SUT Journal of Mathematics
Vol. 39, No. 1 (2003), 1–14

Spherical t-designs and the Bernstein Theorem

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(Received October 23, 2001; Revised June 19, 2002)

Abstract. In this paper we study the construction problem of spherical t-designs as a generalization of the Chebyshev problem of Gauss type quadrature formula. It is known that spherical t-designs can be constructed if we can construct interval t-designs for the weight function $(1-x)^{\frac{d-2}{2}}$. We shall show the exsitence and non-existence theorems of some interval t-desings.

AMS 2000 *Mathematics Subject Classification.* 05B30, 65D30.

Key words and phrases. Spherical designs, Gauss type quadrature formula, the Gegenbauer polynomials.

§1. Introduction

Spherical t-designs on d-dimensional sphere S^d are defined in [4] by P.Delsarte, J.M.Goethals and J.J.Seidel. There are existence theorems and sporadic examples but the general and explicit construction is unknown. B.Bajnok shows in [3] that spherical t-designs can be constructed if we can construct interval t-designs for the weight function $(1-x)^{\frac{d-2}{2}}$. To construct spherical t-designs along Bajnok's method, we have to find the finite subset $\{x_1, \cdots, x_n\}$ of $[-1,1]$ which satisfies

$$\frac{1}{\alpha}\int_{-1}^{1} f(x)(1-x^2)^{\frac{d-2}{2}}dx = \frac{1}{n}\sum_{i=1}^{n} f(x_i),\ where\ \alpha = \int_{-1}^{1}(1-x^2)^{\frac{d-2}{2}}dx,$$

for every polynomial $f(x)$ whose degree does not exceed t (We simply call such $\{x_1, \cdots, x_n\}$ an interval t-design). When $d = 2$, this is the Chebyshev problem in Gauss type quadrature formula. Supposing $t = n$, the Chebyshev problem is solvable for $1 \leq t = n \leq 7$ and $t = n = 9$, and unsolvable for $t = n = 8$ and $10 \leq t = n$ (see [5]). In this paper, we study the construction problem of spherical designs as a generalization of the Chebyshev problem. We show that the Chebyshev problem for $d = 3$ is solvable for $1 \leq t = n \leq 3$ and $t = n = 5$, and unsolvable for $t = n = 4$ and $6 \leq t = n$.

§2. The Chebyshev problem in Gauss type quadrature formula

In this section we study interval designs for the case $t = n$. First we state the classical result for $d = 2$; Taking $f(x) = x^k$ $(1 \le k \le t)$ in the definition of the interval design, $x_1, \cdots, x_n$ must satisfy

$$P_k := x_1^k + x_2^k + \cdots + x_n^k = \begin{cases} 0 & if\ k\ is\ odd, \\ \dfrac{n}{k+1} & if\ k\ is\ even. \end{cases}$$

This is necessary and sufficient. From the Newton formula

$$P_k - P_{k-1}S_1 + P_{k-2}S_2 - \cdots + (-1)^k k S_k = 0,$$

where $S_k = S_k(n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ is the fundamental symmetric of degree k in n variables, it follows

$$\begin{cases} S_{2k+1} = 0, \\ S_{2k} = -\dfrac{1}{2k}(P_{2k} + P_{2k-2}S_2 + P_{2k-4}S_4 + \cdots + P_2 S_{2k-2}). \end{cases}$$

For a given t, we shall find an interval t-design $\{x_1, \cdots, x_n\}$ such that $t = n$ as a set of n distinct real roots in $[-1, 1]$ of the polynomial

$$f_n(X) = \begin{cases} X^{2m} + S_2(2m)X^{2m-2} + \cdots + S_{2m}(2m) & if\ n = 2m, \\ X^{2m+1} + S_2(2m+1)X^{2m-1} + \cdots + S_{2m}(2m+1)X & if\ n = 2m+1. \end{cases}$$

The list of such $f_n(X)$ is known:

$$\begin{aligned}
n = t = 2: &\quad f_2(X) = X^2 - \frac{1}{3}, \\
n = t = 3: &\quad f_3(X) = X^3 - \frac{1}{2}X, \\
n = t = 4: &\quad f_4(X) = X^4 - \frac{2}{3}X^2 + \frac{1}{45}, \\
n = t = 5: &\quad f_5(X) = X^5 - \frac{5}{6}X^3 + \frac{7}{72}X, \\
n = t = 6: &\quad f_6(X) = X^6 - X^4 + \frac{1}{5}X^2 - \frac{1}{105}, \\
n = t = 7: &\quad f_7(X) = X^7 - \frac{7}{6}X^5 + \frac{119}{360}X^3 - \frac{149}{6480}X, \\
n = t = 9: &\quad f_9(X) = X^9 - \frac{3}{2}X^7 + \frac{27}{40}X^5 - \frac{57}{560}X^3 + \frac{53}{22400}X.
\end{aligned}$$

Each set of the roots of the above polynomial is the interval 3,3,5,5,7,7,9-design for $d = 2$, respectively.

In a similar way we study for the case $d \ge 3$. If k is an odd integer,

$x^k(1-x^2)^{\frac{d-2}{2}}$ is an odd function. Thus

$$P_{2m+1} = \frac{n}{\alpha}\int_{-1}^{1} x^{2m+1}(1-x^2)^{\frac{d-2}{2}}dx = 0.$$

If k is an even integer, by integration by parts

$$P_{2m} = \frac{n}{\alpha}\int_{-1}^{1} x^{2m}(1-x^2)^{\frac{d-2}{2}}dx = n\prod_{l=1}^{m}\frac{2l-1}{d+2l-1}.$$

Then we can find some interval designs as the set of n real roots in $[-1,1]$ of the following $f_n(X)$:

$$\begin{aligned} n=t=2:&\quad f_2(X) = X^2 - \frac{1}{d+1}, \\ n=t=3:&\quad f_3(X) = X^3 - \frac{3}{2(d+1)}X, \\ n=t=5:&\quad f_5(X) = X^5 - \frac{5}{2(d+1)}X^3 - \frac{5(d-9)}{8(d+1)^2(d+3)}X. \end{aligned}$$

Hence we have the following theorem.

Theorem 2.1. *We have following interval t-designs such that $t=n$ for $d \geq 3$.*

$$\begin{aligned} n=t=2:&\ \left\{\pm\sqrt{\frac{1}{d+1}}\right\}. \\ n=t=3:&\ \left\{0,\ \ \pm\sqrt{\frac{3}{2d+2}}\right\}. \\ n=t=5:&\ \left\{0, \pm\frac{1}{2}\sqrt{\frac{5d+15\pm\sqrt{5(d+3)(7d-3)}}{(d+1)(d+3)}}\right\}, \\ &\qquad\qquad\qquad\qquad \textit{if and only if } 3 \leq d \leq 8. \end{aligned}$$

As for the case $n=t=5$, we remark that $0 < 5d+15-\sqrt{5(d+3)(7d-3)}$ holds when $d \leq 8$.

§3. An application of Sturm's Theorem

In the previous section, we solve some algebraic equations and find interval t-designs under the assumption $t=n$. In this section we study the same problem without the assumption $t=n$.

When $n < t$ and the values of $P_1, \cdots, P_n, \cdots, P_t$ are given, we can determine the values of $S_1, \cdots, S_t$ by the Newton formula. Thus the coefficients of $f_n(X)$ are determined by the first n conditions on $P_1, \cdots, P_n$. It implies interval t-designs such that $t > n$ is unique if it exists. We will find such t-design by showing that the roots of $f_n(X) = 0$ satisfy the other additional conditions on $P_{n+1}, \cdots, P_t$. When $n > t$ and the values of $P_1, \cdots, P_t$ are given, the values of $S_1, \cdots, S_t$ are determined by the Newton formula. It means that only the coefficients of $X^{n-1}, X^{n-2}, \cdots, X^{n-t}$ are fixed but there are no any restrictions on the other cofficients. If we can take cofficients of some lower terms in such a way that $f_n(X) = 0$ has n distinct real roots in $[-1, 1]$, then the set of the roots presents an interval t-design. Although it is difficult to find an algebraic representation of such roots, the degree of such polynomial gives an upper bound of n for a fixed t.

In the particular case $t = n-1$, we can modify the constant term of $f_n(X)$. Then we determine the range of the constant term for some t by using Strum's theorem. Now we recall Sturm's theorem ; Let $f(X)$ be a polynomial with real cofficients and $f'(X)$ its derivative. Apply the Euclidean algorithm to $f(X)$ and $f'(X)$ with changing the sign of the remainders and put

$$F_{m-1} = q_m F_m - F_{m+1} \qquad (m = 1, 2, \cdots)$$

where $F_0 = f(X)$ and $F_1 = f'(X)$. With seeing

$$F_0(a),\ F_1(a),\ F_2(a),\ \cdots,\ F_k(a)$$

from left to right, count how many times the sign is changed and denote the number of the times by $V(a)$. Then Sturm's theorem says

$$V(a) - V(b) = \#\{\ x \in (a, b] : f(x) = 0\ \}.$$

For example, we apply Sturm's theorem to $X^6 - X^4 + \frac{1}{5}X^2 + \delta$ where δ is a parameter for the constant term of $f_6(X)$ for $d = 2$. We put X for X^2 and determine the range of δ such that $F_0(X) = X^3 - X^2 + \frac{1}{5}X + \delta = 0$ has 3 real roots in $(0, 1]$.

	$F_0(X)$	$F_1(X)$	$F_2(X)$	$F_3(X)$
Evaluate at $X = 0$	δ	$\frac{1}{5}$	$-\frac{1}{45} - \delta$	$\frac{9}{80} + \frac{45}{8}\delta - \frac{6075}{16}\delta^2$
Evaluate at $X = 1$	$\frac{1}{5} + \delta$	$\frac{6}{5}$	$\frac{1}{15} - \delta$	$\frac{9}{80} + \frac{45}{8}\delta - \frac{6075}{16}\delta^2$

By the above table, we see that $V(0) - V(1) = 3$ holds for $\frac{1}{135} - \frac{4}{675}\sqrt{10} < \delta < 0$. In short, for an arbitary δ in this range, an interval 5-design is given by

$$\left\{ \pm\sqrt{\frac{1}{3} + \left(\alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}}\right)},\ \pm\sqrt{\frac{1}{3} + \left(\omega\alpha^{\frac{1}{3}} + \omega^2\beta^{\frac{1}{3}}\right)},\ \pm\sqrt{\frac{1}{3} + \left(\omega^2\alpha^{\frac{1}{3}} + \omega\beta^{\frac{1}{3}}\right)} \right\}.$$

$$\left(\ \{\alpha, \beta\} = \left\{ \frac{2 - 270\delta}{540} \pm \frac{1}{450}\sqrt{50625\delta^2 - 750\delta - 15} \right\}\ \right)$$

This is the set of 6 real roots of $X^6 - X^4 + \frac{1}{5}X^2 + \delta = 0$ in $[-1, 1]$. Similarly we have the following theorem.

Theorem 3.1. *Each set of the roots of the following polynomial is an interval 4,5,6,7,8,9-design for $d = 2$, respectively.*

$$n = 5, t = 4 \,:\, X^5 - \frac{5}{6}X^3 + \frac{7}{72}X + \delta, \qquad \left(0 < |\delta| < \frac{1}{5400}\sqrt{39750 - 2790\sqrt{155}} \right).$$

$$n = 6, t = 5 \,:\, X^6 - X^4 + \frac{1}{5}X^2 + \delta, \qquad \left(\frac{1}{135} - \frac{4}{675}\sqrt{10} < \delta < 0 \right).$$

$$n = 7, t = 6 \,:\, X^7 - \frac{7}{6}X^5 + \frac{119}{360}X^3 - \frac{149}{6480}X + \frac{1}{5000}.$$

$$n = 8, t = 7 \,:\, X^8 - \frac{4}{3}X^6 + \frac{22}{45}X^4 - \frac{148}{2835}X^2 + \delta,$$

$$\left(-3\left(\lambda^{\frac{2}{3}}\omega^2 + \frac{4}{135}\right)^2 - 3\left(\overline{\lambda^{\frac{2}{3}}\omega^2} + \frac{4}{135}\right)^2 - \frac{16}{14175} < \delta < -\frac{43}{42525} \right).$$

$$n = 9, t = 8 \,:\, X^9 - \frac{3}{2}X^7 + \frac{27}{40}X^5 - \frac{57}{560}X^3 + \frac{53}{22400}X + \frac{1}{100000}.$$

$$n = 10, t = 9 \,:\, X^{10} - \frac{5}{3}X^8 + \frac{8}{9}X^6 - \frac{100}{567}X^4 + \frac{17}{1701}X^2 - \frac{1}{15000}.$$

For $t = 6, 8$ and 9, the exact range of the constant term is unknown. But it is easy to know by using computer whether all the roots are real and included in $[-1, 1]$ whenever the constant term is fixed. It is true for these polynomials that all the roots are real and included in $[-1, 1]$ even if the constant term is changed a little. For $d \geq 3$, we obtain the similar results.

Theorem 3.2. *Let $d \geq 3$. We obtain interval t-designs from the following polynomials.*

$$n = 4, t = 3 \,:\, X^4 - \frac{2}{d+1}X^2 + \delta, \qquad \left(0 < \delta < \frac{1}{(d+1)^2} \right).$$

$$n = 5, t = 3 \,:\, X^5 - \frac{5}{2(d+1)}X^3 + \delta X, \qquad \left(0 < \delta < \frac{25}{16(d^2 + 2d + 1)} \right).$$

$$n = 5, t = 4 \,:\, X^5 - \frac{5}{2(d+1)}X^3 - \frac{5d - 45}{8(d+3)(d+1)^2}X + \delta,$$

$$\left(0 < |\delta| < \frac{9-5d+\sqrt{(d+3)(11d+9)}}{8}\sqrt{\frac{3d+9-\sqrt{(d+3)(11d+9)}}{(1+d)^5(d+3)^3}} \ (3 \leq\ d \leq\ 8) \right).$$

$$n = 6, t = 5 \,:\, X^6 - \frac{3}{(d+1)}X^4 + \frac{9}{(d+1)^2(d+3)}X^2 + \delta,$$

$$\left(\frac{(d+3)(2d-3) - 2d\sqrt{(d+3)d}}{(d+3)^2(1+d)^3} < \delta < 0 \right).$$

We modify not only the constant term but cofficients of some lower terms, and find the following polynomials such that the set of its roots is an interval t-design. It is true also for these polynomials that all the roots are real and included in $[-1,1]$ even if the constant term is changed a little.

Proposition 3.1. *Each set of the roots of the following polynomial is an interval 7-design for $d=3$, 11,13-design for $d=2$, respectively.*

$$X^9 - \frac{9}{8}X^7 + \frac{45}{128}X^5 - \frac{39}{1024}X^3 + \frac{1}{1000}X,$$

$$X^{14} - \frac{7}{3}X^{12} + \frac{91}{45}X^{10} - \frac{331}{405}X^8 + \frac{959}{6075}X^6 - \frac{2723}{200475}X^4 + \frac{1}{2500}X^2 - \frac{7}{100000000},$$

$$X^{18} - 3X^{16} + \frac{18}{5}X^{14} - \frac{78}{35}X^{12} + \frac{134}{175}X^{10} - \frac{282}{1925}X^8 + \frac{12954}{875875}X^6 - \frac{341}{500000}X^4 + \frac{51}{5000000}X^2 - \frac{1}{1250000000}.$$

Now, the supplementary result of Theorem 2.1 is obtained by applying Sturm's theorem for $f_n(X)$ when $d \geq 3$.

Theorem 3.3. *Let $d \geq 3$. Interval designs do not exist if $n = t = 4, 6, 7$.*

Proof. The results are followed from the below tables.

$f_4(X) = X^4 - \frac{2}{d+1}X^2 - \frac{d-3}{(d+1)^2(d+3)}$

	$F_0(X)$	$F_1(X)$	$F_2(X)$
Evaluate at $X=0$	$\frac{3-d}{(d+1)^2(d+3)}$	$\frac{-2}{d+1}$	$\frac{2d}{(d+1)^2(d+3)}$
Evaluate at $X=1$	$\frac{d(d^2+3d-2)}{(d+1)^2(d+3)}$	$\frac{2}{d+1}$	$\frac{2d}{(d+1)^2(d+3)}$

$f_6(X) = X^6 - \frac{3}{d+1}X^4 + \frac{9}{(d+1)^2(d+3)}X^2 - \frac{3(2d^2-5d+5)}{(d+1)^3(d+3)(d+5)}$

	$F_0(X)$	$F_1(X)$	$F_2(X)$	$F_3(X)$
$X=0$	$\frac{-3(2d^2-5d+5)}{(d+1)^3(d+3)(d+5))}$	$\frac{9}{(d+1)^2(d+3)}$	$\frac{6d(d-3)}{(d+1)^3(d+3)(d+5)}$	$\frac{-9(5d^3+2d^2-35d+16)}{(d+1)^2(d+3)(d+5)^2}$
$X=1$	$\frac{d^5+8d^4+12d^3-23d^2+8d}{(d+1)^3(d+3)(d+5)}$	$\frac{3d(d^2+3d-1)}{(d+1)^2(d+3))}$	$\frac{2d(d^2+9d-4)}{(d+1)^3(d+3)(d+5))}$	$\frac{-9(2d^2-35d+16+5d^3)}{(d+1)^2(d+3)(d+5)^2}$

$f_7(X) = X^7 - \frac{7}{2(d+1)}X^5 + \frac{7(d+15)}{8(d+1)^2(d+3)}X^3 - \frac{7(43d^2-124d+225)}{48(d+1)^3(d+3)(d+5)}X$

	$F_0(X)$	$F_1(X)$	$F_2(X)$	$F_3(X)$
$X=0$	$\frac{-7(43d^2-124d+225)}{48(d+1)^3(d+3)(d+5)}$	$\frac{7(d+15)}{8(d+1)^2(d+3)}$	$\frac{7(61d^2-256d+75)}{72(d+1)^3(d+3)(d+5)}$	$*1$
$X=1$	$*2$	$\frac{24d^3+64d^2-49d+9}{8(d+1)^2(d+3)}$	$\frac{7(22d^3+187d^2-182d+45)}{72(d+1)^3(d+3)(d+5)}$	$*1$

$$* 1 = -\tfrac{3}{8}\tfrac{13987d^5-11123d^4-107558d^3+147750d^2-67005d+10125}{(d+1)^2(d+3)(11d-3)^2(d+5)^2}$$

$$* 2 = \tfrac{1}{48}\tfrac{48d^5+360d^4+378d^3-1435d^2+1018d-225}{(d+1)^3(d+3)(d+5)}$$

§4. The Bernstein Theorem for $d = 3$

The Gauss type quadrature formula in $[-1, 1]$ for an weight function $w(x) = (1-x^2)^{\frac{d-2}{2}}$ is

$$(*) \qquad \int_{-1}^{1} f(x)w(x)dx = \sum_{i=1}^{m} A_i f(\zeta_i), \quad (\deg f \leq 2m-1)$$

where $\zeta_1, \cdots, \zeta_m$ are zeros of the m^{th} degree Gegenbauer polynomial $C_m^{\nu}(x)$ for $\nu = \frac{d-1}{2}$ and

$$A_i = \frac{(2m+d-3)(m+d-2)r_{m-1}}{m(1-\zeta_i^2)\{C_m^{\nu(1)}(\zeta_i)\}^2}. \quad \left(r_{m-1} = \int_{-1}^{1} C_{m-1}^{\nu}(x)^2 w(x)dx \right)$$

(see [1].) By using properties of the Legendre polynomials, which are the Gagenbauer polynomials of $d = 2$, S.N.Bernstein proved that the Chebyshev problem was unsolvable for $10 \leq t = n$. In this section we show that the Chebyshev problem for $d = 3$ is unsolvable for $6 \leq t = n$.

Lemma 4.1. *If the quadrature formula* $\int_{-1}^{1} f(x)w(x)dx = \frac{\alpha}{n}\sum_{i=1}^{n} f(x_i)$ *holds for every polynomial* $f(x)$ *such that* $\deg f = t \leq 2m-1$ *where* $m < n$*, then, enumerating* x_i *in order of size,*

$$x_n > \zeta_m$$

where ζ_m *is the largest root of the Gegenbauer polynomial* $C_m^{\nu}(x)$ *for* $\nu = \frac{d-1}{2}$.

Proof. Let $F(x)=\dfrac{C_m^\nu(x)^2}{x-\zeta_m}$. Since the polynomial $\dfrac{C_m^\nu(x)}{x-\zeta_m}$ is of degree $m-1$, it is orthogonal to $C_m^\nu(x)$. We therefore have

$$\int_{-1}^{1} F(x)(1-x^2)^{\frac{d-2}{2}}dx=\int_{-1}^{1}\frac{C_m^\nu(x)}{x-\zeta_m}\cdot C_m^\nu(x)\cdot(1-x^2)^{\frac{d-2}{2}}dx=0.$$

Since $\deg F=2m-1$, we may take $f(x)=F(x)$ in the assumption. Thus

$$\frac{\alpha}{n}\sum_{i=1}^{n}F(x_i)=\int_{-1}^{1}F(x)(1-x^2)^{\frac{d-2}{2}}dx=0,$$

and hence

$$\sum_{i=1}^{n}F(x_i)=0.$$

Because $m<n$, all x_i do not satisfy $C_m^\nu(x_i)\neq 0$. Therefore there exist positive and negative terms in the last sum. Since $F(x_i)>0$ holds only for $x_i>\zeta_m$, we see $x_n>\zeta_m$. ■

Lemma 4.2. *If the quadrature formula* $\int_{-1}^{1}f(x)w(x)dx=\frac{\alpha}{n}\sum_{i=1}^{n}f(x_i)$ *holds for every polynomial* $f(x)$ *such that* $\deg f=t\le 2m-1$ *where* $m<n$*, then*

$$A_m>\frac{\alpha}{n}\qquad\left(\alpha=\int_{-1}^{1}(1-x^2)^{\frac{d-2}{2}}dx\right).$$

Proof. Let $F(x)=\left\{\dfrac{C_m^\nu(x)}{(x-\zeta_m)C_m^{\nu(1)}(\zeta_m)}\right\}^2$. Since the polynomial $F(x)$ is of degree $2m-2$, we may take $f(x)=F(x)$ in the assumption. Then we have

$$\int_{-1}^{1}F(x)w(x)dx=\frac{\alpha}{n}\sum_{i=1}^{n}F(x_i). \tag{4.1}$$

Now, we obtain by the quadrature formula $(*)$

$$\int_{-1}^{1}F(x)w(x)dx=\sum_{i=1}^{m}A_iF(\zeta_i)=A_m, \tag{4.2}$$

because $F(\zeta_i)=\begin{cases}1 & (i=m)\\ 0 & (1\le i<m)\end{cases}$. From (4.1) and (4.2), it follows that $\frac{\alpha}{n}\sum_{i=1}^{n}F(x_i)=A_m$. Since $F(x)$ is a positive function, we see

$$\frac{\alpha}{n}F(x_n)\le A_m. \tag{4.3}$$

Now, $F(x) = \{C_m^{\nu(1)}(\zeta_m)\}^2 \prod_{i=1}^{m-1}(x-\zeta_i)^2$ is an increasing function for $x > \zeta_m$. We see $F(x_n) > F(\zeta_m) = 1$ since $x_n > \zeta_m$ by Lemma 4.1. It finally follows from (4.3) that

$$\frac{\alpha}{n} < A_m.$$

■

Lemma 4.3.

$$C_m^{\nu(1)}(\zeta_m)\left\{1+\frac{d}{3}+\frac{d^2}{24}-\frac{(m-1)(m+d)}{6}\frac{1-\zeta_m}{1+\zeta_m}\right\} > \left\{1-\frac{(1-\zeta_m)^4 m^4(m+d-1)^4}{24(d+6)(d+4)(d+2)d}\right\}\frac{C_m^{\nu}(1)}{1-\zeta_m}.$$

Proof. Making use of Taylor's series of $C_m^{\nu}(x)$ at $x = \zeta_m$ with three terms and the integral forms of the remainder;

$$C_m^{\nu}(x) = C_m^{\nu(1)}(\zeta_m)(x-\zeta_m) + \frac{1}{2}C_m^{\nu(2)}(\zeta_m)(x-\zeta_m)^2 + \frac{1}{6}C_m^{\nu(3)}(\zeta_m)(x-\zeta_m)^3 + \frac{1}{6}\int_{\zeta_m}^{x} C_m^{\nu(4)}(u)(x-u)^3 du. \tag{4.4}$$

Differentiate Gegenbauer's differential equation

$$(1-x^2)C_m^{\nu(2)} - dxC_m^{\nu(1)} + m(m+d-1)C_m^{\nu} = 0$$

k times, and we have

$$(1-x^2)C_m^{\nu(2+k)} - (d+2k)xC_m^{\nu(k+1)} + (m-k)(m+d+k-1)C_m^{\nu(k)} = 0,$$

where we denote by $C_m^{\nu(k)}$ the k^{th} derivative. Setting here $x = 1$, we obtain

$$C_m^{\nu(k+1)}(1) = \frac{(m-k)(m+d+k-1)}{d+2k}C_m^{\nu(k)}(1) = \frac{\Gamma(m+1)}{\Gamma(m-k)}\frac{\Gamma(m+d+k)}{\Gamma(m+d-1)}\frac{C_m^{\nu}(1)}{\prod_{i=0}^{k}(d+2i)}. \tag{4.5}$$

Setting $x = \zeta_m$ for $k = 0$, we find

$$C_m^{\nu(2)}(\zeta_m) = \frac{\zeta_m}{1-\zeta_m^2}dC_m^{\nu(1)}(\zeta_m) \tag{4.6}$$

and for $k = 1$

$$C_m^{\nu(3)}(\zeta_m) = \frac{1}{1-\zeta_m^2}\left(\frac{\zeta^2}{1-\zeta^2}(d+2)d - (m-1)(m+d)\right) C_m^{\nu(1)}(\zeta_m). \tag{4.7}$$

By using (4.6) and (4.7), (4.4) for $x = 1$ becomes

$$\begin{aligned} C_m^{\nu}(1) = C_m^{\nu(1)}(\zeta_m)(1-\zeta_m)\Bigg(1 + \frac{d}{2}\frac{\zeta_m}{1+\zeta_m} + \frac{(d+2)d}{6}\left(\frac{\zeta_m}{1+\zeta_m}\right)^2 \\ - \frac{(m-1)(m+d)}{6}\frac{1-\zeta_m}{1+\zeta_m}\Bigg) + \frac{1}{6}\int_{\zeta_m}^1 C_m^{\nu(4)}(u)(1-u)^3 du. \end{aligned}$$

Since $1 < \zeta_m$, we have $\frac{\zeta_m}{1+\zeta_m} < \frac{1}{2}$. Since $C_m^{\nu(3)}$ is an monotonically increasing function on $[\zeta_m, 1]$, we have

$$\begin{aligned} \frac{1}{6}\int_{\zeta_m}^1 C_m^{\nu(4)}(u)(1-u)^3 du &< \frac{1}{24}(1-\zeta_m)^4 C_m^{\nu(4)}(1) \\ &= \frac{1}{24}(1-\zeta_m)^4 \frac{\Gamma(m+1)}{\Gamma(m-3)}\frac{\Gamma(m+d+3)}{\Gamma(m+d-1)}\frac{C_m^{\nu}(1)}{(d+6)(d+4)(d+2)d}. \end{aligned}$$

Since

$$(m-k)(m+d+k-1) = m(m+d-1) - k(d+k-1) < m(m+d-1)$$

for a positive integer k, we obtain

$$\frac{\Gamma(m+1)}{\Gamma(m-3)}\frac{\Gamma(m+d+3)}{\Gamma(m+d-1)} < m^4(m+d-1)^4.$$

Therefore

$$\frac{1}{6}\int_{\zeta_m}^1 C_m^{\nu(4)}(u)(1-u)^3 du < \frac{1}{24}(1-\zeta_m)^4 \frac{m^4(m+d-1)^4}{(d+6)(d+4)(d+2)d} C_m^{\nu}(1).$$

Using these inequalties, we prove the lemma ■

Lemma 4.4. *When $d = 3$, the largest root ζ_m of $C_m^{\nu}(x) = 0$ satisfies the inequality*

$$\frac{3}{m(m+2)} < 1 - \zeta_m < \frac{8}{m(m+2)}.$$

Proof. We remark that $C_m^\nu(x)$ is a downwards convex function in $[\zeta_m, 1]$. Hence we know by (4.5) in the proof of Lemma 4.4

$$\frac{C_m^\nu(1)}{1-\zeta_m} < C_m^{\nu(1)}(1) = \frac{m(m+d-1)}{d} C_m^\nu(1).$$

The lower bound is just for $d = 3$.

Next we give the upper bound of $1-\zeta_m$. Gegenbauer's differential equation can be written by

$$\frac{d}{dx}\left((1-x^2)C_m^{\nu(1)}\right) = (d-2)xC_m^{\nu(1)} - m(m+d-1)C_m^\nu.$$

Integrate both hands from ζ_m to 1. Integrating first term in the righthand by parts, we have

$$(1-\zeta_m^2)C_m^{\nu(1)} = -(d-2)C_m^\nu(1) + (m+1)(m+d-2)\int_{\zeta_m}^1 C_m^\nu(x)dx. \tag{4.8}$$

Since $C_m^\nu(x)$ is a downwards convex function in $[\zeta_m, 1]$, the last integral is larger than the area of the triangle surrounded by $y = C_m^{\nu(1)}(1)(x-1) + C_m^\nu(1)$, $x = 1$ and x-axis. Thus we know

$$\int_{\zeta_m}^1 C_m^\nu(x)dx > \frac{1}{2}\frac{C_m^\nu(1)^2}{C_m^{\nu(1)}(1)} = \frac{1}{2}\frac{d}{m(m+d-1)}C_m^\nu(1).$$

Using this estimate to (4.8), we get

$$(1-\zeta_m^2)C_m^{\nu(1)}(\zeta_m) > \frac{1}{2}\left\{4-d+\frac{d(d-2)}{m(m+d-1)}\right\}C_m^\nu(1).$$

If $d = 3$, we have the inequality

$$\frac{2m(m+2)}{m^2+2m+3}(1-\zeta_m^2)C_m^{\nu(1)}(\zeta_m) > C_m^\nu(1). \tag{4.9}$$

On the other hand, we can find another lower bound of $\int_{\zeta_m}^1 C_m^\nu(x)dx$ by taking first two terms of Taylor's series of $C_m^\nu(x)$ at $x = \zeta_m$. Because $C_m^{\nu(k)}(\zeta_m) > 0$ for $k \geq 1$, we see

$$\begin{aligned}\int_{\zeta_m}^1 C_m^\nu(x)dx &= \sum_{k=1}^m \int_{\zeta_m}^1 \frac{C_m^{\nu(k)}(\zeta_m)}{k!}(x-\zeta_m)^k dx = \sum_{k=1}^m \frac{C_m^{\nu(k)}(\zeta_m)}{(k+1)!}(1-\zeta_m)^{k+1} \\ &> \frac{1}{2}C_m^{\nu(1)}(\zeta_m)(1-\zeta_m)^2 + \frac{1}{6}C_m^{\nu(2)}(\zeta_m)(1-\zeta_m)^3.\end{aligned}$$

From (4.6) in the proof of Lemma 4.4 and $\dfrac{1}{1+\zeta_m} > \dfrac{1}{2}$, it follows

$$\int_{\zeta_m}^{1} C_m^\nu dx > \frac{1}{2} C_m^{\nu(1)}(\zeta_m)(1-\zeta_m)^2(1+\frac{d}{6}\zeta_m).$$

Using this estimate to (4.8), we get

$$\begin{aligned}(1-\zeta_m^2)&C_m^{\nu(1)} + (d-2)C_m^\nu(1)\\ &> \frac{1}{2}(m+1)(m+d-2)C_m^{\nu(1)}(\zeta_m)(1-\zeta_m)^2(1+\frac{d}{6}\zeta_m).\end{aligned}$$

When $d = 3$, we thereby have the inequality

$$(1-\zeta_m^2)C_m^{\nu(1)} + C_m^\nu(1) > \frac{1}{4}(m+1)^2 C_m^{\nu(1)}(\zeta_m)(1-\zeta_m)^2(2+\zeta_m). \tag{4.10}$$

Combining (4.9) and (4.10), we have

$$\frac{1}{4}(m+1)^2(1-\zeta_m)(2+\zeta_m) < (1+\zeta_m)\left(1+\frac{2m(m+2)}{m^2+2m+3}\right).$$

Under the condition $\zeta_m > 0$, this inequality means

$$\begin{aligned}\zeta_m &> \frac{1}{2}\frac{-m^2-2m-15+\sqrt{(3(m+1)^2+2)^2+128}}{m^2+2m+3}\\ &> \frac{1}{2}\frac{-m^2-2m-15+3(m+1)^2+2}{m^2+2m+3} = 1-\frac{8}{m^2+2m+3}.\end{aligned}$$

Since $m(m+2) < m^2+2m+3$, finally we have

$$1-\zeta_m < \frac{8}{m(m+2)}.$$

■

Theorem 4.1. *For $6 \le t = n$, there does not exist a finite subset $\{x_1, \cdots, x_n\}$ of $[-1,1]$ which satisfies*

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}dx = \frac{\alpha}{n}\sum_{i=1}^{n} f(x_i), \text{ where } \alpha = \int_{-1}^{1}\sqrt{1-x^2}dx,$$

for every polynomial $f(x)$ whose degree does not exceed t.

Proof. Let $d = 3$. By Lemma 4.3, Lemma 4.4 and $C_m^\nu(1) = \binom{m+d-2}{m} = m+1$, we obtain

$$\frac{1}{C_m^{\nu(1)}(\zeta_m)^2} < \frac{8037225}{5396329}\frac{(34m^2+68m-45)^2}{(m+1)^2m^2(m+2)^2(2m^2+4m-3)^2}.$$

Hence

$$A_m = \frac{(2m+d-3)(m+d-2)r_{m-1}}{m(1-\zeta_m^2)C_m^{\nu(1)}(\zeta_m)^2} = \frac{(m+1)\pi}{(1-\zeta_m^2)C_m^{\nu(1)}(\zeta_m)^2}$$
$$< \frac{2679075}{10792658}\frac{\pi(34m^2+68m-45)^2}{(m+1)(2m^2+4m-3)^2(m^2+2m-4)}.$$

Note that $r_{m-1} = \frac{\pi}{2}$ for $d = 3$. By Lemma 4.2 for $\alpha = \frac{\pi}{2}$, we obtain

$$\frac{\pi}{2n} < A_m.$$

Suppose that n is an odd integer and put $n = 2m-1$. Then the upper and lower bounds of A_m yield the inequality

$$\frac{\pi}{2n} < \frac{10716300}{5396329}\frac{\pi(17n^2+102n-5)^2}{(n+3)(n^2+6n-1)^2(n^2+6n-11)}.$$

Hence

$$0 > 5396329n^7 + 113322909n^6 - 5389968379n^5 - 72466523295n^4$$
$$- 220771255585n^3 + 15401846187n^2 + 1638905587n - 178078857.$$

The last inequality does not hold when $31 \leq n$ (The largest root of the polynomial in n of degree 7 in the righthand is about $29.613\cdots$).

Suppose n is an even integer and put $n = 2m-2$. Similary we have the inequality

$$\frac{\pi}{2n} < \frac{10716300}{5396329}\frac{\pi(17n^2+136n+114)^2}{(n+4)(n^2+8n+6)^2(n^2+8n-4)}.$$

Hence

$$0 > 5396329n^7 + 151097212n^6 - 4596708016n^5 - 91333628640n^4$$
$$- 462977360460n^3 - 654307450384n^2 - 281387331312n - 3108285504.$$

The last inequality does not hold when $30 \leq n$ (The largest root of the polynomial in n of the degree 7 in the righthand is about $28.029\cdots$)

It is easily checked on computer that there does not exist an interval t-design if $t = n \leq 29$. Therefore we prove the theorem. ∎

Acknowledgment

It is a pleasure to express my sincere gratitude to the referee for his valuable comments.

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