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On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups

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Abstract. We discuss the existence of an orthogonal basis consisting of decomposable vectors for some symmetry classes of tensors associated with certain subgroups of the full symmetric group. The dimensions of these symmetry classes of tensors are also computed.

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§1. Introduction

Denote by S_n the symmetric group on $\{1, 2, \ldots, n\}$. Let V be a unitary complex vector space of dimension m. Suppose n is an integer ≥ 2. Let $\ {}^nV$ be the n-th tensor power of V, and write $v \ := v_1 \quad v_2 \quad \cdots \quad v_n$ for the tensor product of the indicated vectors.

For $\sigma \in S_n$, there is a (unique) linear operator $P(\sigma^{-1})$ on $\ {}^nV$ which has the effect $P(\sigma^{-1})(v_1 \quad v_2 \quad \cdots \quad v_n) := v_{\sigma(1)} \quad v_{\sigma(2)} \quad \cdots \quad v_{\sigma(n)}$, for all $v_1, v_2, \ldots, v_n \in V$. Let G be a subgroup of S_n and λ be an irreducible complex character of G. We define $T(G, \lambda)$ as a linear operator on $\ {}^nV$ with the following definition

$$T(G, \lambda) := \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma). \tag{1.1}$$

With respect to the induced inner product in $\ {}^nV$, $T(G, \lambda)$ is an orthogonal projection onto its range $V_\lambda^n(G)$, (see [3], [8]). Let $I(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that $\{T(G,\lambda) | \lambda \in I(G)\}$ is a set of annihilating idempotents which sum to the identity.

The image of $v \ := v_1 \quad v_2 \quad \cdots \quad v_n$ under $T(G,\lambda)$ is denoted by $v^\lambda := v_1 * v_2 * \cdots * v_n$ and it is called a decomposable tensor. $V_\lambda^n(G)$ is called the symmetry class of tensors associated with G and λ, and the dimension of $V_\lambda^n(G)$ is

$$\dim V_\lambda^n(G) = \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) m^{c(\sigma)} \tag{1.2}$$

where $c(\sigma)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of σ, (see [7]). With respect to the induced inner product in $\ {}^nV$, and the orthogonal relations for characters we have

$$ {}^nV = \bigoplus_{\chi \in I(G)} V_\chi^n(G) \tag{1.3}$$

which is an orthogonal direct sum.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $1 \cdot \ \alpha_i \cdot \ m$, so α is a mapping from a set of n elements into a set of m elements. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha := \alpha \circ \sigma^{-1}$, $\sigma \in G$, which is a composition of two functions. Let $G_\alpha := \{\sigma \in G \mid \sigma \cdot \alpha = \alpha\}$ be the stabilizer of α, and $O(\alpha) = \{\sigma \cdot \alpha \mid \sigma \in G\}$ be the orbit with representative α. In this setting we have $G_{\sigma \cdot \alpha} = \sigma G_\alpha \sigma^{-1}$, for all $\sigma \in G$.

Let Δ be a system of distinct representatives of the orbits of G acting on Γ_m^n and define

$$\overline{\Delta} = \{\alpha \in \Delta | \sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0\}. \tag{1.4}$$

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of V. With respect to the induced inner product, one easily obtains the condition $e_\gamma^\lambda := e_{\gamma_1} * e_{\gamma_2} \cdots * e_{\gamma_n} \neq 0$ if and only if $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \overline{\Delta}$. Moreover we have:

$$< e_\alpha^\lambda | e_\beta^\lambda >= \begin{cases} \frac{\lambda(1)}{|G|} \sum_{\sigma \in G_\beta} \lambda(\sigma\tau^{-1}), & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in \text{G}, \\ 0, & \text{if } \text{O}(\alpha) \neq \text{O}(\beta). \end{cases} \tag{1.5}$$

For $\gamma \in \overline{\Delta}$, $V_\gamma^\lambda = \langle e_{\sigma \cdot \gamma}^\lambda | \sigma \in G \rangle$ is called the orbital subspace of $V_\lambda^n(G)$. In [3], Freese proved that

$$\dim V_\gamma^\lambda = \frac{\lambda(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \lambda(\sigma). \tag{1.6}$$

In particular, if λ is a linear character of G, then $\dim V_\gamma^\lambda = 1$ for all $\gamma \in \overline{\Delta}$. By the definition of V_γ^λ, it follows that

$$V_\lambda^n(G) = \bigoplus_{\gamma\in\overline{\Delta}} V_\gamma^\lambda \tag{1.7}$$

is an orthogonal direct sum.

If $\alpha = \sigma\cdot\gamma$ and $\beta = \tau\cdot\gamma$, then $\sigma\tau^{-1}\cdot\beta = \alpha$, therefore using formula (1.5), we have:

$$< e_{\sigma\cdot\gamma}^\lambda \mid e_{\tau\cdot\gamma}^\lambda >= \frac{\lambda(1)}{|G|} \sum_{\pi\in\tau G_\gamma \sigma^{-1}} \lambda(\pi). \tag{1.8}$$

An orthogonal basis of the form $\{e_\gamma^\lambda \mid \gamma \in S\}$, where S is a subset of Γ_m^n, is called an orthogonal basis of decomposable symmetrized tensor for $V_\lambda^n(G)$, in this case we say that $V_\lambda^n(G)$ has an O-basis. By (1.7) $V_\lambda^n(G)$ has an O-basis if an only if V_γ^λ has an O-basis for all $\gamma \in \overline{\Delta}$. In particular, if λ is a linear character, since $\dim V_\gamma^\lambda = 1$, for all $\gamma \in \overline{\Delta}$, then V_γ^λ has an O-basis which implies that $V_\lambda^n(G)$ has an O-basis.

Several papers are devoted in investigation of the existence of an O-basis for $V_\lambda^n(G)$, for example [9]. In [5] a necessary and sufficient condition for the existence of an O-basis for $V_\lambda^n(G)$ is given, where G is a cyclic or a dihedral group. Also in [1] a necessary and sufficient condition for the existence of an O-basis for $V_\lambda^n(G)$ is given, where G is the dicyclic group, i.e. a group generated by two elements a and b such that $a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}$ and denoted by T_{4n} in [6]; and in [2] a necessary and sufficient condition for the existence of an O-basis for the symmetry classes of tensors associated with the direct and central product of some permutation groups is given. In this paper we study the symmetry classes of tensors associated with the groups U_{6n} and V_{8n}, which are defined by generators and relations in [6]. We investigate the problem of finding necessary and sufficient conditions for the existence of an O-basis for the above mentioned groups. We also find the dimensions of the symmetry classes of tensors associated with them.

§2. The Group U_{6n}

The group U_{6n}, $n \geq 1$, is defined in [6] as a group generated by the elements a and b such that $a^{2n} = b^3 = 1$, $a^{-1}ba = b^{-1}$, i.e., $U_{6n} := \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$. It is obvious that $\langle b\rangle$ is a normal subgroup of U_{6n} and $U_{6n} = \langle b\rangle : \langle a\rangle \cong \mathbb{Z}_3 : \mathbb{Z}_{2n}$, which is isomorphism to the semi-direct product of a cyclic group of order 3 by a cyclic group of order $2n$. This group is of order

$6n$, and its elements are of the form $U_{6n} = \{a^r, a^r b, a^r b^2 \mid 0 \cdot\ \ r < 2n\}$. It is not hard to see that U_{6n} has $3n$ conjugacy classes which are

$$\{a^{2r}\}, \{a^{2r}b, a^{2r}b^2\}, \{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\},\ r = 0, 1, \dots, n-1,$$

and the character table of U_{6n} is:

Table I
The character table of U_{6n}

$\lvert C_{U_{6n}}(\sigma)\rvert$	$6n$	$6n$	$3n$	$3n$	$2n$	$2n$
σ	1	a^{2r}	b	$a^{2r}b$	a	a^{2r+1}
χ_j	1	ω^{2rj}	1	ω^{2rj}	ω^j	$\omega^{(2r+1)j}$
ψ_k	2	$2\omega^{2rk}$	-1	$-\omega^{2rk}$	0	0

$$\omega = \exp(\frac{2\pi i}{2n}),\ \ 1 \cdot\ \ r \cdot\ \ n-1,\ 0 \cdot\ \ j \cdot\ \ 2n-1,\ 0 \cdot\ \ k \cdot\ \ n-1.$$

From the above table we see that U_{6n} has $2n$ linear characters $\chi_j, 0 \cdot\ \ j \cdot\ \ 2n-1$, and n non-linear irreducible characters $\psi_k, 0 \cdot\ \ k \cdot\ \ n-1$ of degree 2. Now we will embed this group in a suitable symmetric group. If $(1\ 2\ \cdots\ 2n)(2n{+}1\ \ 2n{+}2)$ and $(2n{+}1\ \ 2n{+}2\ \ 2n{+}3)$ are permutations in S_{2n+3}, then it can be verified that the mapping $a \mapsto (1\ \ 2\ \ \cdots\ 2n)(2n+1\ \ 2n+2)$, $b \mapsto (2n+1\ \ 2n+2\ \ 2n+3)$, embeds U_{6n} in S_{2n+3}. Now considering U_{6n} as a subgroup of S_{2n+3} we find the dimensions of the symmetry classes of tensors associated with the group U_{6n}.

Theorem 1. *Let $G = U_{6n},\ n \geq 1$ and let V be an m-dimensional inner product space. Then considering G as a subgroup of S_{2n+3} as above, we have the following formulae for the dimensions of the symmetry classes of tensors associated with U_{6n}.*

$$\dim V_{\chi_j}^{2n+3}(G) = \tfrac{m}{6n}\left[(m^2+2)\sum_{l=0}^{n-1}\omega^{2lj}m^{(2l,2n)} + 3m\sum_{l=0}^{n-1}\omega^{(2l+1)j}m^{(2l+1,2n)}\right],$$

$$\dim V_{\psi_k}^{2n+3}(G) = \tfrac{2m(m^2-1)}{3n}\sum_{l=0}^{n-1}\omega^{2lk}m^{2(l,n)},$$

$$0 \cdot\ \ j \cdot\ \ 2n-1,\ 0 \cdot\ \ k \cdot\ \ n-1,$$

where $(0,n) := n$, and (k,n) denotes the greatest common divisor of k and n, and $\omega = \exp\left(\frac{2\pi i}{2n}\right)$.

Proof. Recall that for a permutation τ we let $c(\tau)$ denote the number of cycles in the cycle structure of τ including cycles of length one. Note that if τ is a cycle of length s and $(t,s) = d$, then τ^t has d cycles of length s/d and therefore $c(\tau^t) = d + c(\tau) - 1$ so $c(1) = 2n+3$, $c(a^{2r}) = (2r, 2n) + 3$,

$c(a^{2r}b) = (2r, 2n)+1$ and $c(a^{2r+1}) = (2r+1, 2n)+2$. Now using the character table of U_{6n} and the formula (1.2), the theorem holds. □

Now we discuss the existence of an O-basis associated with the group U_{6n}. Let V be an m-dimensional unitary space over the complex field, if $m = 1$, then $\dim\ {}^{2n+3}V = 1$, so $\dim V_{\lambda}^{2n+3}(U_{6n}) = 0$ or 1, therefore it is trivial that in the case of $\dim V = 1$ an O-basis for every $\lambda \in I(U_{6n})$ exists. Therefore we assume that $m \geq 2$. As before, if χ is a linear character of G, since the orbital subspaces have dimension $\cdot$ 1, then the symmetry class of tensors associated with G and χ has an O-basis. Therefore we will consider non-linear irreducible complex characters of U_{6n}, i.e. the characters ψ_k $0 \cdot k \cdot n-1$.

Note that if $n = 1$, then $U_6 \cong S$ where - $= \{3,4,5\}$ and $S = \langle a_1 = (3\ 4), b_1 = (3\ 4\ 5)\rangle$. In this case, we can consider ψ given by $\psi(a_1^r b_1^s) := \psi_0(a^r b^s)$ as a nonlinear irreducible character of S . we have

$$V_{\psi_0}^5(U_6) = V \quad V \quad V_{\psi}^3(S\ ).$$

Since S is 2-transitive by [4], $V_{\psi}^3(S\ )$ does not have an O-basis, therefore $V_{\psi_0}^5(U_6)$ does not have an O-basis.

Theorem 2. *Let $G = U_{6n}$, $n \geq 1$, and $\psi = \psi_k$, $0 \cdot k \cdot n-1$ and let $m = \dim V \geq 2$. Then $V_{\psi}^{2n+3}(G)$ is non zero and does not have an O-basis.*

Proof. Take $\gamma := (1, \overbrace{2, 2, \ldots, 2}^{(2n+1)-\text{times}}, 1) \in \Gamma_m^{2n+3}$. Since G is generated by the permutations $a = (1\ 2\ \cdots\ 2n)(2n+1\ 2n+2)$ and $b = (2n+1\ 2n+2\ 2n+3)$ we can conclude that $G_\gamma = 1$, and by equation (1.6),

$$\dim V_{\gamma}^{\psi} = \frac{2}{1} \cdot 2 = 4.$$

Therefore $V_{\psi}^{2n+3}(G)$ is non zero. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of V. Now, by the equation (1.8), we have:

$$< e_{\mu\cdot\gamma}^{\psi} | e_{\tau\cdot\gamma}^{\psi} >= \frac{2}{6n}\psi(\tau\mu^{-1}).$$

Since $\tau, \mu \in U_{6n}$, we have $\tau = a^j b^s$ and $\mu = a^k b^t$, for some j, k, s and t in $\mathbb{Z}$, then using the formula (1.8), we obtain:

$$< e_{\mu\cdot\gamma}^{\psi} | e_{\tau\cdot\gamma}^{\psi} >= 0 \Leftrightarrow j+k \text{ is odd and } s \equiv t\ (mod\ 3).$$

Therefore from the set $\{e_{\sigma\cdot\gamma}^{\psi} | \sigma \in G\}$ we can choose at most two orthogonal vector, but $\dim V_{\gamma}^{\psi} = 4$, hence V_{γ}^{ψ} does not have an O-basis. Whence $V_{\psi}^{2n+3}(U_{6n})$ does have an O-basis. □

§3. The Group V_{8n}

In this section we define the group V_{8n}, and we will study the existence of an O-basis for the symmetry classes of tensors associated with this group and irreducible characters of V_{8n}. The dimensions of these symmetrized tensor spaces are also given.

Let n be a positive integer. The group V_{8n} is defined in [6] for n odd. But one can define it for arbitrary n as follows

$$V_{8n} := \langle a, b \mid a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b \rangle.$$

This group has order $8n$ and in the following we will discuss its conjugacy classes and irreducible complex characters of V_{8n}. Since our discuss in the cases of n even or odd differs, therefore first we will assume that n is odd. To describe the conjugacy classes and the irreducible characters of V_{8n} from [6] we see that V_{8n} has $2n+3$ conjugacy classes which are

$$\begin{array}{ll} \{1\}, \{b^2\}, \{a^{2r+1}, a^{-2r-1}b^2\}, & r = 0, \ldots, n-1 \\ \{a^{2s}, a^{-2s}\}, \{a^{2s}b^2, a^{-2s}b^2\}, & s = 1, \ldots, \frac{n-1}{2} \\ \{a^j b^k : j \text{ even}, \ \mathrm{k} = 1 \text{ or } 3\}, & \text{and} \\ \{a^j b^k : j \text{ odd}, \ \mathrm{k} = 1 \text{ or } 3\}. & \end{array}$$

The irreducible complex character table of V_{8n} has four linear characters $\chi_1, \chi_2, \chi_3, \chi_4$, and n characters ψ_j, $0 \cdot j \cdot n-1$, of degree 2, and a further $n-1$ characters ϕ_j, $1 \cdot j \cdot n-1$, of degree 2 as follows:

Table II

The character table of V_{8n} n odd

$\lvert C_{V_{8n}}(\sigma)\rvert$	$8n$	$8n$	$4n$	$4n$	$4n$	4	4
σ	1	b^2	a^{2r+1}	a^{2s}	$a^{2s}b^2$	b	ab
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1
χ_3	1	1	-1	1	1	1	-1
χ_4	1	1	-1	1	1	-1	1
ψ_j $0\cdot j \cdot n-1$	2	-2	$\omega^{2(2r+1)j} - \omega^{-2(2r+1)j}$	$\omega^{4sj} + \omega^{-4sj}$	$-\omega^{4sj} - \omega^{-4sj}$	0	0
ϕ_j $1\cdot j \cdot n-1$	2	2	$\omega^{(2r+1)j} + \omega^{-(2r+1)j}$	$\omega^{2sj} + \omega^{-2sj}$	$\omega^{2sj} + \omega^{-2sj}$	0	0

$$\omega = \exp\left(\frac{2\pi \mathrm{i}}{2\mathrm{n}}\right), \quad 0 \cdot \mathrm{r} \cdot \mathrm{n}-1, \quad 1 \cdot \mathrm{s} \cdot \frac{\mathrm{n}-1}{2}.$$

Now we embed V_{8n} in a suitable symmetric group. It is easy to see that

$a \longmapsto (1\ \ 2\ \cdots\ \ 2n)(2n+1\ \ 2n+2\ \cdots\ \ 4n)$ and

$b \longmapsto (1\ \ 2\ \ 2n+1\ \ 2n+2)\prod_{k=2}^{\frac{n+1}{2}}[(2k-1\ \ 2(n-k)+4\ \ 2(n+k)-1\ \ 2(2n-k)+4)$
$(2k\ \ 2(2n-k)+3\ \ 2(n+k)\ \ 2(n-k)+3)]$

gives an embedding of V_{8n} in S_{4n} so we assume that V_{8n} is a subgroup of S_{4n}. We need the following **observation** for the proof of the next theorem. Suppose that t is an odd positive number and consider the disjoint sets $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. Let $x = (a_1\ a_2\ \cdots\ a_t)$ and $y = (b_1\ b_2\ \cdots\ b_t)$ be two cycles permuting elements of A and B respectively, and let $z = (a_1\ b_1)(a_2\ b_2)\cdots(a_t\ b_t)$ be a permutation on $A \cup B$. Then the permutation xyz is a cycle of length $2t$, and

$$xyz = (a_1\ b_2\ a_3\ \cdots\ a_t\ b_1\ a_2\ b_3\ \cdots\ b_t).$$

In the following theorem we find the dimensions of the symmetry classes of tensors associated with the group V_{8n}.

Theorem 3. *Let $G = V_{8n}$, n odd, and let V be an m-dimensional inner product space. Then considering G as a subgroup of S_{4n}, we have the following:*

$$\dim V_{\chi_1}^{4n}(V_{8n}) = \tfrac{1}{4n}\Bigg[\sum_{k=0}^{n-1} m^{2(2k+1,2n)} + \sum_{k=0}^{\frac{n-1}{2}}(m^{2(2k,2n)} + m^{(2k,2n)}) + nm^n + nm^{2n+1} - \frac{m^{4n}+m^{2n}}{2}\Bigg],$$

$$\dim V_{\chi_2}^{4n}(V_{8n}) = \tfrac{1}{4n}\Bigg[\sum_{k=0}^{n-1} m^{2(2k+1,2n)} + \sum_{k=0}^{\frac{n-1}{2}}(m^{2(2k,2n)} + m^{(2k,2n)}) - nm^n - nm^{2n+1} - \frac{m^{4n}+m^{2n}}{2}\Bigg],$$

$$\dim V_{\chi_3}^{4n}(V_{8n}) = \tfrac{1}{4n}\Bigg[\sum_{k=0}^{n-1} -m^{2(2k+1,2n)} + \sum_{k=0}^{\frac{n-1}{2}}(m^{2(2k,2n)} + m^{(2k,2n)}) + nm^n - nm^{2n+1} - \frac{m^{4n}+m^{2n}}{2}\Bigg],$$

$$\dim V_{\chi_4}^{4n}(V_{8n}) = \tfrac{1}{4n}\Bigg[\sum_{k=0}^{n-1} -m^{2(2k+1,2n)} + \sum_{k=0}^{\frac{n-1}{2}}(m^{2(2k,2n)} + m^{(2k,2n)}) - nm^n + nm^{2n+1} - \frac{m^{4n}+m^{2n}}{2}\Bigg],$$

$$\dim V_{\psi_j}^{4n}(V_{8n}) = \frac{1}{n}\left[\frac{m^{2n}(m^{2n}-1)}{2} + \sum_{k=1}^{\frac{n-1}{2}} m^{(2k,2n)}(m^{(2k,2n)}-1)\cos\frac{4k\pi j}{n}\right],$$

$0 \cdot\ j \cdot\ n-1.$

$$\dim V_{\phi_j}^{4n}(V_{8n}) = \frac{1}{n}\left[\frac{m^{2n}(m^{2n}+1)}{2} + \sum_{k=0}^{n-1} m^{2(2k+1,2n)}\cos(\frac{(2k+1)\pi j}{n}) + \sum_{k=1}^{\frac{n-1}{2}}(m^{2(2k,2n)}+m^{(2k,2n)})\cos(\frac{2k\pi j}{n})\right],$$

$1 \cdot\ j \cdot\ n-1.$

Here, $(0,n) := n$, and (k,n) denotes the greatest common divisor k and n.

Proof. As before, we know that if τ is a cycle of length s, then τ^t has (t,s) cycles and therefore $c(\tau^t) = (t,s) + c(\tau) - 1$, where $c(\tau)$ denotes the number of cycles in the cycle structure of τ including cycles of length one. So $c(1) = 4n$, $c(b^2) = 2n$, $c(a^{2k+1}) = 2(2k+1,2n)$, $c(a^{2k}) = 2(2k,2n)$ and $c(b) = n$. Since the order of ab is 2 by calculation we obtain the only fixed points of ab are $n+1$ and $3n+1$, hence $c(ab) = \frac{4n-2}{2} + 2 = 2n+1$. Also we have $b^2 = (1\ 2n{+}1)(2\ 2n{+}2)\cdots(2n\ 4n)$ and the permutation a is a product of two disjoint cycles $(1\ 2\ \cdots\ 2n)$ and $(2n{+}1\ 2n{+}2\ \cdots\ 4n)$. Since n is odd, by previous observation, one can show that $c(a^{2k}b^2) = \frac{1}{2}c(a^{2k}) = (2k,2n)$. Using the character table of V_{8n}, and the formula (1.2) the theorem follows. □

Now we discuss the existence of an O-basis associated with the group V_{8n} n odd. As before, let V be an m-dimensional unitary space and $m \geq 2$. If $n = 1$, $V_8 \cong D_8$, the dihedral group, and by [5] the symmetry classes of tensor associated V_8 has an O-basis.

Theorem 4. *Let $G = V_{8n}$, n odd, $n \neq 1$, $\phi = \phi_j$, $1 \cdot\ j \cdot\ n-1$. Assume that $m = \dim V \geq 2$. Then $V_\phi^{4n}(G)$ is non-zero and does not have an O-basis.*

Proof. Take $\gamma := (\overbrace{1,2,2,2,\ldots,2}^{2n-\text{times}},\overbrace{1,1,2,2,\ldots,2}^{2n-\text{times}}) = (\gamma_1,\gamma_2,\ldots,\gamma_{4n}) \in \Gamma_m^{4n}$, by the structure of permutations a and b in G, i.e., $a = (1\ 2\ \cdots\ 2n)(2n+1\ 2n+2\ \cdots\ 4n)$ and

$$\begin{array}{rccccc}
b = & (& 1 & 2 & 2n+1 & 2n+2 &) \\
 & (& 3 & 2n & 2n+3 & 4n &) \\
 & (& 4 & 4n-1 & 2n+4 & 2n-1 &) \\
 & (& 5 & 2n-2 & 2n+5 & 4n-2 &) \\
 & & \vdots & \vdots & \vdots & \vdots & \\
 & (& n-1 & 3n+4 & 3n-1 & n+4 &) \\
 & (& n & n+3 & 3n & 3n+3 &) \\
 & (& n+1 & 3n+2 & 3n+1 & n+2 &)
\end{array}$$

one can conclude that $\langle a\rangle \cap G_\gamma = 1$. Since $n \neq 1$, therefore $\langle b\rangle \cap G_\gamma = 1$. Since $(a^r b)^{-1}(1) \neq 1, 2n+1, 2n+2$ for $r = 1, 2, \ldots, 2n-2$ and $(a^{-1}b)^{-1}(2n+2) = 2n \neq 1, 2n+1, 2n+2$, hence $a^r b \notin G_\gamma$ $\forall r \in \mathbb{Z}$. Since $(a^r b^{-1})^{-1}(1) \neq 1, 2n+1, 2n+2$ for $r = 1, 2, \ldots, 2n-2$ and $(a^{-1}b^{-1})^{-1}(2n+2) = ba(2n+2) = b(2n+3) = 4n \neq 1, 2n+1, 2n+2$, hence $a^r b^{-1} \notin G_\gamma$ $\forall r \in \mathbb{Z}$.
Also $(a^r b^2)^{-1}(1) \neq 1, 2n+1, 2n+2$ for $r = 1, 2, \ldots, 2n-2$ and $(a^{-1}b^2)^{-1}(2n+1) = b^2 a(2n+1) = b^2(2n+2) = 2 \neq 1, 2n+1, 2n+2$, hence $a^r b^2 \notin G_\gamma$ $\forall r \in \mathbb{Z}$. Conclude that $G_\gamma = 1$. Therefore $\dim V_\gamma^\phi = 4$. Hence $V_\phi^{4n}(G)$ is non zero.

Now we let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of V and calculate the inner product $< e^\phi_{\mu\cdot\gamma} | e^\phi_{\tau\cdot\gamma} >$ for all $\mu, \tau \in V_{8n}$. Let $\mu = a^k b^s$ and $\tau = a^r b^t$, then we have

$$< e^\phi_{\mu\cdot\gamma} | e^\phi_{\tau\cdot\gamma} >= \frac{2}{8n}\phi(\tau\mu^{-1}) = \frac{1}{4n}\phi(a^k b^{s-t} a^{-r})$$

therefore from table II we get $< e^\phi_{\mu\cdot\gamma} | e^\phi_{\tau\cdot\gamma} >= 0 \Leftrightarrow |s-t|$ is odd. Hence we can choose at most two orthogonal vector from the set $\{e^\phi_{\sigma\cdot\gamma} | \sigma \in G\}$, therefore V_γ^ϕ does not have an O-basis. Hence $V_\phi^{4n}(V_{8n})$ does not have an O-basis. □

Theorem 5. *Let $G = V_{8n}$, n odd, and let $\psi = \psi_j$ $0 \cdot j \cdot n-1$. Assume that $m = \dim V \geq 2$, then $V_\psi^{4n}(G)$ has an O-basis.*

Proof. Let H be a subgroup of G, $\psi = \psi_j$ $0 \cdot j \cdot n-1$. Since $\psi_j(1) = 2$, we have $< \psi \downarrow_H | 1_H >= 0, 1$ or 2. If $< \psi \downarrow_H | 1_H >= 1$, then there is a linear non identity character χ of H such that $\chi = \psi \downarrow_H -1$. Since χ is a linear character, $|\chi(h)| = 1$, for all $h \in H$. First we find the general form of the elements of H using only the condition $|\chi(h)| = |\psi(h) - 1| = 1$.

Since $\psi(b^2) - 1 = -2 - 1 = -3$, we have $b^2 \notin H$, i.e. $H \cap \langle b\rangle = 1$. As before (n, j) denotes the greatest common divisor of n and j. Let $(n, j) = d$, one can conclude that the elements of H can only be chosen from the set:

$$\{\ a^{tn/d}, a^{(2t+1)n/d}b^2, a^t b, a^t b^{-1} \mid t \in \mathbb{Z}\ \}.$$

Because from table II we see that $\psi(a^{(2t+1)n/d}) = \psi(a^{(2t+1)n/d}b^2) = \psi(a^t b^{\pm 1}) = 0$ and $\psi(a^{2tn/d}) = 2$. Since $\sum_{\sigma\in H}\psi(\sigma) = |H|$, therefore half of elements of H must be of the form $a^{2tn/d}$, where $t \in \mathbb{Z}$. Hence the group H is a subgroup of $\langle a^{n/d}\rangle$ or $\langle a^{2n/d},\ a^{(2k+1)}b^{\pm 1}\rangle$, for some $k \in \mathbb{Z}$.

Now instead of H we consider G_γ as a subgroup of G. If $< \psi \downarrow_{G\gamma} | 1_{G_\gamma} >= 1$, we have $\dim V_\gamma^\psi = 2$ and by the above remark $G_\gamma \cdot \langle a^{n/d}\rangle$, then $\{\ e_\gamma, e_{b\cdot\gamma}\ \}$ is an O-basis for V_γ^ψ. If $G_\gamma \cdot \langle a^{2n/d}, a^{(2k+1)}b\rangle$ or $G_\gamma \cdot \langle a^{2n/d}, a^{(2k+1)}b^{-1}\rangle$, then $\{\ e_\gamma, e_{a\cdot\gamma}\ \}$ is an O-basis for V_γ^ψ.

If $< \psi \downarrow_H | 1_H >= 2$, then H is a subgroup of $\ker\psi$. Note that $\cos(\frac{\pi}{n}4sj) = \pm 1$ if and only if $\frac{\pi}{n}4sj = k\pi$ for some $k \in \mathbb{Z}$, if and only if $\frac{n}{d} | 4s$, therefore $\ker\psi \cdot \langle a^{4n/d}\rangle$. In this case $\dim V_\gamma^\psi = 4$ and the set $\{e_\gamma, e_{a^{n/d}\cdot\gamma}, e_{b\cdot\gamma}, e_{a^{n/d}b\cdot\gamma}\ \}$

is an O-basis of V_γ^ψ. The theorem now follows. Note that the order of $a^{\frac{2n}{d}}$ is d and $\psi(a^{2r+1}) = -\psi(a^{-(2r+1)}), \quad r \in \mathbb{Z}$. $\square$

Now we assume that n is even. In this case the group $G = V_{8n}$ has $2n+6$ conjugacy classes which are:

$$\begin{array}{ll}
\{1\}, \{b^2\}, \{a^n\}, \{a^n b^2\}, & \\
\{a^{2r+1}, a^{-(2r+1)}b^2\}, & r = 0, 1, \ldots, n-1 \\
\{a^{2s}, a^{-2s}\}, \{a^{2s}b^2, a^{-2s}b^2\}, & s = 1, \ldots, n/2-1 \\
\{a^{2k}b^{(-1)^k} | 0 \le k \le n-1\}, & \\
\{a^{2k}b^{(-1)^{k+1}} | 0 \le k \le n-1\}, & \\
\{a^{2k+1}b^{(-1)^k} | 0 \le k \le n-1\}, & \\
\{a^{2k+1}b^{(-1)^{k+1}} | 0 \le k \le n-1\}. &
\end{array}$$

The derived subgroup of G is $\langle a^2 b^2 \rangle$, hence G has eight linear characters $\chi_1, \chi_2, \ldots, \chi_8$. Since $H = \langle b^2 \rangle$ is a normal subgroup of G and $G/H \cong D_{4n}$, we obtained $n-1$ irreducible characters $\psi_j \; 1 \le j \le n-1$, of degree 2. Since b^2 is not in the derived subgroup and $(b^2)^2 = 1$, there exists a linear character χ_2 such that $\chi_2(b^2) = -1$. The product of the linear character χ_2 with ψ_j, gives further $n-1$ irreducible characters $\psi_j \cdot \chi_2, \; 1 \le j \le n-1$, of degree 2. Since character values in cases $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$ differ, therefore we distinguish these cases and give the character table of V_{8n} in Table III and Table IV respectively.

The embedding of G in $S_{4n}, \; n =$even, is different from the case $n =$odd. In this case if we take the following permutations in S_{4n},

$a \mapsto (1 \; 2 \; \cdots \; 2n)(2n+1 \; 2n+2 \; \cdots \; 4n)$, and

$$b \mapsto (1 \; 2 \; 2n+1 \; 2n+2)[\prod_{k=2}^{n/2}(2k-1 \; 2(n-k)+4 \; 2(n+k)-1 \; 2(2n-k)+4)$$
$$(2k \; 2(2n-k)+3 \; 2(n+k) \; 2(n-k)+3)](n+1 \; n+2 \; 3n+1 \; 3n+2),$$

then we see that we have a monomorphism of G into S_{4n}. So we assume that G is a subgroup of S_{4n}. Take $\Omega = \{\{1, 2n+1\}, \{2, 2n+2\}, \ldots, \{2n, 4n\}\}$. The group G/H acts on Ω by $\sigma H \cdot \{i, 2n+i\} := \{\sigma(i), \sigma(2n+i)\}$ and this action is faithful. We put $i := \{i, 2n+i\}, 1 \le i \le 2n$, and consider G/H as a subgroup of S_{2n}, therefore the cycle structure of aH and bH on Ω are as follows:

$$aH := (1 \; 2 \; \cdots \; 2n) \text{ and } bH := (1 \; 2)(3 \; 2n)(4 \; 2n-1)\cdots(n \; n+3)(n+1 \; n+2).$$

Therefore we consider $D_{4n} := \langle aH, bH \rangle$ as a subgroup of S_{2n}.

Let $G = V_{8n}, \; n$ even, and $H = \{1, b^2\}, \; \psi = \psi_j, \; 1 \le j \le n-1$. Since $H \le ker\psi$ the character $\overline{\psi}(\sigma H) = \psi(\sigma)$ is an irreducible character of $G/H = D_{4n}$. Let W be a p-dimensional inner product space, $p \geq 2$. Let $\gamma = (1, 1, 2, 2, \ldots, 2)$ be in Γ_p^{2n} and note that $bH \in (D_{4n})_\gamma$ special one can conclude that $(D_{4n})_\gamma =$

$\{1, bH\}$. Similar to the proof of (Theorem 3.1, [5]), if $W_{\gamma}^{\overline{\psi_j}}(D_{4n})$ has an O-basis, then $\overline{\psi}_j(a^k) = 2\cos\frac{2\pi jk}{2n} = 0$ for some k in $\mathbb{Z}$. In other words $\frac{2\pi jk}{2n} = (2l+1)\frac{\pi}{2}$ for some integer l. This implies $2j_2$ divide n, where $j = j_2 j_{2'}$, $j_{2'}$ odd and j_2 a power of 2, i.e.,

$$W_{\gamma}^{\overline{\psi_j}}(D_{4n}) \text{ has an } O\text{−basis } \Rightarrow 2n \equiv 0 \pmod{4j_2}. \tag{3.1}$$

Theorem 6. *Let* $G = V_{8n}$ *and assume that* dim V ≥ 2. *Then* $\;^{4n}V$ *has an O-basis if and only if n is a power of 2.*

Proof. For $\psi = \psi_j, 1 \cdot \; j \cdot \; n-1$ and $(\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \beta_1, \beta_2, \ldots, \beta_{2n}) \in \Gamma_m^{4n}$, we may assume that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{2n})$ are elements of Γ_m^{2n} and therefore we will set $(\alpha, \beta) := (\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \beta_1, \beta_2, \ldots, \beta_{2n})$. In this setting we have:

$$\begin{aligned}
e_{(\alpha,\beta)}^{\psi} &= \frac{\psi(1)}{|G|}\sum_{\sigma\in G}\psi(\sigma)e_{\sigma\cdot(\alpha,\beta)} \\
&= \frac{\psi(1)}{|G|}\frac{1}{2}\left[\sum_{\sigma\in G}\psi(\sigma)e_{\sigma\cdot(\alpha,\beta)} + \sum_{\sigma\in G}\psi(\sigma b^2)e_{\sigma b^2\cdot(\alpha,\beta)}\right] \\
&= \frac{\psi(1)}{|G|}\frac{1}{2}\sum_{\sigma\in G}\psi(\sigma)\left[e_{\sigma\cdot\alpha} \quad e_{\sigma\cdot\beta} + e_{\sigma\cdot\beta} \quad e_{\sigma\cdot\alpha}\right] \\
&= \frac{\psi(1)}{|G|}\frac{1}{2}\sum_{\overline{\sigma}\in G/H} 2\psi(\overline{\sigma})\left[e_{\overline{\sigma}\cdot\alpha} \quad e_{\overline{\sigma}\cdot\beta} + e_{\overline{\sigma}\cdot\beta} \quad e_{\overline{\sigma}\cdot\alpha}\right] \\
&= \frac{\overline{\psi}(1)}{|G/H|}\sum_{\overline{\sigma}\in G/H}\psi(\overline{\sigma})\frac{1}{2}\left[e_{\overline{\sigma}\cdot\alpha} \quad e_{\overline{\sigma}\cdot\beta} + e_{\overline{\sigma}\cdot\beta} \quad e_{\overline{\sigma}\cdot\alpha}\right]
\end{aligned}$$

specially, when $\alpha = \beta$, we have

$$e_{(\alpha,\alpha)}^{\psi} = \frac{\overline{\psi}(1)}{|G/H|}\sum_{\overline{\sigma}\in G/H}\psi(\overline{\sigma})e_{\overline{\sigma}\cdot\alpha} \quad e_{\overline{\sigma}\cdot\alpha}.$$

Hence $< e_{(\alpha,\alpha)}^{\psi} \mid e_{(\beta,\beta)}^{\psi} >= 0$ if and only if $< e_{\alpha}^{\overline{\psi}} \mid e_{\beta}^{\overline{\psi}} >= 0$, moreover

$$\dim V_{(\alpha,\alpha)}^{\psi} = \frac{\psi(1)}{|G_{(\alpha,\alpha)}|}\sum_{\sigma\in G_{(\alpha,\alpha)}}\psi(\sigma) = \frac{\overline{\psi}(H)}{|(G/H)_{\alpha}|}\sum_{\overline{\sigma}\in(G/H)_{\alpha}}\overline{\psi}(\sigma) = \dim V_{\alpha}^{\overline{\psi}}.$$

Therefore $V_{\alpha}^{\overline{\psi}}$ has an O-basis if and only if $V_{(\alpha,\alpha)}^{\psi}$ has an O-basis. Note that $b^2 \in G_{(\alpha,\alpha)}$ and the stabilizer of α under $G/\langle b^2\rangle$ is $G_{(\alpha,\alpha)}/\langle b^2\rangle$.

If $V_{\psi}^{4n}(G)$ has an O-basis, then for every $\gamma \in \Gamma_m^{2n}$, $V_{(\gamma,\gamma)}^{4n}$ has an O-basis. Hence by the above remark we obtain an O-basis for $V_{\gamma}^{\overline{\psi}}$. So $V_{\overline{\psi}}^{2n}(G/H)$ has an O-basis. If $\psi = \psi_j$ and $j = j_2 j_{2'}$ where $j_{2'}$ is the odd part of j and j_2 is a power of 2, then by formula (3.1) we have $2n \equiv 0 \pmod{4j_2}$, which holds all j from $1, 2, \ldots, n-1$. This implies that n is a power of 2.

Conversely, assume that n is a power of 2, and $\psi = \psi_j$ (*or* $\psi_j \cdot \chi_2$). Let $\gamma = (\alpha, \beta) \in \overline{\Delta}$, where $\alpha, \beta \in \Gamma_m^{2n}$. If $\alpha = \beta$, by the above remark $V_\gamma^{\psi_j}$ has an O-basis and

$$\begin{aligned}
e_{(\alpha,\alpha)}^{\psi\cdot\chi_2} &= \tfrac{\psi(1)\cdot\chi_2(1)}{|G|}\sum_{\sigma\in G}\psi(\sigma)\chi_2(\sigma)e_{\sigma\cdot(\alpha,\alpha)} \\
&= \tfrac{\psi(1)}{|G|}\tfrac{1}{2}\left[\sum_{\sigma\in G}\psi(\sigma)\chi_2(\sigma)e_{\sigma\cdot(\alpha,\alpha)} + \sum_{\sigma\in G}\psi(\sigma b^2)\chi_2(\sigma b^2)e_{\sigma b^2\cdot(\alpha,\alpha)}\right] \\
&= \tfrac{\psi(1)}{|G|}\tfrac{1}{2}\sum_{\sigma\in G}\psi(\sigma)\chi_2(\sigma)\left[e_{\sigma\cdot\alpha}\quad e_{\sigma\cdot\alpha} - e_{\sigma\cdot\alpha}\quad e_{\sigma\cdot\alpha}\right] = 0,
\end{aligned}$$

hence $V_\gamma^{\psi_j\cdot\chi_2} = 0$.

If $\alpha \neq \beta$, then $b^2 \in G_\gamma$ which implies that $\langle b\rangle \bigcap G_\gamma = 1$. As in the proof of Theorem 5, $< \psi \downarrow_{G_\gamma} \mid 1_{G_\gamma} >= 0, 1$ or 2.

If $< \psi \downarrow_{G_\gamma} \mid 1_{G_\gamma} >= 2$, then $G_\gamma \cdot \ ker\psi$ and by the formula (1.6), $\dim V_\gamma^\psi = 4$. Using the character table of V_{8n}, Tables III and IV, we obtain $G \cdot \ \langle a^{\frac{2n}{j_2}}\rangle$. In this case, using formula (1.8) and Tables III and IV one can show that the set $\{e_\gamma, e_{b\cdot\gamma}, e_{a^{\frac{n}{2j_2}}\cdot\gamma}, e_{a^{\frac{n}{2j_2}}b^{-1}\cdot\gamma}\}$ is an orthogonal basis for $V_\gamma^\psi(G)$. Hence $V_\gamma^\psi(G)$ has an O-basis.

If $< \psi \downarrow_{G_\gamma} \mid 1_{G_\gamma} >= 1$, then $\psi \downarrow_{G_\gamma} -1$ is a non-identity linear character of G_γ, and the norm $|\psi(x) - 1| = 1$, i.e. $\psi(x) = 0$ or 2 for all $x \in G_\gamma$. By formula (1.6), $\dim V_\gamma^\psi = 2$. From here on we must deal with the cases $\psi = \psi_j$ and $\psi_j \cdot \chi_2$ separately. First assume that $\psi = \psi_j$, $1 \cdot\ j \cdot\ n-1$. The elements of G_γ are of the form $\{a^{(2t+1)\frac{n}{2j_2}}, a^{(2t+1)\frac{n}{2j_2}}b^2, a^t b^{\pm 1} \mid t \in \mathbb{Z}\}$ on which the value of ψ is zero, and $\{a^{2t\frac{n}{j_2}}, a^{2t\frac{n}{j_2}}b^2 \mid t \in \mathbb{Z}\}$ on which the value of ψ is 2. The equality $\sum_{\sigma\in G_\gamma}\psi(\sigma) = |G_\gamma|$ implies that the values of ψ on exactly half of elements of G_γ must be zero. If $a^t b^{\pm 1} \in G_\gamma$, then $(a^t b^{\pm 1})^2 = 1$ or b^2, and since $b^2 \notin G_\gamma$ therefore t must be odd. Therefore $G_\gamma \cdot \ \langle a^{\frac{2n}{j_2}}\rangle \cdot K$ or $\langle a^{\frac{2n}{j_2}}b^2\rangle \cdot K$, where $K = \langle a^{\frac{n}{2j_2}}\rangle$ or $\langle a^{\frac{n}{2j_2}}b^2\rangle$ or $\langle a^{(2t+1)}b^{\pm 1}\rangle$. In the case $K = \langle a^{(2t+1)}b^{\pm 1}\rangle$, the set $\{e_\gamma, e_{a^{\frac{n}{2j_2}}\cdot\gamma}\}$ and in other cases $\{e_\gamma, e_{b\cdot\gamma}\}$ is an orthogonal for V_γ^ψ. Hence V_γ^ψ has an O-basis.

If $\psi = \psi_j \cdot \chi_2$, similar to the previous case, G_γ is a subgroup of the form $\langle a^{\frac{n}{j_2}}\rangle \cdot K$ where $K = \langle a^{(2t+1)}b^{\pm 1}\rangle$ or $\langle a^{(2t+1)\frac{n}{2j_2}}\rangle$ or $\langle a^{(2t+1)\frac{n}{2j_2}}b^2\rangle$. If $K = \langle a^{(2t+1)}b^{\pm 1}\rangle$ take the set $\{e_\gamma, e_{a^{\frac{n}{2j_2}}\cdot\gamma}\}$ and the other cases the set $\{e_\gamma, e_{b\cdot\gamma}\}$ are the orthogonal basis for V_γ^ψ. □

Theorem 7. *Let $G = V_{8n}$, n even. Assume $m = \dim \mathrm{V} \geq 2$. Then the dimensions of symmetry classes of tensors associated with G and the irreducible characters of G are:*

$$\dim V_{\chi_1}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} + nm^{2n+2} + (3+n)m^{2n} + 2nm^n + 2\sum_{k=0}^{n-1} m^{2(2k+1,2n)} + 4\sum_{s=1}^{n/2-1} m^{2(2s,2n)} \right\},$$

$$\dim V_{\chi_2}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} - nm^{2n+2} + (n-1)m^{2n} \right\},$$

$$\dim V_{\chi_3}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} + nm^{2n+2} + (3+n)m^{2n} - 2nm^n + 4\sum_{s=1}^{n/2-1} m^{2(2s,2n)} - 2\sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},$$

$$\dim V_{\chi_4}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} - nm^{2n+2} + (n-1)m^{2n} \right\},$$

$$\dim V_{\chi_5}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} - nm^{2n+2} + (3-n)m^{2n} - 2nm^n + 4\sum_{s=1}^{n/2-1} m^{2(2s,2n)} + 2\sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},$$

$$\dim V_{\chi_6}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} + nm^{2n+2} - (n+1)m^{2n} \right\},$$

$$\dim V_{\chi_7}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} - nm^{2n+2} - (n-3)m^{2n} + 2nm^n + 4\sum_{s=1}^{n/2-1} m^{2(2s,2n)} - 2\sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},$$

$$\dim V_{\chi_8}^{4n}(G) = \frac{1}{8n}\left\{ m^{4n} + nm^{2n+2} - (n+1)m^{2n} \right\},$$

$$\dim V^{4n}_{\psi_j}(G) = \tfrac{1}{4n}\Bigg\{2m^{4n} + 2(1+2(-1)^j)m^{2n} + 8\sum_{s=1}^{n/2-1} m^{2(2s,2n)}\cos\frac{2\pi sj}{n} + 4\sum_{k=0}^{n-1} m^{2(2k+1,2n)}\cos\frac{\pi j(2k+1)}{n}\Bigg\},$$

$$\dim V^{4n}_{\psi_j\cdot\chi_2}(G) = \tfrac{1}{4n}\left\{\left(2m^{4n} - 2m^{2n}\right)\right\},$$

where $1 \cdot \ j \cdot \ n-1$.

Proof. Similarly to the proof of the Theorem 3, note that $c(1) = 4n$, $c(b^2) = c(a^n) = c(a^n b^2) = 2n,\ c(a^r) = 2(r, 2n),\ c(a^{4s}b^2) = 2(4s, 2n)$, $c(a^{4t+2}b^2) = 2(4t+2, 2n),\ c(b^{\pm 1}) = n,\ c(ab) = 2n$ and $c(ab^{-1}) = 2n+2$. □

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Table III
The character table of V_{8n}, $n \equiv 0 \pmod 4$,

$\lvert Cl(\sigma)\rvert$	1	1	1	1	2	2	2	2	2	2	n	n	n	n
σ	1	b^2	a^n	$a^n b^2$	a^{4k+1}	a^{4k+3}	a^{4s}	a^{4t+2}	$a^{4s}b^2$	$a^{4t+2}b^2$	b	b^{-1}	ab	ab^{-1}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	i	-i	1	-1	-1	1	-i	i	1	-1
χ_3	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
χ_4	1	-1	1	-1	-i	i	1	-1	-1	1	i	-i	1	-1
χ_5	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_6	1	-1	1	-1	i	-i	1	-1	-1	1	i	-i	-1	1
χ_7	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
χ_8	1	-1	1	-1	-i	i	1	-1	-1	1	-i	i	-1	1
ψ_j	2	2	$2(-1)^j$	$2(-1)^j$	$\alpha^{j(4k+1)}$	$\alpha^{j(4k+3)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0
$\psi_j \cdot \chi_2$	2	-2	$2(-1)^j$	$-2(-1)^j$	$i\alpha^{j(4k+1)}$	$-i\alpha^{j(4k+3)}$	$\alpha^{j(4s)}$	$-\alpha^{j(4t+2)}$	$-\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0

$\alpha^{jr} = \omega^{jr} + \omega^{-jr} = 2\cos(\frac{\pi jr}{n})$, $\omega = \exp\left(\frac{2\pi i}{2n}\right)$;
$0 \cdot\ k \cdot\ n/2-1,\ \ 1 \cdot\ s \cdot\ n/4-1,\ \ 0 \cdot\ t \cdot\ n/4-1,\ \ 1 \cdot\ j \cdot\ n-1.$

Table IV
The character table of V_{8n}, $n \equiv 2 \pmod 4$,

$\lvert Cl(\sigma)\rvert$	1	1	1	1	2	2	2	2	2	2	n	n	n	n
σ	1	b^2	a^n	$a^n b^2$	a^{4k+1}	a^{4k+3}	a^{4s}	a^{4t+2}	$a^{4s}b^2$	$a^{4t+2}b^2$	b	b^{-1}	ab	ab^{-1}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	i	-i	1	-1	-1	1	-i	i	1	-1
χ_3	1	1	1	1	-1	-1	1	1	1	1	-1	-1	1	1
χ_4	1	-1	-1	1	-i	i	1	-1	-1	1	i	-i	1	-1
χ_5	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
χ_6	1	-1	-1	1	i	-i	1	-1	-1	1	i	-i	-1	-1
χ_7	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1
χ_8	1	-1	-1	1	-i	i	1	-1	-1	1	-i	i	-1	1
ψ_j	2	2	$2(-1)^j$	$2(-1)^j$	$\alpha^{j(4k+1)}$	$\alpha^{j(4k+3)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	$\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0
$\psi_j \cdot \chi_2$	2	-2	$-2(-1)^j$	$2(-1)^j$	$i\alpha^{j(4k+1)}$	$-i\alpha^{j(4k+3)}$	$\alpha^{j(4s)}$	$-\alpha^{j(4t+2)}$	$-\alpha^{j(4s)}$	$\alpha^{j(4t+2)}$	0	0	0	0

$\alpha^{jr} = \omega^{jr} + \omega^{-jr} = 2\cos(\frac{\pi jr}{n})$, $\omega = \exp\left(\frac{2\pi i}{2n}\right)$;

$0 \cdot\ k \cdot\ n/2-1,\ \ 1 \cdot\ s \cdot\ n/4-1,\ \ 0 \cdot\ t \cdot\ n/4-1,\ \ 1 \cdot\ j \cdot\ n-1.$

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