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Skew-symmetric Hadamard matrices and association schemes

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Abstract. In this paper, we give a method to construct an association scheme from another association scheme under some assumptions. To make a new scheme, we construct a skew-symmetric Hadamard matrix.

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§1. Introduction

We know that many association schemes are constructed by finite groups, but we also know that many association schemes are not. For example, there exist eighteen association schemes of order 23 and of class 2 with intransitive automorphisms. We do not know how to construct them theoretically. They are constructed by a computer.

We consider a non-symmetric association scheme of class 2. Then we can construct a skew-symmetric Hadamard matrix. By the Hadamard matrix, we can construct other association schemes. It may be isomorphic to the original one, but we investigate when they are isomorphic, and later give an example such that they are non-isomorphic.

Our method is like switching operation in graph theory. But under our assumptions, switching cannot generate association schemes, except the trivial cases, switching for no point or all points. Roughly speaking, we add one point to the association scheme and consider the switching relations.

§2. Association schemes

First we define association schemes. For a matrix M, we use the notation M_{ij} as the (i, j)-entry, and ${}^tM$ as the transposed matrix of M.

Let X be a finite set of the cardinality n, and let R_i be a subset of $X \times X$, $i = 0, \cdots, d$. Define $n \times n$ matrices A_i indexed by X as

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } (x,y) \in R_i, \\ 0, & \text{if } (x,y) \notin R_i, \end{cases}$$

for $i = 0, \cdots, d$. Then $(X, \{R_i\}_{i=0,\cdots,d})$ is called an *association scheme* if the following conditions hold :

(1) $A_0 = I$ (the identity matrix),

(2) $\sum_{i=0}^{d} A_i = J$ (the all one matrix),

(3) ${}^tA_i = A_{i'}$ for some $i' \in \{0, \cdots, d\}$,

(4) and $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for some p_{ij}^k.

We call n the *order*, d the *class*, and A_i an adjacency matrix of the association scheme. By the condition (4) in the definition, A_i has the same row (column) sums, and we call it the *valency* of A_i. Actually, the valency of A_i is equal to $p_{ii'}^0$ in (4).

To denote an association scheme, we use the *relation matrix*

$$A = \sum_{i=0}^{d} iA_i.$$

For convenience of our notation, we say that the matrix A is an association scheme.

We define the automorphism group of an association scheme A. Let $\mathcal{S}_n$ be the group consisting of all permutation matrices of degree n. Define the automorphism group of A by

$$\mathrm{Aut}(A) = \{P \in \mathcal{S}_n \mid P^{-1}AP = A\}.$$

It is easy to check that P is in $\mathrm{Aut}(A)$ if and only if $P^{-1}A_iP = A_i$ for all i.

Let A and B be association schemes of same order n. We define that A and B are *isomorphic* if there exists a permutation matrix P of degree n and a permutation ρ on $\{0, \cdots, d\}$ such that $P^{-1}A^\rho P = B$, where A^ρ is the entry wise action of ρ. If ρ is the identity, we say A and B are *strongly isomorphic*.

In this paper, we only consider the case that $n \equiv 3 \pmod 4$, $d = 2$, and the valency of A_1 and A_2 are equal. In this case, A_1 must be non-symmetric matrix, so ${}^tA_1 = A_2$. The reason of this assumption is as follows.

We are concerning to the case that n is a prime. We know some examples of association schemes of prime order with intransitive automorphism groups, and all of them satisfying the above conditions, except when $n \equiv 1 \pmod 4$ and A_1 and A_2 are symmetric. We do not know how to construct such schemes in general, so we want to consider this case.

In our assumption, the valencies are $\{1, k, k\}$, where $k = (n-1)/2$. For details about association schemes, we refer to the book of Bannai and Ito [1].

§3. Hadamard matrices

In this section, we define Hadamard matrices. Let H be an $m \times m$ matrix with all entries in $\{\pm 1\}$. We call H a *Hadamard matrix* if

$$^tHH = mI.$$

This means that distinct row (column) vectors of H are orthogonal. The next theorem is well-known.

Theorem 3.1 ([2, Lemma 9.4]). *If an $m \times m$ matrix H is a Hadamard matrix, then $m = 1, 2$, or $m \equiv 0 \pmod 4$.*

It is conjectured that there exists a Hadamard matrix of size $4a$ for any integer a, but this is still open.

A Hadamard matrix H is called *skew-symmetric* if $H_{ij} = -H_{ji}$ for all $i \neq j$ and $H_{ii} = 1$ for all i.

Now we define the automorphism group of a skew-symmetric Hadamard matrix, though it is not a standard definition. ¿From here, we always assume that H is a skew-symmetric Hadamard matrix. Let $\mathcal{S}_m$ be the group consisting of all permutation matrices of degree m, and let $\mathcal{D}_m$ be the group consisting of all diagonal matrices with all diagonal entries in $\{\pm 1\}$. Define $\mathcal{S}_m^{\pm}$ be the group generated by $\mathcal{S}_m$ and $\mathcal{D}_m$. Then $\mathcal{S}_m^{\pm}$ is a semidirect product of $\mathcal{S}_m$ and $\mathcal{D}_m$. If $Q \in \mathcal{S}_m^{\pm}$, then $Q^{-1}HQ$ is also a skew-symmetric Hadamard matrix. We define

$$\mathrm{Aut}^{\pm}(H) = \{Q \in \mathcal{S}_m^{\pm} \mid Q^{-1}HQ = H\}.$$

There exists a natural epimorphism from $\mathcal{S}_m^{\pm}$ to $\mathcal{S}_m$. We call the image of $\mathrm{Aut}^{\pm}(H)$ by this epimorphism as the automorphism group of H, and denote it by $\mathrm{Aut}(H)$. In general, it is easy to see that $\mathrm{Aut}^{\pm}(H) \cong \mathrm{Aut}(H)$.

We fix i. We say H is *normalized* at i if $H_{ij} = 1$ for all j and $H_{ji} = -1$ for all $j \neq i$. For any H and any i, we can normalize H at i as follows. Consider the diagonal matrix D with

$$D_{jj} = \begin{cases} 1, & \text{if } H_{ij} = 1, \\ -1, & \text{if } H_{ij} = -1. \end{cases}$$

Then $D^{-1}HD$ is normalized at i.

§4. Construction of new association schemes

Let A be an association scheme. Recall our assumption. The order is $n \equiv 3 \pmod 4$, the class is $d = 2$, the valencies are $\{1, k, k\}$ where $k = (n-1)/2$, and ${}^tA_1 = A_2$. Then we can conclude that

$$\begin{aligned} {A_1}^2 &= \frac{k-1}{2} A_1 + \frac{k+1}{2} A_2, \\ A_1A_2 = A_2A_1 &= kA_0 + \frac{k-1}{2} A_1 + \frac{k-1}{2} A_2, \\ {A_2}^2 &= \frac{k+1}{2} A_1 + \frac{k-1}{2} A_2 \end{aligned}$$

by [1, II. Proposition 2.2].

Now we define a skew-symmetric Hadamard matrix. Put

$$H = \left(\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline -1 & & & \\ \vdots & & A_0 + A_1 - A_2 & \\ -1 & & & \end{array}\right).$$

Proposition 4.1. *The matrix H defined above is a skew-symmetric Hadamard matrix.*

Proof. By the direct calculation, ${}^tHH = (n+1)I$. Since ${}^tA_1 = A_2$, H is skew-symmetric. □

We consider the normalization of H at some i, and remove i-th row and column of it. We write this matrix as $H^{(i)}$. Define an $n \times n$ matrix $A^{(i)}$ by

$${A^{(i)}}_{jk} = \begin{cases} 0, & \text{if } j = k, \\ 1, & \text{if } j \neq k \text{ and } {H^{(i)}}_{jk} = 1, \\ 2, & \text{if } j \neq k \text{ and } {H^{(i)}}_{jk} = -1. \end{cases}$$

Then we have the following.

Proposition 4.2. *The matrix $A^{(i)}$ is an association scheme.*

Proof. Let A'_0, A'_1, A'_2 be adjacency matrices of $A^{(i)}$. We consider that A'_i is indexed by $\{1, \cdots, n+1\} \setminus \{i\}$. It is enough to check the condition (4) in the definition.

We consider the a-th and b-th rows of H, where $a \neq i$, $b \neq i$, and $a \neq b$. Put

$$\begin{array}{rclrcl}\alpha & = & \sharp\,\{j \mid H_{aj}=1,\ H_{bj}=1\}, & \beta & = & \sharp\,\{j \mid H_{aj}=1,\ H_{bj}=-1\},\\ \gamma & = & \sharp\,\{j \mid H_{aj}=-1,\ H_{bj}=1\}, & \delta & = & \sharp\,\{j \mid H_{aj}=-1,\ H_{bj}=-1\}.\end{array}$$

Since the a-th row is orthogonal to the i-th row $(1,\cdots,1)$, we have $\alpha+\beta=\gamma+\delta$, and similarly $\alpha+\gamma=\beta+\delta$. Also the a-th and b-th rows are orthogonal, so $\alpha+\delta=\beta+\gamma$. These equations imply that $\alpha=\beta=\gamma=\delta$, and obviously $\alpha+\beta+\gamma+\delta=n+1$, so they equal $(n+1)/4$.

Now we consider the (a,b)-entry of ${A'_1}^2$, where $a \neq i$ and $b \neq i$. We have

$$\begin{array}{rcl}({A'_1}^2)_{ab} & = & \sharp\,\{j \mid H_{aj}=1,\ H_{jb}=1,\ j \neq i,a,b\}\\ & = & \sharp\,\{j \mid H_{aj}=1,\ H_{bj}=-1,\ j \neq i,a,b\}.\end{array}$$

If $a=b$, then $({A'_1}^2)_{aa}=0$. Suppose $a \neq b$. For $j=i,a,b$,

$$(H_{ai},H_{bi})=(-1,-1),\quad (H_{aa},H_{ba})=(1,\varepsilon),\quad (H_{ab},H_{bb})=(-\varepsilon,1),$$

where $\varepsilon=1$ or -1. Thus

$$({A'_1}^2)_{ab}=\begin{cases}(n-3)/4, & \text{if } H_{ab}=1,\\ (n+1)/4, & \text{if } H_{ab}=-1.\end{cases}$$

This means that ${A'_1}^2$ is a linear combination of A'_0, A'_1, and A'_2. Similarly ${A'_2}^2$ is also a linear combination of them.

Next we consider the (a,b)-entry of $A'_1A'_2$, where $a \neq i$ and $b \neq i$, by the similar argument above. We have

$$\begin{array}{rcl}(A'_1A'_2)_{ab} & = & \sharp\,\{j \mid H_{aj}=1,\ H_{jb}=-1,\ j \neq i,a,b\}\\ & = & \sharp\,\{j \mid H_{aj}=1,\ H_{bj}=1,\ j \neq i,a,b\}.\end{array}$$

If $a=b$, then $(A'_1A'_2)_{aa}=\sharp\,\{j \mid H_{aj}=1,\ j \neq i,a\}=(n-1)/2$. Suppose $a \neq b$. Then, considering the case $j=i,a,b$, we have

$$(A'_1A'_2)_{ab}=\begin{cases}(n+1)/4, & \text{if } H_{ab}=1,\\ (n-1)/4, & \text{if } H_{ab}=-1.\end{cases}$$

Thus $A'_1A'_2$ is a linear combination of A'_0, A'_1, and A'_2, and similarly so is $A'_2A'_1$. The result follows. □

Next we consider when $A^{(i)}$ and $A^{(j)}$ are isomorphic.

Proposition 4.3. *If there exists $\sigma \in \mathrm{Aut}(H)$ such that $i^\sigma=j$, then $A^{(i)}$ and $A^{(j)}$ are strongly isomorphic.*

Proof. We may assume that H is normalized at i. Then the permutation matrix P defined by σ gives an isomorphism from $A^{(i)}$ to $A^{(j)}$ □

Also, we have the following.

Proposition 4.4. *If $A^{(i)}$ and $A^{(j)}$ are strongly isomorphic, then there exists $\sigma \in \mathrm{Aut}(H)$ such that $i^\sigma = j$.*

Proof. We may assume that H is normalized at i. There exists a permutation matrix P such that $P^{-1}A^{(i)}P = A^{(j)}$. We can regard P as a bijection $\{1, \cdots, n+1\} \setminus \{i\} \to \{1, \cdots, n+1\} \setminus \{j\}$. We define $P' \in \mathcal{S}_{n+1}$ by $k^{P'} = k^P$ if $k \neq i$ and $i^{P'} = j$. Then $P' \in \mathrm{Aut}^{\pm}(H)$, and the result holds. □

By this proposition, if H has a transitive automorphism group, then we can get only one association scheme by this method. Actually, in the next section, we will show that the Hadamard matrix obtained by a cyclotomic scheme has a transitive automorphism group.

Of course, our construction is applicable for an arbitrary skew-symmetric Hadamard matrix.

§5. Examples

In this section, we introduce the cyclotomic scheme of class 2 with a prime power order. We shall show that the Hadamard matrix obtained by it has a transitive automorphism group. So we can not construct new association schemes from it. Later we will give an example of a skew-symmetric Hadamard matrix with an intransitive automorphism group.

Let F be the finite field of order $q = p^e$, $q \equiv 3 \pmod 4$. Put $F^\times$ the multiplicative group of F, and $(F^\times)^2$ the set of squares in $F^\times$. Since $q \equiv 3 \pmod 4$, $x \in (F^\times)^2$ if and only if x has an odd order in $F^\times$. We define $r : F^\times \to \{\pm 1\}$ by $r(x) = 1$ if $x \in (F^\times)^2$ and otherwise $r(x) = -1$. Then we have $r(x)r(y) = r(xy)$, $r(-x) = -r(x)$, and $r(x) = r(x^{-1})$.

We define an association scheme. Define relations on F by $(x, x) \in R_0$, $(x, y) \in R_1$ if $u(x - y) = 1$, and otherwise $(x, y) \in R_2$. Then it is well known that this is an association scheme, which is called the *cyclotomic scheme* of class 2. This satisfies all our assumptions, and it is easy to see that this association scheme has a transitive automorphism group.

We define the skew-symmetric Hadamard matrix H by the cyclotomic scheme of class 2. For convenience, we consider the matrix is indexed by $F \cup \{\infty\}$. To show that the automorphism group of H is transitive, it is enough to show that ∞ is not a fixed point. So we shall give an automorphism of H which moves ∞. Now

$$H_{xy}=\begin{cases} 1, & \text{if } x=y \text{ or } x=\infty,\\ -1, & \text{if } x\in F \text{ and } y=\infty,\\ r(x-y), & \text{if } x\neq y,\ x,y\in F.\end{cases}$$

We consider the normalization at 0. Define a diagonal matrix D by

$$D_{xx}=\begin{cases} -r(x), & \text{if } x\in F^{\times},\\ 1, & \text{if } x=0,\\ -1, & \text{if } x=\infty.\end{cases}$$

Then $K=D^{-1}HD$ is a Hadamard matrix normalized at 0. Now

$$\begin{aligned} K_{\infty y} &= D_{\infty\infty}D_{yy}H_{\infty y}=r(y),\\ K_{x\infty} &= D_{xx}D_{\infty\infty}H_{x\infty}=-r(x),\\ K_{xy} &= D_{xx}D_{yy}H_{xy}=r(x)r(y)r(x-y),\end{aligned}$$

for $x,y\in F^{\times}$, $x\neq y$.

We define a permutation σ on $F\cup\{\infty\}$. Let $x^{\sigma}=-x^{-1}$ for $x\in F^{\times}$, $0^{\sigma}=\infty$, and $\infty^{\sigma}=0$. Put P the permutation matrix given by σ, and $L=P^{-1}D^{-1}HDP$. Then L is normalized at ∞ and

$$\begin{aligned} L_{0y} &= K_{\infty(-y^{-1})}=r(-y^{-1})=-r(y),\\ L_{x0} &= K_{(-x^{-1})\infty}=-r(-x^{-1})=r(x),\\ L_{xy} &= K_{(-x^{-1})(-y^{-1})}=r(x^{-1})r(y^{-1})r(-x^{-1}+y^{-1})\\ &= r(x)r(y)r(-x^{-1}+y^{-1})=r(xy(-x^{-1}+y^{-1}))=r(x-y).\end{aligned}$$

This means $L=H$, so $\sigma\in \mathrm{Aut}(H)$ and σ moves ∞.

Proposition 5.1. *The skew-symmetric Hadamard matrix defined by a cyclotomic scheme of class* 2 *has a transitive automorphism group.*

Now we give an example of a skew-symmetric Hadamard matrix H of degree

24 with intransitive automorphism group.

$$
\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
-&1&1&1&1&1&1&1&1&1&1&1&1&-&-&-&-&-&-&-&-&-&-&-\\
-&-&1&1&1&1&1&1&-&-&-&-&-&1&1&1&1&1&1&-&-&-&-&-\\
-&-&-&1&1&1&1&1&1&-&-&-&-&1&-&-&-&-&-&1&1&1&1&1\\
-&-&-&-&1&1&1&-&-&1&1&1&-&-&1&1&1&-&-&1&1&1&-&-\\
-&-&-&-&-&1&1&1&-&1&1&-&1&-&1&-&-&1&1&1&-&-&1&1\\
-&-&-&-&-&-&1&1&1&1&-&1&1&1&-&1&1&1&-&-&1&-&1&-\\
-&-&-&-&1&-&-&1&1&-&1&1&1&1&1&1&-&-&1&-&-&1&-&1\\
-&-&1&-&1&1&-&-&1&1&-&1&-&-&-&-&1&1&1&-&-&1&1&1\\
-&-&1&1&-&-&-&1&-&1&1&1&-&1&-&-&1&-&1&1&1&-&-&1\\
-&-&1&1&-&-&1&-&1&-&1&-&1&-&1&-&1&-&1&-&1&1&1&-\\
-&-&1&1&-&1&-&-&-&-&1&1&1&1&-&1&-&1&-&1&-&1&1&-\\
-&-&1&1&1&-&-&-&1&1&-&-&1&-&1&1&-&1&-&1&1&-&-&1\\
-&1&-&-&1&1&-&-&1&-&1&-&1&1&-&-&1&1&1&1&1&-&-&-\\
-&1&-&1&-&-&1&-&1&1&-&1&-&1&1&-&-&1&1&1&-&1&-&-\\
-&1&-&1&-&1&-&-&1&1&1&-&-&1&1&1&1&-&-&-&-&-&1&1\\
-&1&-&1&-&1&-&1&-&-&-&1&1&-&1&-&1&1&-&-&1&1&-&1\\
-&1&-&1&1&-&-&1&-&1&1&-&-&-&-&1&-&1&1&-&1&1&1&-\\
-&1&-&1&1&-&1&-&-&-&-&1&1&-&-&1&1&-&1&1&-&-&1&1\\
-&1&1&-&-&-&1&1&1&-&1&-&-&-&-&1&1&1&-&1&-&1&-&1\\
-&1&1&-&-&1&-&1&1&-&-&1&-&-&1&1&-&-&1&1&1&-&1&-\\
-&1&1&-&-&1&1&-&-&1&-&-&1&1&-&1&-&-&1&-&1&1&-&1\\
-&1&1&-&1&-&-&1&-&1&-&-&1&1&1&-&1&-&-&1&-&1&1&-\\
-&1&1&-&1&-&1&-&-&-&1&1&-&1&1&-&-&1&-&-&1&-&1&1
\end{pmatrix}
$$

Here we write "$-$" instead of -1. The automorphism group of H is not transitive, it is not even half-transitive. It has eight orbits, so we can get eight strongly isomorphism classes of association schemes. To get them, consider $A^{(1)}$, $A^{(4)}$, $A^{(5)}$, $A^{(6)}$, $A^{(10)}$, $A^{(11)}$, $A^{(12)}$, and $A^{(13)}$. But some of them are isomorphic : $A^{(1)} \cong A^{(10)}$, $A^{(4)} \cong A^{(13)}$ $A^{(5)} \cong A^{(11)}$, and $A^{(6)} \cong A^{(12)}$. So we can get four isomorphism classes of association schemes from H.

By the computer classification, there exist exactly eight equivalence classes of skew-symmetric Hadamard matrices of degree 24 (here we say that two Hadamard matrices H and H' are equivalent if there exists $Q \in \mathcal{S}_{24}^{\pm}$ such that $Q^{-1}HQ = H'$ or $Q^{-1}HQ = {}^{t}H'$), and exactly nineteen isomorphism classes of association scheme of order 23 which satisfies our assumption.

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