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p-BASIS OF A REGULAR SEMI-LOCAL RING

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Abstract. For a Frobenius-Sandwich of regular semi-local rings $R \subset R' \subset R^p$, R/R' has a p-basis if and only if R has constant rank as R'-module.

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Let p be always a prime number, S a commutative ring with identity of characteristic p and $S^p = \{x^p \mid x \in S\}$. Let S' be a subring of S. A subset Γ of S is said to be p-independent over S' if, for any subset $b_1, \ldots, b_n$ of Γ, the set of monomials $b_1^{e_1} \cdots b_n^{e_n} (0 \leq e_i < p)$ is linearly independent over $S'[S^p]$. A subset Γ of S is called a p-basis of S over S' (or a p-basis of S/S') if it is p-independent over S' and $S = S^p[S', \Gamma]$. Let L be a finite projective S-module. If $\mathrm{rg}_q(L) = n$ for any prime ideal q of S, we shall say that L has constant rank n (cf. [1, Chap. II, §5, 3, Def. 2], or [3]). For the another terminology used in this paper, we refer to [4].

The main result of this paper, Corollary 1 of Theorem is a generalization of the following;

Let R be a regular local ring of characteristic $p > 0$ and let $R'(\supset R^p)$ be a regular subring of R such that R is a finite R'-module. Then R has a p-basis over R' (Theorem of [2]).

To prove our main result, we have to prove the following theorem, a slight modification of Theorem 3.7 of [3, Chap. 1, §3];

Theorem. *Let R be a semi-local ring of characteristic p and let R' be a subring of R such that R' contains R^p and such that R is a finite R'-module. Then the following conditions are equivalent:*

(1) *R/R' has a p-basis which consists of n elements.*

(2) *For any prime ideal q of R, $R_q/R'_{q'}$ has a p-basis which consists of n elements, where $q' = R' \cap q$.*

(3) *For any maximal ideal M of R, $R_M/R'_{M'}$ has a p-basis which consists of n elements, where $M' = R' \cap M$.*

Proof. (1) $\Longrightarrow$ (2). For any prime ideal q of R, we put $q' = R' \cap q$. If $\{a_1, \ldots, a_n\}$ is a p-basis of R over R', $\{\phi(a_1), \ldots, \phi(a_n)\}$ is a p-basis of R_q over $R'_{q'}$, where ϕ is a canonical map $R \longrightarrow R_q$. Therefore $R_q/R'_{q'}$ has a p-basis which consists of n elements.

(2) $\Longrightarrow$ (3). It is trivial that (2) implies (3).

(3) $\Longrightarrow$ (1). Let $M_1, \ldots, M_r$ be the maximal ideals of the semi-local ring R, and let $M'_i = M_i \cap R'$ $(1 \leq i \leq r)$. Then $M'_1, \ldots, M'_r$ are the maximal ideals of R'. By the assumption, $R_{M_i}/R'_{M'_i}$ has a p-basis $\Gamma_i = \{a_{i_1}, \ldots, a_{i_n}\}$ contained in R, for $i = 1, 2, ..., r$. By the Chinese Remainder theorem, we have the following isomorphism:

$$R/(M'_1 \cap \cdots \cap M'_r)R \simeq R/M'_1R \times \cdots \times R/M'_rR.$$

From this isomorphism, there exist elements $a_1, \ldots, a_n$ of R such that $a_j \equiv a_{i_j}$ (mod M'_iR) for $i = 1, ..., r, j = 1, ..., n$. The set of p^n monomials $\Lambda_i := \{a_{i_1}^{e_{i_1}} \cdots a_{i_n}^{e_{i_n}} : 1 \leq e_{i_j} \leq p-1\}$ is a free basis in the $R'_{M'_i}$-module R_{M_i}. Then $\bar{\Lambda}_i := \{\bar{a}_{i_1}^{e_{i_1}} \cdots \bar{a}_{i_n}^{e_{i_n}} : 1 \leq e_{i_j} \leq p-1\}$ is also a free basis in the $R'_{M'_i}/M'_iR'_{M'_i}$-module $R_{M_i}/M'_iR_{M_i}$, where $\bar{a}_{i_j}$ is the class of a_{i_j} modulo $M'_iR_{M_i}$. So $\bar{\Lambda} = \{\bar{a}_1^{e_1} \cdots \bar{a}_n^{e_n} : 1 \leq e_i \leq p-1\}$ is a free basis in the same $R'_{M'_i}/M'_iR'_{M'_i}$-module $R_{M_i}/M'_iR_{M_i}$, where $\bar{a}_i$ is the class of a_i modulo $M'_iR_{M_i}$. Hence $\Lambda = \{a_1^{e_1} \cdots a_n^{e_n} : 1 \leq e_i \leq p-1\}$ is a free basis in the $R'_{M'_i}$-module R_{M_i} for $i = 1, ..., r$. Therefore Λ is a free basis of R'-module R, by [1, Chap. II, §3, 3, Theorem 1]. It follows that the set $\{a_1, \ldots, a_n\}$ is a p-basis of R/R'. □

Corollary 1. *Let R be a regular semi-local ring of characteristic p and let R' be a regular subring of R such that R' contains R^p and such that R is a finite R'-module. Then the following conditions are equivalent:*

(1) *R has a p-basis over R'.*

(2) *R'-module R has constant rank.*

Proof. For any maximal ideal M of R, we put $M' = R' \cap M$. It is sufficient to see that (2) implies (1). Since R_M and $R'_{M'}$ are regular local rings and R_M is a finite $R'_{M'}$-module, R_M has a p-basis Γ_M over $R'_{M'}$, in virtue of Theorem of [2]. By the assumption, Γ_M consists of n elements free from the choice of M. It follows from the Theorem that R has a p-basis over R'. □

Corollary 2. *Let R be a regular semi-local ring of characteristic p such that R is a finite R^p-module. If R has constant rank as R^p-module, R has a p-basis over R^p.*

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