbitrary Lagrangian for g. Then we obtain

$$\begin{aligned}0 = (\mathcal{L}_X g)\left(\hat{Y},\hat{Z}\right) &= X^c g\left(\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right)\\ &= X^c Y^v Z^v \widetilde{L} - [X^c,Y^v]Z^v\widetilde{L} - Y^v[X^c,Z^v]\widetilde{L} = Y^v Z^v X^c \widetilde{L}\end{aligned}$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$. It follows that $X^c\widetilde{L}$ is an affine function on each fibre. Now we define a new Lagrangian L by

$$L := \widetilde{L} - \widetilde{L}\circ i_1 \circ \tau - d\widetilde{L}\circ i_2.$$

It is easy to see that the difference of $\widetilde{L}$ and L is also a fibrewise affine function, thus their Hessians are the same, i.e., g. We compute the action of X^c on the difference $\widetilde{L} - L$ over an induced chart $(\tau^{-1}(U), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ in TM by a chart $(U, (u^i)_{i=1}^n)$ in M:

$$\begin{aligned}&X^c\left(\widetilde{L}\circ i_1\circ\tau + d\widetilde{L}\circ i_2\right) = \left[X\left(\widetilde{L}\circ i_1\right)\right]^v + X^c\left(d\widetilde{L}\circ i_2\right)\\ &= \left[X^i \frac{\partial\left(\widetilde{L}\circ i_1\right)}{\partial u^i}\right]^v + (X^i)^v\frac{\partial}{\partial x^i}\left(d\widetilde{L}\circ i_2\right) + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}\left(d\widetilde{L}\circ i_2\right)\\ &= (X^i)^v\left(\frac{\partial\widetilde{L}}{\partial x^i}\circ\bar{\tau}\right) + (X^i)^v y^j\left(\frac{\partial^2\widetilde{L}}{\partial x^i\partial y^j}\circ\bar{\tau}\right) + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\left(\frac{\partial\widetilde{L}}{\partial y^i}\circ\bar{\tau}\right).\end{aligned}$$

This is a fibrewise affine function, just like $X^c\widetilde{L}$. To show that they are equal, it is enough to check that they coincide on the zero section and so do their linear parts on each fibre. The expression of $X^c\widetilde{L}$ over our induced chart is

$$X^c\tilde{L} = (X^i)^v\frac{\partial\widetilde{L}}{\partial x^i} + y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\frac{\partial\widetilde{L}}{\partial y^i}.$$

Thus, $X^cL = X^c\widetilde{L} - X^c\left(\widetilde{L} - L\right)$ vanishes indeed on the zero section:

$$X^c\widetilde{L}\circ i_1 - X^c\left(\widetilde{L} - L\right)\circ i_1 = X^i\left(\frac{\partial\widetilde{L}}{\partial x^i}\circ i_1\right) - X^i\left(\frac{\partial\widetilde{L}}{\partial x^i}\circ i_1\right) = 0,$$

whereas the linear part of X^cL is

$$y^i\left(\frac{\partial}{\partial y^i}X^c\widetilde{L}\right)\circ\bar{\tau} - (X^i)^v y^j\left(\frac{\partial^2\widetilde{L}}{\partial x^i\partial y^j}\circ\bar{\tau}\right) - y^j\left(\frac{\partial X^i}{\partial u^j}\right)^v\left(\frac{\partial\widetilde{L}}{\partial y^i}\circ\bar{\tau}\right) = 0.$$

(2) *Sufficiency*

$$
\begin{aligned}
&(\mathcal{L}_X g)\left(\hat{Y},\hat{Z}\right) = X^c g\left(\hat{Y},\hat{Z}\right) - g\left(\widehat{[X,Y]},\hat{Z}\right) - g\left(\hat{Y},\widehat{[X,Z]}\right) \\
&= X^c Y^v Z^v L - [X^c,Y^v] Z^v L - Y^v [X^c,Z^v] L = Y^v Z^v X^c L = 0.
\end{aligned}
$$

□

Corollary 6.4. *If (M,E) is a Finsler manifold with Finslerian metric $g = \nabla^v\nabla^v E$, then a vector field X on M is a Killing vector field of g if and only if $X^c E = 0$.*

§7. Translations

In this section we shall work on a manifold endowed with a weakly normal and Miron regular metric. It can be shown (see [10]) that in this case, the absolute energy E is a Finsler energy function. Then E can be extended continuously to the zero section. We shall denote by $\xi \in \mathfrak{X}(\overset{\circ}{T}M)$ the canonical spray of the Finsler manifold (M,E) determined by the relation $(dd_J E)(\xi,\eta) = -\eta E$ for $\eta \in \mathfrak{X}(\overset{\circ}{T}M)$. It is well-known that there is a canonical Ehresmann connection on a Finsler manifold called the *Barthel connection* [21]. In this section we shall use this connection and the corresponding metric covariant derivative D. Then, for any vector field X on M,

$$
X^h = \frac{1}{2}(X^c + [X^v,\xi]), \qquad X^h E = 0,
$$

and $\xi = \mathcal{H}\delta$ is horizontal.

Definition 7.1. *A Killing vector field X of g is called a* translation *if every non-constant integral curve of X is a geodesic of the Finsler manifold (M,E).*

For classical results on translations of Riemannian manifolds, see [2, 16, 26]. Now we generalize the important *conservation lemma* from Riemannian geometry ([14], p. 252) as follows.

Proposition 7.2. *If $X \in \mathfrak{X}(M)$ is a Killing vector field, and $c : I \to M$ is a geodesic of E, then the function*

$$
t \in I \mapsto g_{\dot{c}(t)}(X(c(t)),\dot{c}(t))
$$

is constant.

Proof. Let us denote the function in question by f. The curve $\dot{c}$ is an integral curve of ξ, thus we have

$$f' = \xi g\left(\hat{X}, \delta\right) \circ \dot{c}.$$

Using (3) in section 3 and the relation $X^c E = 0$, we obtain

$$\begin{aligned}\xi g\left(\hat{X}, \delta\right) &= \xi \vartheta_g(X^c) = \xi(d_J E)(X^c) = \xi X^v E = -X^c E - X^v \xi E + \xi X^v E \\ &= -2X^h E = 0,\end{aligned}$$

and therefore $f' = 0$, which implies that f is constant. $\square$

Proposition 7.3. *Let X be a Killing vector field of g. Then X is a translation if and only if the function*

$$p \in M \mapsto E(X_p)$$

is constant.

Proof.

(1) *Necessity*
Suppose that X is a translation. If $X = 0$, the statement is obvious. Hence we assume that there is a point $q \in M$ such that $X_q \neq 0$. We define the following subset of M:

$$V := \{p \in M | E(X_p) = E(X_q)\}.$$

We shall show that $V = M$. First, $V \neq \emptyset$, since $q \in V$. Furthermore, V is closed, since it is the inverse image of the closed set $\{E(X_q)\} \subset \mathbb{R}$ under the function

$$f : p \in M \mapsto f(p) := E(X_p).$$

Thus it remains only to show that V is open.

To see this, take a point $p \in V$. By the straightening-out theorem (see e.g. [1]), there is a chart $(U, (u^i)_{i=1}^n)$ around p such that $X \upharpoonright U = \frac{\partial}{\partial u^1}$. Consider an integral curve $c : I \to M$ of X, which is, by the definition of translations, a geodesic as well. Its components $c^i := u^i \circ c$ have the following form:

$$c^1(t) = c^1(0) + t, \qquad c^i(t) = c^i(0) \quad (2 \leqq i \leqq n).$$

On the other hand, c satisfies the differential equations of the geodesics:

$$c^{i\prime\prime} + 2G^i \circ \dot{c} = 0,$$

where

$$G^i = \frac{1}{2} g^{ij} \left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j} \right),$$

and (g^{ij}) is the inverse matrix of (g_{ij}). Putting these together, we infer that $G^i \circ \frac{\partial}{\partial u^1} = 0$ on U. Since the matrix (g^{ij}) is non-degenerate, this implies that

$$\begin{aligned} 0 &= \left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j} \right) \circ \frac{\partial}{\partial u^1} = \left(\frac{\partial^2 E}{\partial x^1 \partial y^j} - \frac{\partial E}{\partial x^j} \right) \circ \frac{\partial}{\partial u^1} \\ &= -\frac{\partial E}{\partial x^j} \circ \frac{\partial}{\partial u^1} = -\frac{\partial}{\partial u^j} \left(E \circ \frac{\partial}{\partial u^1} \right), \end{aligned}$$

which, in turn, implies that the function $E \circ \frac{\partial}{\partial u^1}$ is constant on U. Hence $p \in V$ is contained together with an open neighbourhood in V. We conclude that $V = M$.

(2) Sufficiency
If the function $f : p \in M \mapsto f(p) := E(X_p)$ is constant, then, in a chart similar to that in the previous part, it can be seen that the integral curves of X are geodesics as well. □

§8. Some special cases

8.1. Randers manifolds

Let (M, α) be a Riemannian manifold and β a one-form on M. We recall from section 2 that the tensor $\hat{\alpha}$ along τ and the function $\bar{\beta}$ on TM are given by

$$\hat{\alpha}_v(w_1, w_2) = \alpha_p(w_1, w_2), \qquad \bar{\beta}(v) = \beta_p(v) \qquad (v, w_1, w_2 \in T_pM, p \in M).$$

We define the following functions on TM:

$$F_\alpha(v) := \sqrt{\alpha_{\tau(v)}(v, v)} \quad (v \in TM), \quad F := F_\alpha + \bar{\beta}, \quad E := \frac{1}{2} F^2.$$

Then F and E are smooth on $\overset{\circ}{T}M$.

Due to the non-degeneracy of α, there is a unique vector field $\beta^\sharp$ on M such that $\beta(Y) = \alpha(\beta^\sharp, Y)$ for any vector field Y on M (Riesz' lemma). Conversely, if X is a vector field on M, then we have a one-form $X^\flat$ such that $X^\flat(Y) = \alpha(X, Y)$ for any vector field Y.

If $\|\beta^\sharp\| < 1$, (M, E) is a Finsler manifold, called the *Randers manifold* obtained from the Riemannian manifold (M, α) by the perturbation with the one-form β.

Lemma 8.1 ([11]). *Let (M, E) be the Randers manifold arising from the Riemannian manifold (M, α) by perturbation with β such that $\|\beta^\sharp\| < 1$. Then the metric tensor g of (M, E) takes the form*

$$g = \frac{F}{F_\alpha}\hat{\alpha} - \frac{\bar{\beta}}{F_\alpha^3}\bar{\alpha} \otimes \bar{\alpha} + \frac{1}{F_\alpha}\bar{\alpha} \odot \hat{\beta} + \hat{\beta} \otimes \hat{\beta},$$

where $\odot$ stands for the symmetric product.

In his paper [9], M. Matsumoto proved that $\beta^\sharp$ is a Killing vector field of the Randers manifold if and only if it is a Killing vector field of the original Riemannian manifold (M, α) as well. Now we use the results of section 5 to give a new proof of the sufficiency of this condition:

Proposition 8.2. *Suppose that (M, α) is a Riemannian manifold, and $X \in \mathfrak{X}(M)$ is a Killing vector field of (M, α) such that $\|X\| < 1$. Let $\beta := X^\flat$, $F := F_\alpha + \bar{\beta}$ and $E = \frac{1}{2}F^2$. Then X is a Killing vector field of the Randers manifold (M, E).*

Proof. First, suppose that $X(p) \neq 0$ at $p \in M$. Consider a chart $(U, (u^i)_{i=1}^n)$ around p and the induced chart $(\tau^{-1}(U), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on TM. Let $i, j \in \{1, \dots, n\}$ be arbitrary, then

$$\begin{aligned}&(\mathcal{L}_X g)\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) \\ &= X^c g\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) + g\left(\mathcal{L}_X \widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) + g\left(\widehat{\frac{\partial}{\partial u^i}}, \mathcal{L}_X \widehat{\frac{\partial}{\partial u^j}}\right).\end{aligned}$$

By the straightening-out theorem, we can choose a chart such that $X = \frac{\partial}{\partial u^1}$. Then the last two terms vanish since, e.g.,

$$\mathcal{L}_X \widehat{\frac{\partial}{\partial u^i}} = \widehat{\left[X, \frac{\partial}{\partial u^i}\right]} = \widehat{\left[\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^i}\right]} = 0.$$

It remains to show that the first term also vanishes. We have the following coordinate expressions:

$$\hat{\alpha}\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) = \alpha_{ij}^v, \qquad \hat{\beta}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \beta_i^v, \qquad \bar{\alpha}\left(\widehat{\frac{\partial}{\partial u^i}}\right) = \alpha_{ij}^v y^j,$$
$$\bar{\beta} = \beta_i^v y^i, \qquad F_\alpha = \sqrt{\alpha_{ij}^v y^i y^j}, \qquad F = \sqrt{\alpha_{ij}^v y^i y^j} + \beta_i^v y^i.$$

We substitute the expression in the preceding lemma for g:

$$(*)\qquad (\mathcal{L}_X g)\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right) = \frac{\partial}{\partial x^1} g\left(\widehat{\frac{\partial}{\partial u^i}}, \widehat{\frac{\partial}{\partial u^j}}\right)$$

$$= \frac{\partial}{\partial x^1}\left[\left(1 + \frac{\beta^v_k y^k}{(\alpha^v_{lm} y^l y^m)^{1/2}}\right)\alpha^v_{ij} - \frac{\beta^v_k y^k}{(\alpha^v_{lm} y^l y^m)^{3/2}}\alpha^v_{ir}\alpha^v_{js} y^r y^s \right.$$

$$\left. + \frac{1}{(\alpha^v_{lm} y^l y^m)^{1/2}}(\alpha^v_{ir} y^r \beta^v_j + \beta^v_i \alpha^v_{jr} y^r) + \beta^v_i \beta^v_j\right],$$

and

$$\beta_i = \alpha_{ij} X^j = \alpha_{ij}\delta^j_1 = \alpha_{i1}.$$

On the other hand, since X is a Killing vector field of (M, α), we obtain

$$0 = \left(\mathcal{L}_{\frac{\partial}{\partial u^1}}\alpha\right)\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \frac{\partial \alpha_{ij}}{\partial u^1}.$$

Thus we have shown that all functions in the square bracket of $(*)$ have vanishing partial derivatives with respect to x^1, and hence $\mathcal{L}_X g = 0$ on T_pM if $X(p) \neq 0$. On the other hand, if $X(p) = 0$, and there is a series $(p_n)_{n=0}^{\infty}$ such that $p_n \to p$ and $X(p_n) \neq 0$ $(n \in \mathbb{N})$, then $\mathcal{L}_X g$ vanishes on T_pM by continuity. Finally, if there is a neighbourhood of p on which X vanishes, then $\mathcal{L}_X g = 0$ on T_pM automatically. $\square$

8.2. Funk metrics

In this subsection we shall work on an open subset of $\mathbb{R}^n$; D_v will denote the directional derivative with respect to a vector $v \in \mathbb{R}^n$ and D_i the ith partial derivative $(i = 1, \dots, n)$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a *Minkowski functional* [18], i.e., a function satisfying the following conditions:

(1) φ is continuous on $\mathbb{R}^n$ and smooth on $\mathbb{R}^n \setminus \{0\}$;

(2) $\varphi(0) = 0$, and $\varphi(p) > 0$ if $p \neq 0$;

(3) φ is positively homogeneous of degree 1;

(4) the second derivative $\varphi''(p)$ is non-degenerate (and thus necessarily positive definite) if $p \neq 0$.

The set $\Omega := \varphi^{-1}\,[0, 1[$ is the interior of the indicatrix of φ. We shall use the canonical identification $T\Omega \cong \Omega \times \mathbb{R}^n$ and the natural projections $\pi_1 : T\Omega \to \Omega$

and $\pi_2 : T\Omega \to \mathbb{R}^n$. A Finslerian fundamental function $F : T\Omega \to \mathbb{R}$ on Ω is determined by the relation

$$\varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right) = 1 \qquad \text{on } \overset{\circ}{T}\Omega.$$

The Finsler structure determined by F is traditionally called the *Funk metrics* on Ω. The Finsler energy is then $E = \frac{1}{2}F^2$. For more about Funk metrics, see [18].

Proposition 8.3. *With notations and hypotheses as above, for a vector field X on Ω the following conditions are equivalent:*

(1) *X is a Killing vector field of (Ω, F);*

(2) *for every point $p \in \Omega$ and vector $v \in \mathbb{R}^n$ such that $p+v \in \partial\Omega$, the vector $X(p) + D_vX(p)$ is parallel to the tangent hyperplane of $\partial\Omega$ in $p+v$.*

Proof. Let $(u^i)_{i=1}^n$ be the restriction of the canonical coordinate system of $\mathbb{R}^n$ to Ω and $((x^i)_{i=1}^n, (y^i)_{i=1}^n)$ the induced coordinate system on $T\Omega$. If the coordinate expression of X is $X^i \frac{\partial}{\partial u^i}$, its complete lift is

$$X^c = \left(X^i\right)^v \frac{\partial}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}.$$

If we act by X^c on both sides of the relation defining F, we obtain

$$\begin{aligned}
0 &= \left[D_k\varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right)\right] \Bigg[\left(X^i\right)^v \frac{\partial}{\partial x^i}\left(x^k + \frac{y^k}{F}\right) \\
&\qquad\qquad + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \frac{\partial}{\partial y^i}\left(x^k + \frac{y^k}{F}\right)\Bigg] \\
&= \left[D_k\varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right)\right] \Bigg[\left(X^i\right)^v \left(\delta_i^k - \frac{y^k}{F^2}\frac{\partial F}{\partial x^i}\right) \\
&\qquad\qquad + y^j \left(\frac{\partial X^i}{\partial u^j}\right)^v \left(\frac{\delta_i^k}{F} - \frac{y^k}{F^2}\frac{\partial F}{\partial y^i}\right)\Bigg] \\
&= \left[D_k\varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right)\right] \Bigg[\left(X^k\right)^v + \frac{y^j}{F}\left(\frac{\partial X^k}{\partial u^j}\right)^v \\
&\qquad\qquad - \frac{y^k}{F^2}\left(\left(X^i\right)^v \frac{\partial F}{\partial x^i} + y^j \left(\frac{\partial X^i}{\partial u^i}\right)^v \frac{\partial F}{\partial y^i}\right)\Bigg] \\
&= \left[D_k\varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right)\right] \left[\left(X^k\right)^v + \frac{y^j}{F}\left(\frac{\partial X^k}{\partial u^j}\right)^v - \frac{y^k}{F^2}X^cF\right].
\end{aligned}$$

X is a Killing field if and only if $X^c F = 0$. Furthermore, if $v_p (\neq 0) \in T\Omega$ is arbitrary, and $z := p + \frac{v}{F(v_p)} (\in \partial\Omega)$, then

$$v^k D_k \varphi(z) = \frac{g_z(z, v)}{\varphi(z)} = g_z(z, v) \neq 0.$$

Therefore, it follows that X is a Killing field if and only if

$$(*) \qquad \left[D_k \varphi \circ \left(\pi_1 + \frac{\pi_2}{F}\right)\right] \left[\left(X^k\right)^v + \frac{y^j}{F} \left(\frac{\partial X^k}{\partial u^j}\right)^v\right] = 0.$$

From now on, we suppose that v_p is of the form as in the proposition, i.e., $p + v \in \partial\Omega$. By the homogeneity of F, if $(*)$ is satisfied for such v_p's, it is satisfied for all. In that case, $F(v_p) = 1$, and evaluating $(*)$ at v_p we obtain

$$\begin{aligned}(D_k\varphi)(p+v)(X^k(p) + v^j D_j X^k(p)) &= (D_k\varphi)(p+v)(X^k(p) + D_v X^k(p)) \\ &= \langle \mathrm{grad}\varphi(p+v), X(p) + D_v X(p)\rangle = 0,\end{aligned}$$

or, equivalently, the vector $X(p)+D_v X(p)$ is parallel to the tangent hyperplane of the indicatrix at $p + v$. □

§9. Discussion

It is known that a geodesic on a Riemannian manifold meets a translation at constant angles [2, 16, 26]. In the general case, if g is positive definite, the angle φ of a translation X and a geodesic c may be given by

$$\cos\varphi(t) := \frac{g_{\dot{c}(t)}(X(c(t)), \dot{c}(t))}{\sqrt{g_{\dot{c}(t)}(X(c(t)), X(c(t))) g_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t))}}.$$

The numerator is constant by 7.2, and the second factor in the denominator is constant as well even in the most general case. It follows from 7.3 that in the Riemannian case the first factor is also constant, since then the function $g\left(\hat{X}, \hat{X}\right)$ is constant on each fibre. From our results, however, it does not follow that the first factor is constant in general, even for Finsler manifolds. Therefore, it does not follow that φ is constant. It remains an open question whether there exists any class of metrics in which this angle is constant and which is more general than the Riemannian case.

Moreover, there is a broad class of metrics that have no non-trivial translations at all. For example, the hyperbolic plane does not have any. In Poincaré's upper half-plane model with canonical coordintates (u^1, u^2) the Killing fields have the form

$$X = (\alpha u^1 + \beta u^2 + \gamma)\frac{\partial}{\partial u^1} + \alpha u^2 \frac{\partial}{\partial u^2},$$

with some $\alpha, \beta, \gamma \in \mathbb{R}$. If $\alpha \neq 0$, the integral curves of X are given by

$$c(t) = \left((c_1 + \beta c_2 t)e^{\alpha t} - \frac{\gamma}{\alpha}, c_2 e^{\alpha t} \right),$$

with $c_1, c_2 \in \mathbb{R}$, $c_2 > 0$, which are no geodesics. That is, however, not surprising, since, if the hyperbolic plane had a non-trivial translation, a geodesic quadrangle with angle sum 2π could be constructed, in contradiction with the Gauss – Bonnet theorem.

In summary, we have tried to generalize some theorems of Riemannian geometry and Finsler geometry, and found that those not relying on the notion of translation may be successfully generalized.

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