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Remarks on the Decay Rate for the Energy of the Dissipative Linear Wave Equations in Exterior Domains

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Abstract. Combining the results in Ikehata-Matsuyama [5] with the Nakao inequality ([6], Lemma 2.2), we will derive more precise decay rate like $E(t) \leq C/(1+t)^2$ for the total energy $E(t)$ to the mixed problem of the dissipative linear wave equation in an exterior domain through the multiplier method only.

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§1. Introduction

We are concerned with the following linear wave equation with linear dissipative term

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty), \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N (N \geq 2)$ is an exterior domain with smooth compact boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$. In the sequal $\|\cdot\|$ means the usual $L^2(\Omega)$-norm. The total energy for the equation (1.1) is defined by

$$E(t) = \frac{1}{2}(\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2).$$

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In this paper we discuss the decay rate of $E(t)$. First of all let us mention the related works concerning this problem.

By relying on the spectral analysis Dan-Shibata [2] have obtained the local energy decay estimate depending on the space dimension N:

$$E_R(t) = \frac{1}{2}\int_{\Omega\cap B_R}\{|u_t(t,x)|^2 + |\nabla u(t,x)|^2\}dx \le C(1+t)^{-N} \tag{1.4}$$

for the compactly supported initial data $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$, where $B_R = \{x \in \mathbf{R}^N |\ |x| < R\}$. On the other hand, in Kawashima-Nakao-Ono [6] they have also derived the total energy decay rate faster than $(1+t)^{-1}$ to the Cauchy problem in $\mathbf{R}^N$ of the equation (1.1) by using the Fourier transformation. Simply speaking, however, we can not apply their method to the exterior mixed problem (1.1)-(1.3) because of the existence of the boundary $\partial\Omega$. Furthermore, they demand the slightly restricted assumptions on the initial data: $[u_0, u_1] \in L^r(\mathbf{R}^N) \times L^r(\mathbf{R}^N)(1 \le r \le 2)$. If we take $r = 2$, they have merely derived the usual energy decay rate like $E(t) \le C/(1+t)$ (for another type of equation with strong dissipation, see Ikehata [4]).

The purpose of this paper is to obtain the total energy decay rate faster than $(1+t)^{-1}$ in the framework of L^2-space. It is easy to derive the decay estimate: $E(t) \le C(1+t)^{-1}$ by the usual energy method. But, it seems unknown whether the total energy $E(t)$ to the problem (1.1)-(1.3) with the non-compact support initial data $[u_0, u_1]$ decays faster than $(1+t)^{-1}$ or not without the $L^r(\mathbf{R}^N) \times L^r(\mathbf{R}^N)(1 \le r < 2)$ assumptions of the initial data.

Now let us state our results. First let us define a function $d(x)$ as follows:

$$d(x) = \begin{cases} |x|, & N \ge 3, \\ |x|\log(B|x|), & N = 2, \end{cases} \tag{1.5}$$

where B is a constant such that $B \ge 2\sup\{|x|^{-1};\ x \in \Omega\} > 0$. Then it is well known that the following Hardy type inequality holds (cf. Dan-Shibata [2]).

Lemma 1.1. *Let $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\|u/d(\cdot)\|^2 \le C\|\nabla u\|^2,$$

where $d(x)$ is the function defined by (1.5).

Based on this inequality, Ikehata-Matsuyama [5] have just proved the following L^2-decay property of a weak solution $u(t,x)$ to the "exterior" problem (1.1)-(1.3).

Theorem 1.2. ([5]) *Let $N \ge 2$ and assume that the initial data $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfies $\|d(\cdot)(u_0 + u_1)\| < +\infty$. Then the* *weak solution* $u(t,x) \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ *to the problem* (1.1)-(1.3) *satisfies*

$$(1+t)\|u(t,\cdot)\|^2 \le C(\|u_0\|_{H^1}^2 + \|u_1\|^2 + \|d(\cdot)(u_0+u_1)\|^2),$$

$$(1+t)E(t) \le C(\|u_0\|_{H^1}^2 + \|u_1\|^2)$$

for all $t \ge 0$ *with some constant* $C > 0$ *independent of* $t \in [0,+\infty)$*, where* $\|u_0\|_{H^1}$ *denotes the usual* $H^1(\Omega)$*-norm of* u_0.

Now by applying the Nakao inequality (see [6], Lemma 2.2), our main result reads as follows.

Theorem 1.3. *Let* $N \ge 2$ *and assume that the initial data* $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ *further satisfies* $\|d(\cdot)(u_0+u_1)\| < +\infty$. *Then the associated solution* $u(t,x) \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ *to the problem* (1.1)-(1.3) *satisfies*

$$E(t) \le C/(1+t)^2$$

for all $t \ge 0$ *with some constant* $C > 0$ *depending on* $\|u_0\|_{H^1}, \|u_1\|$ *and* $\|d(\cdot)(u_0+u_1)\|$.

As a corollary, especially, in the case when $N = 2$, one has the extension of the results in [2] (see (1.4)) through the quite simple multiplier method.

Corollary 1.4. *Let* $N = 2$ *and let* $R > 0$ *be arbitrarily fixed such that* $\partial\Omega \subset B_R$. *Assume that the initial data* $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ *further satisfies*

$$supp\, u_0 \cup supp\, u_1 \subset \Omega \cap B_R.$$

Then it holds that

$$E(t) \le C/(1+t)^2$$

with some constant $C > 0$ *depending on* $\|u_0\|_{H^1}, \|u_1\|$ *and* $\|d(\cdot)(u_0+u_1)\|$*, so that the local energy* $E_R(t)$ *also decays with the same rate.*

Moreover, by using Lemma 1.1 (cf. Escobedo-Kavian [3]) replaced by the following Lemma 1.5, one can also deal with the Cauchy problem in $\mathbf{R}^N (N \ge 3)$:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^N, \tag{1.6}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^N. \tag{1.7}$$

Lemma 1.5. *Let* $N \ge 3$ *and* $u \in H^1(\mathbf{R}^N)$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \frac{|u(x)|^2}{(1+|x|)^2}\, dx \le C\|\nabla u\|^2.$$

Our result reads as follows.

Theorem 1.6. *Let $N \geq 3$ and assume that the initial data* $[u_0, u_1] \in H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ *further satisfies* $\|(1+|x|)(u_0+u_1)\| < +\infty$. *Then the weak solution* $u \in C([0,+\infty); H^1(\mathbf{R}^N)) \cap C^1([0,+\infty); L^2(\mathbf{R}^N))$ *to the problem* (1.6)-(1.7) *satisfies*

$$E(t) \leq C/(1+t)^2$$

for all $t \geq 0$ with some constant $C > 0$ depending on $\|u_0\|_{H^1}, \|u_1\|$ *and* $\|(1+|x|)(u_0+u_1)\|$.

In Theorem 1.6, in the framework of L^2-space only, one can obtain the energy decay rate faster than $(1+t)^{-1}$ without the so called $L^p - L^q$ estimate as in Kawashima-Nakao-Ono [6]. Note that the assumption $\|(1+|x|)v\| < +\infty$ does not necessarily imply $v \in L^1(\mathbf{R}^N)$.

Finally, we shall present a mathematical example for which $E(t) = O((1+t)^{-3})$ as $t \to +\infty$.

Let us consider the following mixed problem in an exterior domain $\Omega \subset \mathbf{R}^N (N \geq 2)$ with smooth compact boundary $\partial\Omega$. For simplicity we may assume $0 \notin \bar{\Omega}$.

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{1.8}$$

$$u(0,x) = \phi(x), \quad u_t(0,x) = -\phi(x), \quad x \in \Omega, \tag{1.9}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.10}$$

Our result reads as follows.

Theorem 1.7. *Let $N \geq 2$. Assume that $\phi \in H_0^1(\Omega)$ further satisfies* $\|d(\cdot)\phi\| < +\infty$. *Then the weak solution* $u(t,x) \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ *to the problem* (1.8)-(1.10) *satisfies*

$$E(t) \leq C/(1+t)^3.$$

Remark 1.8. Although the problem treated in Theorem 1.7 may be very rare, at least we have an example of the initial data for which the total energy decays with a rate like $(1+t)^{-3}$.

§2. Proof of Theorems 1.3 and 1.7.

The following fact concerning the well-posedness of the problem (1.1)-(1.3) is well-known, and we shall prove Theorems 1.3 and 1.7 based on this Proposition 2.1 below.

Proposition 2.1. *For each* $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ *there exists a unique solution* $u \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ *to the problem* (1.1)-(1.3) *such that*

$$E(t) + \int_0^t \|u_t(s,\cdot)\|^2\,ds = E(0),$$

$$\frac{d}{dt}(u_t(t,\cdot), u(t,\cdot)) + \|\nabla u(t,\cdot)\|^2 + (u_t(t,\cdot), u(t,\cdot)) = \|u_t(t,\cdot)\|^2.$$

Before going to the proof of our main Theorem 1.3, for the reader's convenience, we shall review the proof of Theorem 1.2 (see [5]).

First we set

$$w(t,x) = \int_0^t u(s,x)\,ds.$$

Then $w \in C^1([0,+\infty); H_0^1(\Omega)) \cap C^2([0,+\infty); L^2(\Omega))$ satisfies

$$w_{tt}(t,x) - \Delta w(t,x) + w_t(t,x) = u_1 + u_0, \quad (t,x) \in (0,\infty) \times \Omega,$$
$$w(0,x) = 0, \quad w_t(0,x) = u_0(x), \quad x \in \Omega,$$
$$w|_{\partial\Omega} = 0, \quad t \in (0,\infty)$$

and

$$\begin{aligned} &\frac{1}{2}\|w_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla w(t,\cdot)\|^2 + \int_0^t \|w_t(s,\cdot)\|^2\,ds \\ = &\ \frac{1}{2}\|u_0\|^2 + \int_0^t (u_1 + u_0, w_t(s,\cdot))\,ds. \end{aligned} \tag{2.1}$$

Since

$$\int_0^t (u_1 + u_0, w_t(s,\cdot))\,ds = \int_0^t \frac{d}{ds}(u_1 + u_0, w(s,\cdot))\,ds = (w(t,\cdot), u_1 + u_0),$$

and

$$(w(t,\cdot), u_1 + u_0) \le \|d(\cdot)(u_1 + u_0)\|\|\frac{w(t,\cdot)}{d(\cdot)}\|,$$

we see from Lemma 1.1 and (2.1) that

$$\begin{aligned} &\frac{1}{2}\|w_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla w(t,\cdot)\|^2 + \int_0^t \|w_t(s,\cdot)\|^2\,ds \\ \le &\ \frac{1}{2}\|u_0\|^2 + \frac{C_1}{\varepsilon}\|d(\cdot)(u_0 + u_1)\|^2 + \frac{C_2\varepsilon}{2}\|\nabla w(t,\cdot)\|^2 \end{aligned}$$

with some constants $C_i > 0 (i = 1, 2)$. Because of $w_t = u$, this inequality implies

$$\frac{1}{2}\|u(t,\cdot)\|^2 + \frac{1}{2}(1 - C_2\varepsilon)\|\nabla w(t,\cdot)\|^2 + \int_0^t \|u(s,\cdot)\|^2\,ds \le \frac{1}{2}\|u_0\|^2 + C_1\|d(\cdot)(u_0 + u_1)\|^2$$

for $\varepsilon > 0$ with some constants $C_i > 0$ $(i = 1, 2)$. Taking $\varepsilon > 0$ so small, we have arrived at the following estimate.

Lemma 2.2. *Under the same assumptions as in Theorem* 1.2, *it holds that*

$$\frac{1}{2}\|u(t,\cdot)\|^2 + \int_0^t \|u(s,\cdot)\|^2\,ds \le \frac{1}{2}\|u_0\|^2 + C_1\|d(\cdot)(u_0+u_1)\|^2 \quad \forall t > 0.$$

On the other hand, one has the following estimate which has been known more or less. For convenience of the readers, we shall sketch its proof briefly again.

Lemma 2.3. *Under the assumptions as in Theorem* 1.2, *one has*

$$(1+t)\|u(t,\cdot)\|^2 \le C_3 + C_4\int_0^t \|u(s,\cdot)\|^2\,ds \quad \forall t > 0,$$

where $C_i > 0$ $(i = 3,4)$ *are some constants.*

Proof of Lemma 2.3. Multiplying the both sides of (1.1) by $tu_t(t,x)$ and integrating it over Ω and $[0,t]$, one has

$$\int_0^t s\frac{d}{ds}(u_s(s,\cdot),u(s,\cdot))ds - \int_0^t s\|u_s(s,\cdot)\|^2 ds$$

$$+\int_0^t s\|\nabla u(s,\cdot)\|^2 ds + \frac{1}{2}\int_0^t s\frac{d}{ds}\|u(s,\cdot)\|^2 ds = 0.$$

Integrating by parts, we see that

$$\int_0^t s\frac{d}{ds}(u_s(s,\cdot),u(s,\cdot))ds = t(u_t(t,\cdot),u(t,\cdot)) - \frac{1}{2}\|u(t,\cdot)\|^2 + \frac{1}{2}\|u_0\|^2$$

and

$$\int_0^t s\frac{d}{ds}\|u(s,\cdot)\|^2 ds = t\|u(t,\cdot)\|^2 - \int_0^t \|u(s,\cdot)\|^2 ds.$$

Thus, we get

$$t(u_t(t,\cdot),u(t,\cdot)) + \frac{1}{2}\|u_0\|^2 + \int_0^t s\|\nabla u(s,\cdot)\|^2 ds + \frac{t}{2}\|u(t,\cdot)\|^2 \tag{2.2}$$

$$= \frac{1}{2}\|u(t,\cdot)\|^2 + \int_0^t s\|u_s(s,\cdot)\|^2 ds + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds.$$

On the other hand, it is easy to prove (cf. [5] or [6]) that

$$\|u(t,\cdot)\|^2 \le I_0, \quad (1+t)E(t) \le E(0) + \beta, \quad \int_0^t E(s)ds \le \beta, \tag{2.3}$$

where we set $\beta = \frac{1}{2}E(0) + \frac{1}{8}I_0$ with $I_0 = 2\|u_0\|^2 + (u_0, u_1) + 8E(0)$. Therefore, integrating by parts with respect to t, one gets

$$\beta \geq \int_0^t E(s)ds = tE(t) - \int_0^t sE'(s)ds = tE(t) + \int_0^t s\|u_s(s,\cdot)\|^2 ds$$

from which it follows that

$$\int_0^t s\|u_s(s,\cdot)\|^2 ds \leq \beta, \tag{2.4}$$

where we have just used the relation $E'(t) = -\|u_t(t,\cdot)\|^2$. Note that

$$-(u_t(t,\cdot), u(t,\cdot)) \leq \|u_t(t,\cdot)\|\|u(t,\cdot)\| \leq \frac{1}{4}\|u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2. \tag{2.5}$$

From (2.2) and (2.5) it follows that

$$\begin{aligned}\frac{1}{2}\|u(t,\cdot)\|^2 &+ \int_0^t s\|u_s(s,\cdot)\|^2 ds + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds \\ &\geq t(u_t(t,\cdot), u(t,\cdot)) + \frac{t}{2}\|u(t,\cdot)\|^2 \\ &\geq \frac{t}{4}\|u(t,\cdot)\|^2 - t\|u_t(t,\cdot)\|^2.\end{aligned}$$

Therefore, we see that

$$t\|u_t(t,\cdot)\|^2 + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds + \int_0^t s\|u_s(s,\cdot)\|^2 ds + \frac{1}{2}\|u(t,\cdot)\|^2 \geq \frac{t}{4}\|u(t,\cdot)\|^2.$$

Since $(1+t)E(t) \leq \beta + E(0)$ implies $t\|u_t(t,\cdot)\|^2 \leq 2(\beta + E(0))$, from (2.4) we have the desired inequality. ∎

Therefore, Theorem 1.2 is an immediate consequence of Lemmas 2.2 and 2.3.

Now, let us prove our main Theorem 1.3. Once we have obtained the L^2-decay property of a solution to the problem (1.1)-(1.3), the result is a direct consequence of the following Nakao inequality.

Lemma 2.4. ([6], Lemma 2.2.) *Let $\phi(t)$ be a nonnegative function on $[0, +\infty)$, satisfying*

$$\sup_{t \leq s \leq t+1} \phi(s)^{1+\alpha} \leq k_0(1+t)^\beta \{\phi(t) - \phi(1+t)\}$$

for some $k_0 > 0$, $\alpha > 0, \beta < 1$. Then $\phi(t)$ has a decay property

$$\phi(t) \leq C_0(1+t)^{-\frac{1-\beta}{\alpha}},$$

where $C_0 > 0$ denotes a positive constant depending on $\phi(0)$ and other known constants.

As in [6], set

$$D(t)^2 = E(t) - E(1+t) = \int_t^{1+t} \|u_s(s,\cdot)\|^2 ds.$$

Then one has

Lemma 2.5. *Let $u(t,x)$ be a solution to the problem* (1.1)-(1.3) *as in Proposition* 2.1. *Under the hypothesis as in Theorem* 1.3, *one has*

$$\sup_{t\le s\le 1+t} E(s) \le C\{D(t)^2 \ + D(t) \sup_{t\le s\le 1+t} \|u(s,\cdot)\|\},$$

with a generous constant $C > 0$.

Proof. Multiplying the equation (1.1) by u_t and integrating it over Ω and $[t, 1+t]$, we get

$$\int_t^{1+t} \|u_s(s,\cdot)\|^2 ds = E(t) - E(t+1) = D(t)^2. \tag{2.6}$$

Applying the mean value theorem to the left-hand side of (2.6), there exist numbers $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i,\cdot)\| \le 2D(t) \quad (i = 1, 2).$$

Next, multiplying the equation (1.1) by u and integrating it over Ω and $[t_1, t_2]$, one has

$$\begin{aligned}
\int_{t_1}^{t_2} \|\nabla u(s,\cdot)\|^2 ds &= \int_{t_1}^{t_2} \|u_s(s,\cdot)\|^2 ds + (u_t(t_1,\cdot), u(t_1,\cdot)) - (u_t(t_2,\cdot), u(t_2,\cdot)) \\
&\quad - \int_{t_1}^{t_2} (u_t(s,\cdot), u(s,\cdot)) ds \\
&\le D(t)^2 + \|u_t(t_2,\cdot)\|\|u(t_2,\cdot)\| + \|u_t(t_1,\cdot)\|\|u(t_1,\cdot)\| \\
&\quad + (\int_{t_1}^{t_2} \|u_s(s,\cdot)\|^2 ds)^{1/2} (\int_{t_1}^{t_2} \|u(s,\cdot)\|^2 ds)^{1/2} \\
&\le D(t)^2 + 4D(t) \sup_{t\le s\le 1+t} \|u(s,\cdot)\| + D(t)(t_2 - t_1)^{1/2} \sup_{t\le s\le 1+t} \|u(s,\cdot)\| \\
&\le D(t)^2 + 5D(t) \sup_{t\le s\le 1+t} \|u(s,\cdot)\| = A(t). \qquad (2.7)
\end{aligned}$$

On the other hand, we also have (see Proposition 2.1)

$$E(t) = E(t_2) + \int_t^{t_2} \|u_s(s,\cdot)\|^2 ds. \tag{2.8}$$

Since $t_2 - t_1 \geq \frac{1}{2}$, we see that

$$\int_{t_1}^{t_2} E(s)ds \geq (t_2 - t_1)E(t_2) \geq \frac{1}{2}E(t_2).$$

That is,

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(s)ds. \tag{2.9}$$

It follows from (2.7)-(2.9) that

$$E(t) \leq 2\int_{t_1}^{t_2} E(s)ds + \int_t^{t+1} \|u_s(s,\cdot)\|^2 ds \leq A(t) + 2D(t)^2.$$

Together with (2.7), we get the desired inequality. ∎

Proof of Theorem 1.3. First we see from Lemma 2.5 that

$$\sup_{t\leq s\leq 1+t} E(s)^2 \leq 2C^2\{D(t)^2 + (\sup_{t\leq s\leq 1+t} \|u(s,\cdot)\|)^2\}D(t)^2. \tag{2.10}$$

On the other hand, it follows from Theorem 1.2 that

$$E(t) \leq C_1(1+t)^{-1}, \quad \|u(t,\cdot)\|^2 \leq C_2(1+t)^{-1}.$$

Therefore (2.10) together with (2.6) implies that

$$\begin{aligned} \sup_{t\leq s\leq 1+t} E(s)^2 &\leq 2C^2\{E(t) + C_2(1+t)^{-1}\}(E(t) - E(t+1)) \\ &\leq 2C^2\{C_1(1+t)^{-1} + C_2(1+t)^{-1}\}(E(t) - E(t+1)) \end{aligned} \tag{2.11}$$

from which it follows that

$$\sup_{t\leq s\leq 1+t} E(s)^2 \leq K_1(1+t)^{-1}(E(t) - E(1+t))$$

with some constant $K_1 > 0$. By applying Lemma 2.4, one has the desired inequality. ∎

Finally, let us prove Theorem 1.7. By applying Theorem 1.3 to the problem (1.8)-(1.10), first we get

$$E(t) = \frac{1}{2}(\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2) \leq C/(1+t)^2$$

for the weak solution $u \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ to the problem (1.8)-(1.10). Setting

$$w(t,x) = \int_0^t u(s,x)ds,$$

$w(t,x)$ also becomes the solution to the problem:

$$w_{tt}(t,x) - \Delta w(t,x) + w_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{2.12}$$

$$w(0,x) = 0, \quad w_t(0,x) = \phi(x), \quad x \in \Omega, \tag{2.13}$$

$$w|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{2.14}$$

Since $\|d(\cdot)(0+\phi)\| = \|d(\cdot)\phi\| < +\infty$, by applying Theorem 1.3 to this problem (2.12)-(2.14) again, it follows that

$$\frac{1}{2}(\|w_t(t,\cdot)\|^2 + \|\nabla w(t,\cdot)\|^2) \leq C/(1+t)^2.$$

Because of $w_t = u$, one has the crucial L^2-decay rate.

Lemma 2.6. *Let $N \geq 2$ and $\phi \in H_0^1(\Omega)$ further satisfies $\|d(\cdot)\phi\| < +\infty$. Then, the weak solution $u(t,x) \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ to the problem* (1.8)-(1.10) *satisfies*

$$\|u(t,\cdot)\|^2 \leq C/(1+t)^2, \quad E(t) \leq C/(1+t)^2.$$

Proof of Theorem 1.7. Based on this lemma 2.6, by repeating the argument as in the proof of Theorem 1.3, we obtain the desired decay estimate. ∎

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