SUPPLEMENT TO: "RELAXING THE I.I.D. ASSUMPTION: ADAPTIVELY MINIMAX OPTIMAL REGRET VIA ROOT-ENTROPIC REGULARIZATION"

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A: PROOFS OF UPPER BOUNDS

The proofs of Theorems 5 and 6 rely on several technical results regarding *online linear optimization* developed in Appendices C and D. In order to simplify notation for FTRL with regularizers that are transformations of the entropy function, we let $\mathrm{FTRL}_H(\psi,\ \beta)$ denote $\mathrm{FTRL}(\{r_t\}_{t\in\mathbb{Z}_+})$ with $r_{0:t} = -\beta(t)[\psi \circ H]$ for any strictly increasing, concave, and twice continuously differentiable function $\psi : [0, \log N] \to \mathbb{R}$ and strictly increasing $\beta : \mathbb{Z}_+ \to \mathbb{R}_+$. The important conclusions from Appendices C and D are summarized in the following result, the proof of which appears in Appendix D.1. The result tells us that the weights played by a player employing the $\mathrm{FTRL}_H(\psi,\ \beta)$ strategy are equivalent to the weights played by Hedge with an implicitly defined, non-deterministic learning rate, and also provides a second-order bound on the quasi-regret incurred.

Theorem A.1. *For every strictly increasing $\beta : \mathbb{Z}_+ \to \mathbb{R}$, and every strictly increasing, concave, and twice continuously differentiable function $\psi : [0, \log N] \to \mathbb{R}$, the solutions to Eq.* (3) *at time t for $\mathrm{FTRL}_H(\psi, \beta)$ given any history and expert predictions satisfy the system of equations*

$$\eta(t+1) = \frac{1}{\beta(t)\cdot[\psi' \circ H](u(t+1))}, \quad u(t+1) = \left(\frac{\exp\{-\eta(t+1)L_i(t)\}}{\sum_{i'\in[N]}\exp\{-\eta(t+1)L_{i'}(t)\}}\right)_{i\in[N]}.$$

Moreover, for any sequence of losses $(\ell(t))_{t\in\mathbb{N}} \subseteq [0,1]^N$, this system has a unique solution satisfying

$$\eta(t+1) \in \left[\frac{1}{\beta(t)\cdot\psi'(0)},\ \frac{1}{\beta(t)\cdot\psi'(\log N)}\right],$$

and there exists a sequence $\{\alpha_t\}_{t\in\mathbb{Z}_+} \subseteq [0,1]$ such that the quasi-regret satisfies

(A.1)

$$\hat{R}_{\mathrm{FTRL}_H}(T) \leq -\beta(T)\psi(0) + \beta(0)[\psi\circ H](u(1)) + \sum_{t=1}^{T}[\beta(t)-\beta(t-1)]\cdot[\psi\circ H](u(t+1))$$
$$+ \sum_{t=1}^{T}\frac{\sqrt{\underset{I\sim v(t+1)}{\mathrm{Var}}\left[\left(\frac{\beta(t)}{\beta(t-1)}-1\right)L_I(t-1)-\ell_I(t)\right]\times\underset{I\sim v(t+1)}{\mathrm{Var}}[\ell_I(t)]}}{\beta(t)\cdot[\psi'\circ H](v(t+1))},$$

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MSC2020 subject classifications: Primary 62L10, 62C20; Secondary 60G25, 62M20, 68Q32, 68T05.

Keywords and phrases: sequential decision theory, prediction with expert advice, adaptive minimax regret, aggregation, robust prediction.

where for each $t \in \mathbb{Z}_+$,

$$v(t+1) = v^{(\alpha_t)}(t+1), \tag{A.2}$$

and for every $t \in \mathbb{N}$ *and* $\alpha \in [0,1]$, *we define*

$$v^{(\alpha)}(t+1) = \operatorname*{arg\,min}_{v \in \mathtt{simp}([N])} \left(\left\langle \alpha L(t) + (1-\alpha)\sqrt{\frac{t+1}{t}} L(t-1),\ v \right\rangle - \sqrt{t+1}\,[\psi \circ H](v) \right).$$

Ultimately, we wish to apply Theorem A.1 to both D.Hedge and FTRL-CARE. Recall that D.Hedge corresponds to

$$\psi(s) = \frac{s}{g(N)},\ \psi'(s) = \frac{1}{g(N)},\ \text{and } \beta(t) = \sqrt{t+1},$$

and therefore, in Eq. (A.1),

$$\frac{1}{\beta(t) \cdot [\psi' \circ H](v(t+1))} = \frac{g(N)}{\sqrt{t+1}}.$$

FTRL-CARE with parameters $c_1, c_2 > 0$ corresponds to

$$\psi(s) = \frac{\sqrt{s+c_2}}{c_1},\ \psi'(s) = \frac{1}{2c_1\sqrt{s+c_2}},\ \text{and } \beta(t) = \sqrt{t+1},$$

and therefore, in Eq. (A.1),

$$\frac{1}{\beta(t) \cdot [\psi' \circ H](v(t+1))} = 2c_1\sqrt{\frac{H(v(t+1)) + c_2}{t+1}}.$$

Both correspond to the choice $\beta(t) = \sqrt{t+1}$, so we focus on this rather than continuing to use a generic $\beta(t)$. We leave ψ as generic for the moment, since the following result equally applies to the algorithms' respective ψ functions. Finally, we wish to move towards proving bounds on the expected regret, which will require taking expectation with respect to a data-generating mechanism π, so we fix a convex $\mathcal{D} \subseteq \mathcal{M}(\hat{\mathcal{Y}}^N \times \mathcal{Y})$ that characterizes the allowable data-generating mechanisms.

In order to control the quasi-regret using Theorem A.1, we need to control the entropy of the FTRL_H weights u as well as the *intermediate weights* v (defined in Eq. (A.2)). The following lemma provides the necessary control, which we prove in Appendix E.1.

Lemma A.1. *For every* $u \in \mathtt{simp}([N])$ *and* $p \in (0,1)$,

$$H(u) \le \frac{2}{e\log 2}\log N_0 + \left(1 + \frac{1}{(1-p)e}\right) \sum_{i \in [N]\setminus \mathcal{I}_0} [u_i]^p. \tag{A.3}$$

Our next lemma bounds the expectation of the second term on the RHS of Eq. (A.3) for the FTRL_H weights. Combined with the previous result, this allows us to bound the expected entropy of the weights. Crucially, the bound on the expected weights that FTRL_H would produce holds regardless of whether the actual prediction policy used is FTRL_H or some other policy, allowing us to control the expected quasi-regret of FTRL_H when a different policy is used to interact with the environment, as in the statements of Theorems 5 and 6.

Lemma A.2. *Letting u denote the weights output by the* $\text{FTRL}_H(\psi,\ t \mapsto \sqrt{t+1})$ *algorithm, for every prediction policy $\hat{\pi}$, $t \in \mathbb{N}$, $p > 0$, and $i \in [N] \setminus \mathcal{I}_0$,*

$$\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\Big[[u_i(t+1)]^p\Big] \le \exp\left\{\frac{p^2}{2(\psi'(0))^2} - \frac{\Delta_0 p\sqrt{t}}{\sqrt{2}(\psi'(0))}\right\},$$

and

$$\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sup_{\alpha\in[0,1]} \Big[[v_i^{(\alpha)}(t+1)]^p\Big] \le \exp\left\{\frac{2p}{\psi'(0)} + \frac{p^2}{2(\psi'(0))^2} - \frac{\Delta_0 p\sqrt{t}}{\sqrt{2}(\psi'(0))}\right\}.$$

The intuition underlying the proof of this result is as follows. First, let $\overline{\eta}(t+1) = \frac{1}{\sqrt{t+1}\cdot\psi'(0)}$. Note that for $(u(t))_{t\in\mathbb{N}}$ and $(\eta(t))_{t\in\mathbb{N}}$ given in Theorem A.1, $\overline{\eta}(t+1) \le \eta(t+1)$ for all $t \in \mathbb{N}\cup\{0\}$. Let $I^*(t) = \arg\min_{i\in[N]} L_i(t)$, so that for any $i \in [N]$, $L_{I^*(t)}(t) \le L_i(t)$. Thus,

$$\Big[u_i(t+1)\Big]^p \le \left(\frac{u_i(t+1)}{u_{I^*(t)}(t+1)}\right)^p \le \min_{i_0\in\mathcal{I}_0} \exp\Big\{-p\ \overline{\eta}(t+1)\,[L_i(t) - L_{i_0}(t)]\Big\}.$$

Applying Theorem 2,

$$\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\Big[u_i(t+1)^p\Big] \le \exp\left\{-t\overline{\eta}(t+1)\Delta_0 p + t\overline{\eta}(t+1)^2\frac{p^2}{2}\right\}.$$

The argument for the intermediate weights is similar. For the complete proof, see Appendix E.2.

By combining Lemma A.2 with Lemma A.1 for $p = 1/2$, and noting that, for all $t \in \mathbb{N}$, $2/(e\log 2) < 17/16$ and $1 + 2/e < 7/4$, it holds that

$$\begin{aligned}&\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} H(u(t+1)) \\ &\qquad \le \frac{17}{16}\log N_0 + \frac{7}{4}(N-N_0)\exp\left\{\frac{1}{8(\psi'(0))^2} - \frac{\Delta_0\sqrt{t}}{2\sqrt{2}(\psi'(0))}\right\},\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}&\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sup_{\alpha\in[0,1]} H(v^{(\alpha)}(t+1)) \\ &\qquad \le \frac{17}{16}\log N_0 + \frac{7}{4}(N-N_0)\exp\left\{\frac{1}{\psi'(0)} + \frac{1}{8(\psi'(0))^2} - \frac{\Delta_0\sqrt{t}}{2\sqrt{2}(\psi'(0))}\right\}.\end{aligned} \tag{A.5}$$

These bounds can now be used for the regularizers specific to D.Hedge and FTRL-CARE. Our approach will be to break up the sums of Eq. (A.1) into the first t_0 rounds and then the remaining rounds for some carefully chosen t_0. Note that t_0 is not a parameter of the algorithm, but rather an artifact of our proof. The rounds after t_0 will be handled using our entropy bounds above, but the early rounds we control with the following worst-case bound. The proof of the following result appears in Appendix E.3. Note that it recovers the correct order of standard adversarial bounds for D.Hedge.

Lemma A.3. *For every $t_0 \in \mathbb{N}$ and sequence of losses $\{\ell(t)\}_{t\in\mathbb{N}} \subseteq [0,1]^N$, the weights played by* $\text{FTRL}_H(\psi,\ t \mapsto \sqrt{t+1})$ *satisfy*

$$\hat{R}_{\text{FTRL}_H}(t_0) \le \left(\psi(\log N) - \psi(0) + \frac{3}{4\psi'(\log N)}\right)\sqrt{t_0+1}\,.$$

The remainder of the proofs of Theorems 5 and 6 can be found in Appendix B, which consists of substituting in the specific expression for ψ to Theorem A.1, Eqs. (A.4) and (A.5), and Lemma A.3. Then, the variance terms are controlled by a worst case bound for $N_0 > 1$, and by Lemma E.2 for $N_0 = 1$, and the summation terms are controlled by an integral comparison (see Lemma E.3). Finally, t_0 is chosen as specified by the statements of Theorems 5 and 6 respectively.

B: ADDITIONAL DETAILS FOR PROOFS OF UPPER BOUNDS

In this section, we complete the argument sketched in Appendix A.

B.1. Details for Theorem 5. Substituting in that D.HEDGE with parameter g corresponds to, for a given $N \in \mathbb{N}$, $\psi(s) = s/g(N)$, Theorem A.1 says that the weights w^{H} lead to quasi-regret bounded by

$$\begin{aligned} \hat{R}_{\mathrm{H}}(T) \le \frac{\log N}{g(N)} + \sum_{t=1}^{T} \frac{\sqrt{t+1}-\sqrt{t}}{g(N)} H(w^{\mathrm{H}}(t+1)) \\ + \sum_{t=1}^{T} \frac{g(N)\sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]}}{\sqrt{t+1}}, \end{aligned} \tag{B.1}$$

where

$$v(t+1) = \operatorname*{arg\,min}_{v\in\mathtt{simp}([N])}\left(\left\langle \alpha_t L(t) + (1-\alpha_t)\frac{\sqrt{t+1}}{\sqrt{t}}L(t-1),\ v\right\rangle - \frac{\sqrt{t+1}}{g(N)}H(v)\right)$$

for some $\alpha_t \in [0,1]$. Then, recalling that $\psi'(s) = 1/g(N)$, we can split up Eq. (B.1) into the rounds before some $t_0 \in \mathbb{N}$ and the rounds after by applying Lemma A.3. That is, when $T \le t_0$, we use the bound of Lemma A.3, and if $T > t_0$ we have

$$\begin{aligned} \hat{R}_{\mathrm{H}}(T) \le \sqrt{t_0+1}\left(\frac{\log N}{g(N)} + \frac{3g(N)}{4}\right) + \sum_{t=t_0+1}^{T} \frac{\sqrt{t+1}-\sqrt{t}}{g(N)} H(w^{\mathrm{H}}(t+1)) \\ + \sum_{t=t_0+1}^{T} \frac{g(N)\sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]}}{\sqrt{t+1}}. \end{aligned} \tag{B.2}$$

Next, substituting ψ and ψ' for D.HEDGE into Eq. (A.4), we get

$$\begin{aligned} &\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} H(w^{\mathrm{H}}(t+1)) \\ &\le \frac{17}{16}\log N_0 + \frac{7}{4}(N-N_0)\exp\left\{\frac{[g(N)]^2}{8}\right\}\exp\left\{-\frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t}\right\}. \end{aligned} \tag{B.3}$$

Thus,

$$
\begin{aligned}
\sum_{t=t_0+1}^{T} \frac{\sqrt{t+1}-\sqrt{t}}{c} & \sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} H(w^{\mathrm{H}}(t+1)) \\
&\leq \frac{17 \log N_0 \left[\sqrt{T+1}-\sqrt{t_0+1}\right]}{16 g(N)} \\
&\qquad + \frac{7(N-N_0)\exp\left\{\frac{[g(N)]^2}{8}\right\}}{8 g(N)} \sum_{t=t_0+1}^{T} \frac{\exp\left\{-\frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t}\right\}}{\sqrt{t}} \\
&\leq \frac{17 \log N_0 \left[\sqrt{T+1}-\sqrt{t_0+1}\right]}{16 g(N)} \\
&\qquad + \frac{7(N-N_0)\exp\left\{\frac{[g(N)]^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t_0}\right\}}{\sqrt{2}\,[g(N)]^2 \Delta_0},
\end{aligned} \tag{B.4}
$$

where the last step comes from applying Lemma E.3 to bound the summation. For the last term of Eq. (B.2), we consider the cases of $N_0 > 1$ and $N_0 = 1$ separately. For both, however, we will use $t_0 = \left\lceil \frac{8(\log(N)+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2} \right\rceil$.

Hedge upper bound: $N_0 > 1$.

If $N_0 > 1$, using Lemma E.1 to bound the variances gives

$$
\begin{aligned}
\sum_{t=t_0+1}^{T} \frac{g(N)}{\sqrt{t+1}} & \sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1) - \ell_I(t)\right] \operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]} \\
&\leq \frac{3g(N)}{8} \sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t+1}} \\
&\leq \frac{3g(N)}{4}\left[\sqrt{T+1}-\sqrt{t_0+1}\right].
\end{aligned} \tag{B.5}
$$

Combining Eqs. (B.2), (B.4) and (B.5) gives that

$$
\begin{aligned}
\sup_{\pi \in \mathscr{P}(\mathcal{D})} & \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{H}}(T) \\
&\leq \sqrt{t_0+1}\left(\frac{\log N}{g(N)} + \frac{3g(N)}{4}\right) + \frac{17 \log N_0 \left[\sqrt{T+1}-\sqrt{t_0+1}\right]}{16 g(N)} \\
&\qquad + \frac{7(N-N_0)\exp\left\{\frac{[g(N)]^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t_0}\right\}}{\sqrt{2}\,[g(N)]^2 \Delta_0} + \frac{3g(N)\left[\sqrt{T+1}-\sqrt{t_0+1}\right]}{4} \\
&= \sqrt{T+1}\left(\frac{3g(N)}{4} + \frac{17\log N_0}{16 g(N)}\right) + \sqrt{t_0+1}\left(\frac{\log N}{g(N)} - \frac{17 \log N_0}{16 g(N)}\right) \\
&\qquad + \frac{7(N-N_0)\exp\left\{\frac{[g(N)]^2}{8} - \frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t_0}\right\}}{\sqrt{2}\,[g(N)]^2 \Delta_0}.
\end{aligned} \tag{B.6}
$$

Substituting t_0 into Eq. (B.6) gives

$$
\begin{aligned}
\sup_{\pi\in\mathscr{P}(\mathcal{D})} & \mathbb{E}_{\pi,\hat\pi}\hat R_{\text{H}}(T) \\
&\le \sqrt{T+1}\left(\frac{3g(N)}{4}+\frac{17}{16g(N)}\log N_0\right) \\
&\quad + \sqrt{\frac{8(\log(N)+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2}+2}\,\left(\frac{\log(N)-\log(N_0)}{g(N)}\right) \\
&\quad + \frac{7(N-N_0)\exp\left\{\frac{[g(N)]^2}{8}-\frac{g(N)\Delta_0}{2\sqrt2}\sqrt{\frac{8(\log(N)+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2}}\right\}}{\sqrt2\,[g(N)]^2\Delta_0} \qquad \text{(B.7)} \\
&\le \sqrt{T}\left(\frac{3g(N)}{4}+\frac{17}{16g(N)}\log(N_0)\right)+\frac{\sqrt2\log(N)}{g(N)}+\frac{3g(N)}{4} \\
&\quad + \frac{2\sqrt2\,[\log(N)]^2}{[g(N)]^2\Delta_0}+\frac{\log(N)}{\sqrt2\,\Delta_0}+\frac{2\sqrt2\log(N)}{g(N)\Delta_0}+\frac{7}{\sqrt2\,[g(N)]^2\Delta_0}.
\end{aligned}
$$

Hedge upper bound: $N_0=1$

If $\mathcal{I}_0=\{i_0\}$, we control the variance terms using Lemma E.2

$$
\begin{aligned}
\mathbb{E}_{\pi,\hat\pi}&\sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt t}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]} \\
&\le \frac94\mathbb{E}_{\pi,\hat\pi}\left[\sum_{i\ne i_0}v_i(t+1)\right].
\end{aligned}
$$

We control this using Lemma A.2 with $p=1$, which gives

$$
\mathbb{E}_{\pi,\hat\pi}\left[\sum_{i\ne i_0}v_i(t+1)\right]\le (N-1)\exp\left\{2g(N)+\frac{[g(N)]^2}{2}\right\}\exp\left\{-\frac{g(N)\Delta_0}{\sqrt2}\sqrt t\right\}.
$$

Thus,

$$
\begin{aligned}
\sup_{\pi\in\mathscr{P}(\mathcal{D})}&\mathbb{E}_{\pi,\hat\pi}\sum_{t=t_0+1}^{T}\frac{g(N)\sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt t}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]}}{\sqrt{t+1}} \\
&\le \frac{9g(N)}{4}(N-1)\exp\left\{2g(N)+\frac{[g(N)]^2}{2}\right\}\sum_{t=t_0+1}^{T}\frac{1}{\sqrt t}\exp\left\{-\frac{g(N)\Delta_0}{\sqrt2}\sqrt t\right\} \qquad \text{(B.8)} \\
&\le \frac{9}{\sqrt2\,\Delta_0}(N-1)\exp\left\{2g(N)+\frac{[g(N)]^2}{2}\right\}\exp\left\{-\frac{g(N)\Delta_0}{\sqrt2}\sqrt{t_0}\right\},
\end{aligned}
$$

where the last step follows from again applying Lemma E.3. Combing Eqs. (B.2), (B.4) and (B.8) gives that when $N_0=1$,

$$\begin{aligned}\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{H}}(T) \leq \sqrt{t_0+1}\Big(\frac{\log N}{g(N)}+\frac{3g(N)}{4}\Big)+\frac{17(\log 1)\Big[\sqrt{T+1}-\sqrt{t_0+1}\Big]}{16g(N)}\\ +\frac{7(N-1)\exp\Big\{\frac{[g(N)]^2}{8}-\frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{t_0}\Big\}}{\sqrt{2}\,[g(N)]^2\Delta_0}\\ +\frac{9(N-1)\exp\Big\{2g(N)+\frac{[g(N)]^2}{2}-\frac{g(N)\Delta_0}{\sqrt{2}}\sqrt{t_0}\Big\}}{\sqrt{2}\,\Delta_0}.\end{aligned} \tag{B.9}$$

Substituting t_0 into Eq. (B.9) gives

$$\begin{aligned}&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{H}}(T)\\ &\leq \sqrt{\frac{8(\log N+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2}+2}\,\Big(\frac{\log N}{g(N)}+\frac{3g(N)}{4}\Big)\\ &\quad+\frac{7\exp\Big\{\frac{[g(N)]^2}{8}-\frac{g(N)\Delta_0}{2\sqrt{2}}\sqrt{\frac{8(\log N+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2}}\Big\}}{\sqrt{2}\,[g(N)]^2\Delta_0}(N-1)\\ &\quad+\frac{9(N-1)\exp\Big\{2g(N)+\frac{[g(N)]^2}{2}-\frac{g(N)\Delta_0}{\sqrt{2}}\sqrt{\frac{8(\log N+[g(N)]^2/4+g(N))^2}{[g(N)]^2\Delta_0^2}}\Big\}}{\sqrt{2}\,\Delta_0}\\ &\leq \frac{2\sqrt{2}(\log N)^2}{[g(N)]^2\Delta_0}+\frac{2\sqrt{2}\log N}{g(N)\Delta_0}+\frac{4\log N}{\sqrt{2}\,\Delta_0}\\ &\quad+\frac{7/[g(N)]^2+9+3[g(N)]^2/4+3g(N)}{\sqrt{2}\,\Delta_0}+\sqrt{2}\Big(\frac{\log N}{g(N)}+\frac{3g(N)}{4}\Big).\end{aligned} \tag{B.10}$$

□

B.2. Details for Theorem 6. This argument follows the same logical structure as the one for Theorem 5. Using that FTRL-CARE with parameters $c_1, c_2 > 0$ corresponds to $\psi(s) = \frac{\sqrt{s+c_2}}{c_1}$, Theorem A.1 says that the weights w^{C} lead to quasi-regret bounded by

$$\begin{aligned}\hat{R}_{\mathrm{C}}(T) \leq -\frac{\sqrt{(T+1)c_2}}{c_1}+\sum_{t=0}^{T}\frac{\sqrt{t+1}-\sqrt{t}}{c_1}\cdot\sqrt{H(w^{\mathrm{C}}(t+1))+c_2}\\ +\sum_{t=1}^{T}\frac{2c_1\sqrt{H(v(t+1))+c_2}}{\sqrt{t+1}}\\ \times\sqrt{\operatorname*{Var}_{I\sim v(t+1)}\Big[\Big(\frac{\sqrt{t+1}}{\sqrt{t}}-1\Big)L_I(t-1)-\ell_I(t)\Big]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]},\end{aligned} \tag{B.11}$$

where

$$v(t+1) = \operatorname*{arg\,min}_{v\in\mathtt{simp}([N])}\left(\left\langle \alpha_t L(t)+(1-\alpha_t)\frac{\sqrt{t+1}}{\sqrt{t}}L(t-1),\ v\right\rangle-\frac{\sqrt{t+1}}{c_1}\sqrt{H(v)+c_2}\right)$$

for some $\alpha_t \in [0,1]$. Then, recalling that $\psi'(s) = \frac{1}{2c_1\sqrt{s+c_2}}$, we can split up Eq. (B.11) into the rounds before some $t_0 \in \mathbb{N}$ and the rounds after by applying Lemma A.3. That is, when $T \le t_0$, we use the bound of Lemma A.3, and if $T > t_0$ we have

$$\begin{aligned}
\hat{R}_{\mathrm{C}}(T) \le{}& \sqrt{(t_0+1)[\log N + c_2]}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big) - \frac{\sqrt{(T+1)c_2}}{c_1} \\
&+ \sum_{t=t_0}^{T} \frac{\sqrt{t+1}-\sqrt{t}}{c_1} \cdot \sqrt{H(w^{\mathrm{C}}(t+1)) + c_2} \\
&+ \sum_{t=t_0+1}^{T} \frac{2c_1\sqrt{H(v(t+1)) + c_2}}{\sqrt{t+1}} \\
&\quad \times \sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]}.
\end{aligned} \tag{B.12}$$

Next, substituting ψ and ψ' for FTRL-CARE into Eq. (A.4), using Jensen's inequality with the concavity of square root, and the fact that $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$ for all $x, y > 0$ gives

$$\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sqrt{H(w^{\mathrm{C}}(t+1)) + c_2} \\
&\quad \le \sqrt{\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} H(w^{\mathrm{C}}(t+1)) + c_2} \\
&\quad \le \sqrt{\frac{17\log N_0}{16} + c_2} + \frac{4\sqrt{(N-N_0)}\exp\left\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{2\sqrt{2}}\right\}}{3}.
\end{aligned} \tag{B.13}$$

Thus,

$$\begin{aligned}
&\sum_{t=t_0+1}^{T} \frac{\sqrt{t+1}-\sqrt{t}}{c_1} \sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sqrt{H(w^{\mathrm{C}}(t+1)) + c_2} \\
&\quad \le \frac{\sqrt{\frac{17}{16}\log N_0 + c_2}\Big[\sqrt{T+1} - \sqrt{t_0+1}\Big]}{c_1} \\
&\qquad + \sum_{t=t_0+1}^{T} \frac{4\sqrt{(N-N_0)}\exp\left\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{t}\right\}}{3c_1\sqrt{t}} \\
&\quad \le \frac{\sqrt{\frac{17}{16}\log N_0 + c_2}\Big[\sqrt{T+1} - \sqrt{t_0+1}\Big]}{c_1} \\
&\qquad + \frac{8\sqrt{2}\sqrt{(N-N_0)}\exp\left\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{t_0}\right\}}{3c_1^2\sqrt{c_2}\,\Delta_0},
\end{aligned} \tag{B.14}$$

where the last step used Lemma E.3. Similarly, we use these same properties and Eq. (A.5) to obtain

$$
\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat\pi}\sqrt{H(v(t+1))+c_2} \\
&\quad\le \sqrt{\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat\pi} H(v(t+1))+c_2} \\
&\quad\le \sqrt{\frac{17\log N_0}{16}+c_2} + \frac{4\sqrt{(N-N_0)}\exp\left\{c_1\sqrt{c_2}+\frac{c_1^2c_2}{4}-\frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{2\sqrt{2}}\right\}}{3}.
\end{aligned}
\tag{B.15}
$$

For the last term of Eq. (B.12), we consider the cases of $N_0 > 1$ and $N_0 = 1$ separately. For both, however, we will use $t_0 = \left\lceil \frac{2[\log N + 3c_1\sqrt{c_2}+\frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2} \right\rceil$ and the constant $C = \max\{c_2, 3c_1\sqrt{c_2}+\frac{5}{4}c_1^2c_2\}$.

FTRL-CARE upper bound: $N_0 > 1$.

If $N_0 > 1$, we again use Lemma E.1 to control the variance terms. Then, using Eq. (B.15) and another application of Lemma E.3,

$$
\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat\pi} \sum_{t=t_0+1}^{T} \frac{2c_1\sqrt{H(v(t+1))+c_2}}{\sqrt{t+1}} \\
&\qquad\times \sqrt{\operatorname*{Var}_{I\sim v(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]} \\
&\quad\le \frac{3c_1}{4}\sum_{t=t_0+1}^{T}\sqrt{\frac{\frac{17}{16}\log N_0+c_2}{t+1}} \\
&\qquad + c_1\sqrt{(N-N_0)}\sum_{t=t_0+1}^{T}\frac{\exp\left\{c_1\sqrt{c_2}+\frac{c_1^2c_2}{4}-\frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{2\sqrt{2}}\right\}}{\sqrt{t}} \\
&\quad\le \frac{3c_1\sqrt{\frac{17}{16}\log N_0+c_2}\left[\sqrt{T+1}-\sqrt{t_0+1}\right]}{2} \\
&\qquad + \frac{8\sqrt{(N-N_0)}\exp\left\{c_1\sqrt{c_2}+\frac{c_1^2c_2}{4}-\frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t_0}}{2\sqrt{2}}\right\}}{\sqrt{2c_2}\,\Delta_0}.
\end{aligned}
\tag{B.16}
$$

Combining Eqs. (B.12), (B.14) and (B.16) gives that

$$
\begin{aligned}
&\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{C}}(T) \\
&\qquad \le \sqrt{(t_0+1)[\log N + c_2]}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big) - \frac{\sqrt{(T+1)c_2}}{c_1} \\
&\qquad\quad + \frac{\sqrt{\frac{17}{16}\log N_0 + c_2}\Big[\sqrt{T+1} - \sqrt{t_0+1}\Big]}{c_1} \\
&\qquad\quad + \frac{3c_1\sqrt{\frac{17}{16}\log N_0 + c_2}\Big[\sqrt{T+1} - \sqrt{t_0+1}\Big]}{2} \\
&\qquad\quad + \frac{16\sqrt{(N-N_0)}\exp\Big\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t_0}}{2\sqrt{2}}\Big\}}{3\sqrt{2}c_1^2\sqrt{c_2}\,\Delta_0} \qquad \text{(B.17)} \\
&\qquad\quad + \frac{8\sqrt{(N-N_0)}\exp\Big\{c_1\sqrt{c_2} + \frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t_0}}{2\sqrt{2}}\Big\}}{\sqrt{2c_2}\,\Delta_0} \\
&\qquad \le \frac{33}{32}\sqrt{(T+1)[\log N_0 + c_2]}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big) \\
&\qquad\quad + \sqrt{(t_0+1)}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big)\Big(\sqrt{\log N + c_2} - \sqrt{\log N_0 + c_2}\Big) \\
&\qquad\quad + \frac{\sqrt{2}\sqrt{(N-N_0)}\exp\Big\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{t_0}\Big\}}{\sqrt{c_2}\,\Delta_0}\Big(\frac{8}{3c_1^2} + 4\exp\{c_1\sqrt{c_2}\}\Big).
\end{aligned}
$$

Substituting t_0 into Eq. (B.17) gives

$$
\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\hat{R}_{\mathrm{C}}(T) \\
&\quad\le \Big(\frac{1}{c_1}+\frac{3c_1}{2}\Big)\Bigg[\frac{33}{32}\sqrt{(T+1)[\log N_0+c_2]} \\
&\qquad\qquad + \sqrt{\Big(\frac{2[\log N+3c_1\sqrt{c_2}+\frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2}+2\Big)[\log N+c_2]}\ \Bigg] \\
&\qquad + \frac{\sqrt{2}\sqrt{(N-N_0)}}{\sqrt{c_2}\,\Delta_0}\Big(\frac{8}{3c_1^2}+4\exp\{c_1\sqrt{c_2}\}\Big) \\
&\qquad\qquad \times\exp\left\{\frac{c_1^2c_2}{4}-\frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{\frac{2[\log N+3c_1\sqrt{c_2}+\frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2}}\right\} \\
&\quad\le \Big(\frac{1}{c_1}+\frac{3c_1}{2}\Big)\Bigg[\frac{33}{32}\sqrt{(T+1)[\log N_0+c_2]} \\
&\qquad\qquad + \frac{\sqrt{2}\,[\log N+C]^{3/2}}{c_1\sqrt{c_2}\,\Delta_0}+\sqrt{2[\log N+c_2]}\Bigg] \\
&\qquad + \frac{\sqrt{2}(8+12c_1^2)}{3c_1^2\sqrt{c_2}\,\Delta_0}.
\end{aligned}
$$

FTRL-CARE upper bound: $N_0 = 1$
If $\mathcal{I}_0=\{i_0\}$, we control the variance terms using Lemma E.2 In particular,

$$
\begin{aligned}
&\mathbb{E}_{\pi,\hat{\pi}}\left[\operatorname*{Var}_{I\sim v(t+1)}\left[\Big(\frac{\sqrt{t+1}}{\sqrt{t}}-1\Big)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim v(t+1)}[\ell_I(t)]\right] \\
&\qquad\le \frac{27}{32}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i\neq i_0}v_i(t+1)\right].
\end{aligned}
$$

We control this using Lemma A.2 with $p=1$, which gives

$$
\sup_{\pi\in\mathscr{P}(\mathcal{D})}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i\neq i_0}v_i(t+1)\right]\le (N-1)\exp\Big\{4c_1\sqrt{c_2}+2c_1^2c_2-\sqrt{2}\,c_1\sqrt{c_2}\Delta_0\sqrt{t}\Big\}.
$$

Thus, using Cauchy-Schwarz and Eq. (B.15) (recalling $\log N_0 = 0$), for any $\pi \in \mathscr{P}(\mathcal{D})$

$$\begin{aligned}
&\mathbb{E}_{\pi,\hat{\pi}} \sqrt{(H(v(t+1)) + c_2) \operatorname*{Var}_{I\sim v(t+1)} \left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \operatorname*{Var}_{I\sim v(t+1)} [\ell_I(t)]} \\
&\quad \le \sqrt{\mathbb{E}_{\pi,\hat{\pi}} H(v(t+1)) + c_2} \\
&\qquad\qquad \times \sqrt{\mathbb{E}_{\pi,\hat{\pi}} \left[\operatorname*{Var}_{I\sim v(t+1)} \left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \operatorname*{Var}_{I\sim v(t+1)} [\ell_I(t)] \right]} \\
&\quad \le \sqrt{\frac{7(N-1)\exp\left\{2c_1\sqrt{c_2} + \frac{c_1^2 c_2}{2} - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{\sqrt{2}}\right\}}{4} + c_2} \\
&\qquad\qquad \times \sqrt{(N-1)} \exp\left\{ 2c_1\sqrt{c_2} + c_1^2 c_2 - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{\sqrt{2}} \right\} \\
&\quad \le \frac{3(N-1)\exp\left\{3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{3c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{t}\right\}}{2} \\
&\qquad\qquad + \sqrt{c_2(N-1)} \exp\left\{ 2c_1\sqrt{c_2} + c_1^2 c_2 - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t}}{\sqrt{2}} \right\} \\
&\quad \le (3/2 + \sqrt{c_2})(N-1)\exp\left\{ 3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{t} \right\}.
\end{aligned}$$

Summing this over t and applying Lemma E.3 gives

$$\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sum_{t=t_0+1}^{T} \frac{2c_1\sqrt{H(v(t+1)) + c_2}}{\sqrt{t+1}} \\
&\qquad \times \sqrt{\operatorname*{Var}_{I\sim v(t+1)} \left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1 \right) L_I(t-1) - \ell_I(t) \right] \operatorname*{Var}_{I\sim v(t+1)} [\ell_I(t)]} \\
&\qquad \le c_1(3 + 2\sqrt{c_2})(N-1)\exp\left\{3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4}\right\} \qquad \text{(B.18)} \\
&\qquad\qquad \times \sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t}} \exp\left\{ -\frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{t} \right\} \\
&\qquad \le \frac{\sqrt{2}(3+2\sqrt{c_2})(N-1)}{\sqrt{c_2}\,\Delta_0} \exp\left\{ 3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{t_0} \right\}.
\end{aligned}$$

Combining Eqs. (B.12), (B.14) and (B.18) gives that for $N_0 = 1$,

$$
\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{C}}(T) \\
&\quad\le \sqrt{(t_0+1)[\log N + c_2]}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big) - \frac{\sqrt{(T+1)c_2}}{c_1} \\
&\qquad + \frac{8\sqrt{2(N-1)}\exp\Big\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{t_0}\Big\}}{3c_1^2\sqrt{c_2}\,\Delta_0} \\
&\qquad + \frac{\sqrt{c_2}}{c_1}\Big[\sqrt{T+1} - \sqrt{t_0+1}\Big] \\
&\qquad + \frac{\sqrt{2}(3+2\sqrt{c_2})(N-1)\exp\Big\{3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{t_0}\Big\}}{\sqrt{c_2}\,\Delta_0} \qquad \text{(B.19)}\\
&\quad\le \sqrt{(t_0+1)[\log N + c_2]}\Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big) \\
&\qquad + \frac{8\sqrt{2(N-1)}\exp\Big\{\frac{c_1^2 c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0\sqrt{t_0}}{2\sqrt{2}}\Big\}}{3c_1^2\sqrt{c_2}\,\Delta_0} \\
&\qquad + \frac{\sqrt{2}(3+2\sqrt{c_2})(N-1)}{\sqrt{c_2}\,\Delta_0}\exp\Big\{3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{t_0}\Big\}.
\end{aligned}
$$

Substituting t_0 into Eq. (B.19) gives

$$
\begin{aligned}
&\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \hat{R}_{\mathrm{C}}(T) \\
&\le \Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big)\sqrt{\Big(\frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2} + 2\Big)[\log N + c_2]} \\
&\quad + \frac{8\sqrt{2(N-1)}}{3c_1^2\sqrt{c_2}\,\Delta_0}\ \exp\left\{\frac{c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{2\sqrt{2}}\sqrt{\frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2}}\right\} \\
&\quad + \frac{\sqrt{2}(3+2\sqrt{c_2})(N-1)}{\sqrt{c_2}\,\Delta_0} \\
&\qquad \times \exp\left\{3c_1\sqrt{c_2} + \frac{5c_1^2c_2}{4} - \frac{c_1\sqrt{c_2}\,\Delta_0}{\sqrt{2}}\sqrt{\frac{2[\log N + 3c_1\sqrt{c_2} + \frac{5}{4}c_1^2c_2]^2}{c_1^2c_2\Delta_0^2}}\right\} \\
&\le \Big(\frac{1}{c_1} + \frac{3c_1}{2}\Big)\left[\frac{\sqrt{2}\,[\log N + C]^{3/2}}{c_1\sqrt{c_2}\,\Delta_0} + \sqrt{2[\log N + c_2]}\right] \\
&\quad + \frac{1}{\sqrt{c_2}\,\Delta_0}\left[\frac{8\sqrt{2}}{3c_1^2} + \sqrt{2}(3+2\sqrt{c_2})\right].
\end{aligned}
$$

□

C: GENERIC FTRL REGRET BOUNDS WITH LOCAL NORMS

C.1. Online linear optimization with FTRL. An *online linear optimization* (OLO) problem in $\mathbb{R}^d$ is defined by a closed *prediction domain* $F \subseteq \mathbb{R}^d$ and a *loss domain* $G \subseteq \mathbb{R}^d$. At each time t, the player selects $\mu(t) \in F$, then observes some $\lambda(t) \in G$ and incurs the loss $\langle \lambda(t),\ \mu(t) \rangle$. For any sequence of losses $\lambda(1), \dots, \lambda(T) \in G$, the player's regret is defined by

$$R_{\texttt{olo}}(T) = \sum_{t=1}^{T} \langle \lambda(t),\ \mu(t) \rangle - \inf_{\mu \in F} \sum_{t=1}^{T} \langle \lambda(t),\ \mu \rangle .$$

There are many ways one could choose $\mu(t)$, but in this work we focus specifically on FTRL, which is a generic method for online linear optimization. The FTRL algorithm is parametrized by F, G, and a sequence of *regularizers* $\{\rho_t : F \to \mathbb{R}\}_{t \in \mathbb{Z}_+}$. For each time $t+1$, a player using the FTRL$(F,\ G,\ (\rho_t)_{t \in \mathbb{Z}_+})$ algorithm outputs

$$\mu(t+1) \in \arg\min_{\mu \in F} \left(\langle \Lambda(t),\ \mu \rangle + \rho_{0:t}(\mu) \right), \tag{C.1}$$

where $\rho_{0:t}(\mu) = \sum_{s=0}^{t} \rho_s(\mu)$ and $\Lambda(t) = \sum_{s=1}^{t} \lambda(s)$.

C.2. OLO FTRL regret bounds. The classical regret bound for FTRL consists of a term that is the difference of losses incurred by consecutive player vectors and a term that looks like the regularizer evaluated at the optimal player vector in hindsight. The former is usually bounded using strong-convexity to obtain a norm of the consecutive weight differences. For tighter control, such as that obtained by Abernethy and Rakhlin [1], this norm may be chosen to be a *local norm*. A local norm with respect to a function f will be of the form $\|x\|_y = \sqrt{\langle x,\ \nabla^2 f(y) x \rangle}$, and has the property that the dual is $\|x\|_{y,\star} = \sqrt{\langle x,\ (\nabla^2 f(y))^{-1} x \rangle}$. The natural choice of function to define the local norm with respect to is the regularizer; however, this is generally more challenging for non-constant regularizers.

Surprisingly, while both local norms and time-dependent regularizers are standard in the FTRL literature, we were unable to find an explicit statement that combines them exactly as we needed. The closest seems to be Theorem 1 of McMahan [9], which requires that the regularizers are strongly convex with respect to a norm and then defines the local norm using the time-dependent strong convexity parameter. This strong-convexity argument is insufficient for our analysis, as the CARE regularizer can be at worst only $1/\sqrt{\log N}$-strongly convex in all settings, and consequently would not lead to the adaptive rates we obtain. We begin with a modification of [10, Lemma 1] to combine local norm bounds with time-dependent regularizer bounds.

LEMMA C.1. *For any F, G, $(\rho_t)_{t \in \mathbb{Z}_+}$, and $(\lambda(t))_{t \in \mathbb{N}} \subseteq G$, the* FTRL$(F,\ G,\ (\rho_t)_{t \in \mathbb{Z}_+})$ *algorithm has regret bounded for all $T \in \mathbb{N}$ by*

$$R_{\texttt{olo}}(T) \leq \rho_{0:T}(\mu_*(T)) - \sum_{t=0}^{T} \rho_t(\mu(t+1)) + \sum_{t=1}^{T} \langle \lambda(t),\ \mu(t) - \mu(t+1) \rangle ,$$

for all $\mu_*(T) \in \arg\min_{\mu \in F} \langle \Lambda(T),\ \mu \rangle$.

PROOF OF LEMMA C.1. This follows from directly modifying the proof of [10, Lemma 1] by not dropping the $\rho_t(\mu(t+1))$ term at the end of [10, Lemma 7]. We reproduce the argument here for completeness.

As shown by Kalai and Vempala [7], and restated in [10, Lemma 6],

$$\sum_{t=0}^{T} f_t(x_\star(t)) \leq \sum_{t=0}^{T} f_t(x_\star(T))$$

for any sequence of functions $(f_t)_{t\in\mathbb{Z}_+}$ and any sequence $x_\star(t) \in \arg\min_x \sum_{s=0}^t f_s(x)$. Thus, by definition of $\mu(t+1)$ minimizing Eq. (C.1),

$$\begin{aligned}\sum_{t=0}^{T} [\langle \lambda(t),\ \mu(t+1)\rangle + \rho_t(\mu(t+1))] &\leq \sum_{t=0}^{T} [\langle \lambda(t),\ \mu(T+1)\rangle + \rho_t(\mu(T+1))] \\ &\leq \sum_{t=0}^{T} [\langle \lambda(t),\ \mu_*(T)\rangle + \rho_t(\mu_*(T))] \\ &= \langle \Lambda(T),\ \mu_*(T)\rangle + \rho_{0:T}(\mu_*(T)).\end{aligned}$$

Rearranging gives that

$$\begin{aligned}R_{\mathtt{olo}}(T) &= \sum_{t=0}^{T} \langle \lambda(t),\ \mu(t)\rangle - \langle \Lambda(T),\ \mu_*(T)\rangle \\ &= \sum_{t=0}^{T} \langle \lambda(t),\ \mu(t) - \mu(t+1)\rangle + \sum_{t=0}^{T} \langle \lambda(t),\ \mu(t+1)\rangle - \langle \Lambda(T),\ \mu_*(T)\rangle \\ &\leq \sum_{t=0}^{T} \langle \lambda(t),\ \mu(t) - \mu(t+1)\rangle + \rho_{0:T}(\mu_*(T)) - \sum_{t=0}^{T} \rho_t(\mu(t+1)).\end{aligned}$$

Finally, the indexing of t in the sums of the lemma statement follows since by convention $\lambda(0) = 0$. □

An alternative to the regret expansion for FTRL from McMahan and Streeter [10] has appeared in more recent literature such as that of Duchi, Hazan and Singer [5], Hazan [6], Orabona [11], Shalev-Shwartz [13]. This alternative analysis can be tighter in certain cases, but requires controlling three terms instead of two. Additionally, it could only lead to improvements in the constants in our case (bounded losses), so we opted for the simpler approach.

C.3. OLO FTRL regret bounds with local norms. Now, we provide a local-norm control on the inner product from Lemma C.1 for time-dependent regularizers which can be defined as a function of time and a constant regularizer. The types of regularizers we will consider are *convex functions of the Legendre type*, as defined by [12, Sec. 26].

DEFINITION C.1 (Essentially smooth, Rockafellar [12], Section 26). *An extended-real-valued function $f : F \to \overline{\mathbb{R}}$ for $F \subseteq \mathbb{R}^d$ is* essentially smooth *on F if it satisfies*

1. interior$(F) \neq \emptyset$,
2. *f is differentiable on* interior(F), *and*
3. *$x \in \partial(F)$ and $\{y_i\}_{i\in\mathbb{N}} \subseteq$ interior(F) with $y_i \to x$ implies $\|\nabla f(y_i)\| \to +\infty$.*

DEFINITION C.2 (Legendre type, Rockafellar [12], Section 26). *A closed convex function $f : F \to \mathbb{R}$ for $F \subseteq \mathbb{R}^d$ is* of the Legendre type *on F if*

1. *f is strictly convex on* interior(F),
2. interior(F) *is convex, and*
3. *f is essentially smooth on F.*

DEFINITION C.3 (Legendre Transform, Rockafellar [12], Section 26). *The* Legendre transform *of a function $f : F \to \mathbb{R}$ for $F \subseteq \mathbb{R}^d$ of the Legendre type on F is the function $f^* : \nabla f(\text{interior}(F)) \to \mathbb{R}$ defined by*

$$f^*(y) = \sup_{x \in F}[\langle x,\ y\rangle - f(x)] = \Big\langle [\nabla f]^{-1}(y),\ y \Big\rangle - f([\nabla f]^{-1}(y)).$$

PROPOSITION C.1 (Rockafellar [12], Theorem 26.5). *If f is a closed convex function of the Legendre type on F for $F \subseteq \mathbb{R}^d$ and $F^* = \nabla f(\text{interior}(F))$, then F^* is convex and f^* is of the Legendre type on F^*,*

$$\nabla f : \text{interior}(F) \to F^\star$$

is a continuous bijection with continuous inverse, and $\nabla[f^*] = [\nabla f]^{-1}$.

COROLLARY C.1. *If $F \subseteq \mathbb{R}^d$ is convex with non-empty interior, and if f is a closed, convex function of the Legendre type on F, then for any y with $-y \in \nabla f(\text{interior}(F))$,*

$$\arg\min_{x \in F} \left(\langle y,\ x\rangle + f(x)\right) = \Big\{[\nabla f]^{-1}(-y)\Big\} = \{[\nabla[f^*]](-y)\} \in \text{interior}(F).$$

PROOF. Since the objective is convex then if a single local minimum occurs in the interior F then it must be the unique optimizer on F. Taking the gradient of the objective, we see that a local minimum occurs when $\nabla f(x) = -y$. Since f is assumed to be of the Legendre type on F then this equation has a unique solution in interior(F) whenever $-y \in \nabla f(\text{interior}(F))$. □

LEMMA C.2. *Suppose that $F \subseteq \mathbb{R}^d$ is convex with non-empty interior, $G \subseteq \mathbb{R}^d$ is arbitrary, and the regularizer ρ_0 is closed, convex, of the Legendre type on F, and twice continuously differentiable on* interior(F)*. For each $t \in \mathbb{N}$, let $\rho_{0:t}(\mu) = \beta(t)\rho_0(\mu)$ for some increasing function $\beta : \mathbb{N} \to \mathbb{R}_+$. Also, for any $y \in G$ and $x \in F$, define the time-dependent local norm by $\|y\|_{t,x}^2 = \langle y,\ \nabla^2\rho_{0:t}(x)y\rangle$, and its dual time-dependent local norm by $\|y\|_{t,x,\star}^2 = \langle y,\ [\nabla^2\rho_{0:t}(x)]^{-1}y\rangle$. Then, for any sequence of losses $(\lambda(t))_{t\in\mathbb{N}} \subseteq G$ such that $(-\frac{1}{\beta(t)}\Lambda(t)) \in [\nabla\rho_0](\text{interior}(F))$ for all $t \in \mathbb{N}$, there exists a sequence $(\alpha_t)_{t\in\mathbb{N}} \subseteq [0,1]$ such that, for all $t \in \mathbb{N}$, the weights $(\mu(t))_{t\in\mathbb{N}}$ output by the* FTRL(F, G, $(\rho_t)_{t\in\mathbb{Z}_+}$) *algorithm satisfy*

$$\langle\lambda(t),\ \mu(t) - \mu(t+1)\rangle \le \frac{1}{\beta(t)} \left\| \left(\frac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t) \right\|_{0,v(t+1),\star} \|\lambda(t)\|_{0,v(t+1),\star},$$

where $v(t+1) = \arg\min_{v\in F}\left(\left\langle \alpha_t\Lambda(t) + (1-\alpha_t)\frac{\beta(t)}{\beta(t-1)}\Lambda(t-1),\ v\right\rangle + \rho_{0:t}(v)\right)$.

REMARK C.1. *In our applications, $[\nabla\rho_0](\text{interior}(F)) = \mathbb{R}^d$ is the whole space, so the assumption*

$$(-\tfrac{1}{\beta(t)}\Lambda(t)) \in [\nabla\rho_0](\text{interior}(F))$$

is benign. ◁

PROOF OF LEMMA C.2. Fix some $t \in \mathbb{N}$ and observe that by Corollary C.1, $\mu(t+1)$ is the unique μ that solves $\nabla\rho_{0:t}(\mu) = -\Lambda(t)$. Thus, applying a first-order Taylor expansion of $[\nabla\rho_{0:t}]^{-1}$ centered at $\nabla\rho_{0:t}(\mu(t))$,

$$\begin{aligned}\mu(t+1) - \mu(t) &= [\nabla\rho_{0:t}]^{-1}(\nabla\rho_{0:t}(\mu(t+1))) - [\nabla\rho_{0:t}]^{-1}(\nabla\rho_{0:t}(\mu(t))) \\ &= [J[\nabla\rho_{0:t}]^{-1}](-\zeta(t))\,[\nabla\rho_{0:t}(\mu(t+1)) - \nabla\rho_{0:t}(\mu(t))],\end{aligned}$$

where J denotes the Jacobian and $-\zeta(t) = \alpha_t \nabla\rho_{0:t}(\mu(t+1)) + (1-\alpha_t)\nabla\rho_{0:t}(\mu(t))$ for some $\alpha_t \in [0,1]$. Using the inverse function theorem on $\nabla\rho_{0:t}$ gives

$$[J[\nabla\rho_{0:t}]^{-1}](-\zeta(t)) = [\nabla^2\rho_{0:t}([\nabla\rho_{0:t}]^{-1}(-\zeta(t)))]^{-1}.$$

Next, observe that

$$\nabla\rho_{0:t}(\mu(t)) = \beta(t)\nabla\rho_0(\mu(t)) = \frac{\beta(t)}{\beta(t-1)}\nabla\rho_{0:t-1}(\mu(t)) = \frac{\beta(t)}{\beta(t-1)}(-\Lambda(t-1)),$$

so $\zeta(t)$ can be viewed as a combination of losses defined by

$$\zeta(t) = \alpha_t\Lambda(t) + (1-\alpha_t)\frac{\beta(t)}{\beta(t-1)}\Lambda(t-1).$$

Therefore, $-\frac{\zeta(t)}{\beta(t)} = \alpha_t\frac{-\Lambda(t)}{\beta(t)} + (1-\alpha_t)\frac{-\Lambda(t-1)}{\beta(t-1)} \in \nabla\rho_0(\text{interior}(F))$ since $\nabla\rho_0(\text{interior}(F))$ is convex (by Proposition C.1). This implies

$$-\zeta(t) \in [\beta(t)\nabla\rho_0](\text{interior}(F)) = \nabla\rho_{0:t}(\text{interior}(F)),$$

so $v(t+1) = [\nabla\rho_{0:t}]^{-1}(-\zeta(t)) \in \text{interior}(F)$ by Corollary C.1. Further,

$$\nabla\rho_{0:t}(\mu(t+1)) - \nabla\rho_{0:t}(\mu(t)) = -\Lambda(t) + \frac{\beta(t)}{\beta(t-1)}\Lambda(t-1) = \left(\frac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t).$$

Combining these results, along with the fact that $\nabla^2\rho_{0:t} = \beta(t)\nabla^2\rho_0$, gives

$$\mu(t+1) - \mu(t) = \frac{1}{\beta(t)}[\nabla^2\rho_0(v(t+1))]^{-1}\left[\left(\frac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t)\right]. \tag{C.2}$$

Next, by Holder's inequality,

$$\begin{aligned}\langle\lambda(t),\ \mu(t) - \mu(t+1)\rangle &\leq \|\mu(t) - \mu(t+1)\|_{t,v(t+1)}\,\|\lambda(t)\|_{t,v(t+1),\star}\\ &= \|\mu(t) - \mu(t+1)\|_{0,v(t+1)}\,\|\lambda(t)\|_{0,v(t+1),\star},\end{aligned}$$

where the last equality follows from the fact that a $\beta(t)$ will factor out of the first norm and a $1/\beta(t)$ will factor out of the second norm. Then, substituting in Eq. (C.2),

$$\begin{aligned}&\|\mu(t) - \mu(t+1)\|^2_{0,v(t+1)}\\ &= \left\langle\nabla^2\rho_0(v(t+1))\,[\mu(t) - \mu(t+1)],\ \mu(t) - \mu(t+1)\right\rangle\\ &= \frac{1}{\beta(t)^2}\left\langle[\nabla^2\rho_0(v(t+1))]^{-1}\left[\left(\tfrac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t)\right],\ \left(\tfrac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t)\right\rangle\\ &= \frac{1}{\beta(t)^2}\left\|\left(\frac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t)\right\|^2_{0,v(t+1),\star}.\end{aligned}$$

Thus,

$$\langle\lambda(t),\ \mu(t) - \mu(t+1)\rangle \leq \frac{1}{\beta(t)}\left\|\left(\frac{\beta(t)}{\beta(t-1)} - 1\right)\Lambda(t-1) - \lambda(t)\right\|_{0,v(t+1),\star}\|\lambda(t)\|_{0,v(t+1),\star}.$$

□

Amir et al. [2] recently made the same observation that closely related bounds have been derived before but not in the explicit form they desire, and they prove a regret bound very similar to Lemmas C.1 and C.2. However, they rely on a Taylor expansion of the regularizer around the weights output by FTRL, while we have used a Taylor expansion of the Legendre dual of the regularizer around the observed losses. This makes it easier for us to ultimately apply Theorem 2 when controlling the bound of Lemma C.2 in expectation. Zimmert and Seldin [14] have a similar expansion in their analysis, and obtain a local norm in the dual space as an intermediate step in the proof of their Lemma 11. However, the object they use this local norm to upper bound is not the same as what we upper bound, and they ultimately use a bound in the primal space to obtain their results.

D: FTRL REGRET BOUNDS ON THE SIMPLEX

When we restrict consideration to proper prediction policies (see Section 6) and focus on controlling the expected regret, then online linear optimization is a generalization of the online prediction problem in Section 2, which is just the case where $F = \mathtt{simp}([N])$, $G = [0,1]^N$, and we are interested in $\mathbb{E}\, R_{\mathtt{olo}}(T)$. To bound the expected regret, we choose an appropriate sequence of regularizers and then apply generic techniques for analyzing FTRL in online linear optimization problems. For clarity and to distinguish between FTRL in the generic online linear optimization setting and in the specific case of online prediction on the simplex, we use $(r_t)_{t\in\mathbb{Z}_+}$ to denote the sequence of regularizers in the latter. Thus, the $\mathrm{FTRL}((r_t)_{t\in\mathbb{Z}_+})$ notation is really shorthand in this case for $\mathrm{FTRL}(\mathtt{simp}([N]), [0,1]^{[N]}, (r_t)_{t\in\mathbb{Z}_+})$.

A significant portion of the heavy-lifting required for Theorem A.1 is done in Appendix C, which proves a very similar result for generic FTRL under some technical constraints. However, we cannot directly apply Lemma C.2 when $F = \mathtt{simp}([N])$, since this set has empty interior. Thus, we need a version of that result tailored to the simplex, which we achieve by a reparametrization of the simplex.

In particular, let $i_1 \in [N]$ be arbitrary, and let $[\hat{N}] = [N] \setminus \{i_1\}$. Let

$$\hat{F} = \left\{ \mu \in [\mathbb{R}_+]^{[\hat{N}]} \text{ s.t. } \sum_{i\in[\hat{N}]} \mu_i \le 1 \right\},$$

and observe that $\mathrm{interior}(\hat{F})$ is non-empty and convex. The canonical bijection $\phi : \mathtt{simp}([N]) \to \hat{F}$ is given by

$$\phi(u) = u_{-i_1}, \quad \text{and} \quad \phi^{-1}(\mu) = \left(\begin{cases} \mu_i & : i \in [\hat{N}] \\ 1 - \langle \mathbf{1},\ \mu \rangle & : i = i_1 \end{cases} \right)_{i\in[N]}$$

where u_{-i} is the vector obtained from u by dropping the coordinate with index i.

For any function $f : \mathtt{simp}([N]) \to \mathcal{Y}$ for some set $\mathcal{Y}$, define $\hat{f} : \hat{F} \to \mathcal{Y}$ by

$$\hat{f}(\mu) = f(\phi^{-1}(\mu)).$$

For example, if we let $H : \mathtt{simp}([N]) \to \mathbb{R}_+$ be the entropy function defined by

$$H(u) = -\sum_{i\in[N]} u_i \log(u_i),$$

then $\hat{H} : \hat{F} \to \mathbb{R}$ is defined by

$$\hat{H}(\mu) = H(\phi^{-1}(\mu)) = -\left(\sum_{i\in[\hat{N}]} \mu_i \log(\mu_i) \right) - (1 - \langle \mathbf{1},\ \mu \rangle) \log(1 - \langle \mathbf{1},\ \mu \rangle).$$

Note that for any sequence of regularizers $(r_t)_{t\in\mathbb{Z}_+}$ on $\mathtt{simp}([N])$ and any sequence of losses $(\lambda(t))_{t\in\mathbb{N}}$ in an arbitrary $G\subseteq\mathbb{R}^{[N]}$, for all $t\in\mathbb{N}$ we have

$$\begin{aligned}\langle\Lambda(t),\ u\rangle + r_{0:t}(u) &= \langle\Lambda_{-i_1}(t),\ u_{-i_1}\rangle + \Lambda_{i_1}(t)(1-\langle\mathbf{1},\ u_{-i_1}\rangle) + \hat{r}_{0:t}(u_{-i_1})\\ &= \Lambda_{i_1}(t) + \langle\Lambda_{-i_1}(t) - \Lambda_{i_1}(t)\mathbf{1},\ u_{-i_1}\rangle + \hat{r}_{0:t}(u_{-i_1}).\end{aligned}$$

Additionally, for any $(b(t))_{t\in\mathbb{N}}\subseteq\mathbb{R}$,

$$\operatorname*{arg\,min}_{u\in\mathtt{simp}([N])}\left(\langle\Lambda(t),\ u\rangle + r_{0:t}(u)\right) = \operatorname*{arg\,min}_{u\in\mathtt{simp}([N])}\left(\langle\Lambda(t)-b(t)\mathbf{1},\ u\rangle + r_{0:t}(u)\right)$$

by the requirement that $u\in\mathtt{simp}([N])$. Similarly, for any sequence $(u(t))_{t\in\mathbb{N}}\subseteq\mathtt{simp}([N])$, the regret is unchanged by shifting the loss vectors. That is,

$$\begin{aligned}&\sum_{t=1}^{T}\langle\lambda(t),\ u(t)\rangle - \inf_{u\in\mathtt{simp}([N])}\sum_{t=1}^{T}\langle\lambda(t),\ u\rangle\\ &\quad=\sum_{t=1}^{T}\langle\lambda(t)-b(t)\mathbf{1},\ u(t)\rangle - \inf_{u\in\mathtt{simp}([N])}\sum_{t=1}^{T}\langle\lambda(t)-b(t),\ u\rangle\,.\end{aligned}$$

Thus, there exist equivalence classes of the outputs from the FTRL($\mathtt{simp}([N])$, G, $(r_t)_{t\in\mathbb{Z}_+}$) algorithm modulo parallel additive shifts of the loss vectors. Further, by transforming the losses via

$$\Phi(\lambda) = \lambda_{-i_1} - \lambda_{i_1}\mathbf{1}\quad\text{ and }\quad\Phi^{+}(\lambda) = \left(\begin{cases}\lambda_i & : i\in[\hat{N}]\\ 0 & : i = i_1\end{cases}\right)_{i\in[N]}$$

and defining $\hat{G} = \{\Phi(\lambda):\lambda\in G\}$, there is a canonical correspondence between the equivalence classes of the outputs from the FTRL($\mathtt{simp}([N])$, G, $(r_t)_{t\in\mathbb{Z}_+}$) algorithm and those of the outputs from the FTRL($\hat{F}$, $\hat{G}$, $(\hat{r}_t)_{t\in\mathbb{Z}_+}$) algorithm. Namely,

$$\operatorname*{arg\,min}_{u\in\mathtt{simp}([N])}\left(\langle\Lambda(t),\ u\rangle + r_{0:t}(u)\right) = \phi^{-1}\left(\operatorname*{arg\,min}_{\mu\in\hat{F}}\left(\langle\Phi(\Lambda(t)),\ \mu\rangle + \hat{r}_{0:t}(\mu)\right)\right).$$

Under this correspondence, if $G=[0,1]^N$, $\hat{R}_{\hat{\pi}}(T) = R_{\mathtt{olo}}(T)$.

COROLLARY D.1. *Consider a regularizer* $r_0:\mathtt{simp}([N])\to\mathbb{R}$ *for which* $\hat{r}_0$ *is closed, convex, of the Legendre type on* $\hat{F}$ *(see Definition C.2), and twice continuously differentiable on* interior$(\hat{F})$. *For each* $t\in\mathbb{N}$, *define* $r_{0:t}(u)=\beta(t)r_0(u)$ *for some increasing function* $\beta:\mathbb{N}\to\mathbb{R}_+$. *Also, for any* $y\in G$ *and* $x\in\mathtt{simp}([N])$, *define the time-dependent local semi-norm by* $\|y\|_{t,x}^2 = \langle\Phi(y),\ \nabla^2\hat{r}_{0:t}(\phi(x))\Phi(y)\rangle$, *and its dual time-dependent local semi-norm by* $\|y\|_{t,x,\star}^2 = \langle\Phi(y),\ [\nabla^2\hat{r}_{0:t}(\phi(x))]^{-1}\Phi(y)\rangle$. *Then, for any sequence of losses* $(\lambda(t))_{t\in\mathbb{N}}\subseteq G$ *such that* $\Phi(-\frac{1}{\beta(t)}\Lambda(t))\in\nabla\hat{r}_0(\text{interior}(\hat{F}))$ *for all* $t\in\mathbb{N}$, *there exists a sequence* $(\alpha_t)_{t\in\mathbb{N}}\subseteq[0,1]$ *such that, for all* $t\in\mathbb{N}$, *the weights* $(u(t))_{t\in\mathbb{N}}$ *output by the* FTRL($\mathtt{simp}([N])$, G, $(r_t)_{t\in\mathbb{Z}_+}$) *algorithm satisfy*

$$\langle\lambda(t),\ u(t)-u(t+1)\rangle \le \frac{1}{\beta(t)}\left\|\left(\frac{\beta(t)}{\beta(t-1)}-1\right)\Lambda(t-1)-\lambda(t)\right\|_{0,v(t+1),\star}\|\lambda(t)\|_{0,v(t+1),\star}\,,$$

where $v(t+1) = \operatorname{arg\,min}_{v\in\mathtt{simp}([N])}\left(\left\langle\alpha_t\Lambda(t) + (1-\alpha_t)\frac{\beta(t)}{\beta(t-1)}\Lambda(t-1),\ v\right\rangle + r_{0:t}(v)\right)$.

Proof of Corollary D.1. For all $t \in \mathbb{N}$, since $u(t), u(t+1) \in \mathtt{simp}([N])$, it holds that for any $\lambda(t) \in G$,

$$\begin{aligned}\langle \lambda(t),\ u(t) - u(t+1)\rangle &= \langle \lambda(t) - \lambda_{i_1}(t)\mathbf{1},\ u(t) - u(t+1)\rangle \\ &= \langle \lambda_{-i_1}(t) - \lambda_{-i_1}(t)\mathbf{1},\ u_{-i_1}(t) - u_{-i_1}(t+1)\rangle \\ &= \langle \Phi(\lambda(t)),\ \phi(u(t)) - \phi(u(t+1))\rangle .\end{aligned}$$

Thus, using that $\phi(u(t))$ are the weights output by the $\mathrm{FTRL}(\hat{F}, \hat{G}, (\hat{r}_t)_{t\in\mathbb{Z}_+})$ algorithm, we can apply Lemma C.2. The result then follows from observing that Φ is linear. □

Lemma D.1. *Suppose $r_0 = -\psi \circ H$ for some $\psi : [0, \log N] \to \mathbb{R}$ that is strictly increasing, concave, and twice continuously differentiable on* $\mathtt{simp}([N])$*. Then $\hat{r}_0$ is closed, strictly convex, twice continuously differentiable on* interior$(\hat{F})$*, and of the Legendre type on $\hat{F}$.*

Moreover, for all $x \in \mathtt{simp}([N])$ and $y \in G$,

$$\|y\|_{0,x,\star}^2 \le \frac{1}{\psi'(H(x))} \operatorname*{Var}_{I\sim x}[y_I].$$

Proof of Lemma D.1. First, note that for $i \neq i' \in [\hat{N}]$ and $\mu \in \hat{F}$,

$$\begin{aligned}-\partial_{\mu_i}\hat{H}(\mu) &= \log(\mu_i) - \log\left(1 - \langle \mathbf{1},\ \mu\rangle\right),\\ -\partial^2_{\mu_i}\hat{H}(\mu) &= \frac{1}{\mu_i} + \frac{1}{1 - \langle \mathbf{1},\ \mu\rangle},\ \text{and}\\ -\partial_{\mu_i}\partial_{\mu_{i'}}\hat{H}(\mu) &= \frac{1}{1 - \langle \mathbf{1},\ \mu\rangle}.\end{aligned}$$

Thus,

$$-\nabla^2\hat{H}(\mu) = \mathrm{diag}(1/\mu) + \frac{1}{1 - \langle \mathbf{1},\ \mu\rangle}\mathbf{1}\mathbf{1}^{\mathrm{T}},$$

which is strictly positive-definite on interior$(\hat{F})$.

Therefore $\hat{H}$ is strictly concave. Since a composition of a strictly concave function with a strictly increasing strictly concave function is strictly concave, $\psi \circ \hat{H}$ is strictly concave, which means $\hat{r}_0$ is strictly convex. Since $\hat{r}_0$ is continuous and finite on $\hat{F}$, and $\hat{F}$ is closed it must also be a closed function, because a proper convex function is closed if it is lower-semi-continuous. The twice continuous differentiability of $\hat{r}_0$ on interior$(\hat{F})$ follows from the twice continuous differentiability of H on $\hat{F}$ and the twice differentiability of ψ.

Since we have already observed that interior$(\hat{F})$ is convex and non-empty, to see that $\hat{r}_0$ is of the Legendre type on $\hat{F}$ we need only verify that $\lim_{n\to\infty}\left\|\nabla\hat{r}_0(\mu^{(n)})\right\| \to \infty$ for any $\left\{\mu^{(n)}\right\}_{n\in\mathbb{N}} \subseteq$ interior$(\hat{F})$ such that $\mu^{(n)} \to \nu \in \partial(\hat{F})$. The gradient of $\hat{r}_0$ is given by

$$\nabla\hat{r}_0(\mu) = -[\psi' \circ \hat{H}(\mu)]\nabla\hat{H}(\mu).$$

Now, notice that if $\nu \in \partial(\hat{F})$, $\hat{H}(\nu) \le \log(N-1)$. Since ψ is strictly increasing and concave on $[0, \log(N)]$, this implies $\psi'(\hat{H}(\nu)) > 0$. At any $\nu \in \partial\hat{F}$, either there exists an $i \in [\hat{N}]$ such that $\nu_i = 0$ or $\langle \mathbf{1},\ \nu\rangle = 1$. In both cases, $\mu^{(n)} \to \nu \in \partial(\hat{F})$ implies $\|\nabla\hat{H}(\mu^{(n)})\| \to +\infty$. Therefore, $\nabla\hat{r}_0(\mu_i) \to \psi'(\hat{H}(\nu)) \cdot (+\infty) = +\infty$, which confirms that $\hat{r}_0$ is of the Legendre type on $\hat{F}$.

To derive the semi-norm formula, first notice that using the Sherman–Morrison–Woodbury formula gives

$$
\begin{aligned}
-[\nabla^2 \hat{H}(\mu)]^{-1} &= \operatorname{diag}(\mu) - \operatorname{diag}(\mu)\mathbf{1}\left((1-\langle \mathbf{1},\ \mu\rangle) + \mathbf{1}^{\mathrm{T}}\operatorname{diag}(\mu)\mathbf{1}\right)^{-1}\mathbf{1}^{\mathrm{T}}\operatorname{diag}(\mu) \\
&= \operatorname{diag}(\mu) - \operatorname{diag}(\mu)\mathbf{1}\mathbf{1}^{\mathrm{T}}\operatorname{diag}(\mu) \\
&= \operatorname{diag}(\mu) - \mu\mu^{\mathrm{T}}.
\end{aligned} \tag{D.1}
$$

Then,

$$
\begin{aligned}
\nabla^2 \hat{r}_0(\mu) &= -[\psi'' \circ \hat{H}(\mu)](\nabla \hat{H}(\mu))(\nabla \hat{H}(\mu))^{\mathrm{T}} - [\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu)) \\
&\succeq -[\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu)),
\end{aligned}
$$

where $A \succeq B$ means $A - B$ is positive semi-definite. Therefore,

$$
\begin{aligned}
[\nabla^2 \hat{r}_0(\mu)]^{-1} &\preceq \left(-[\psi' \circ \hat{H}(\mu)](\nabla^2 \hat{H}(\mu))\right)^{-1} \\
&= \frac{1}{\psi' \circ \hat{H}(\mu)}\left(\operatorname{diag}(\mu) - \mu\mu^{\mathrm{T}}\right).
\end{aligned} \tag{D.2}
$$

Applying Eqs. (D.1) and (D.2) to an arbitrary $x \in \mathtt{simp}([N])$ and $y \in G$ gives

$$
\begin{aligned}
\|y\|_{0,x,\star}^2 &= \left\langle \Phi(y),\ \nabla^2 \hat{r}_0(\phi(x))^{-1}\Phi(y)\right\rangle \\
&= \left\langle y_{-i_1} - y_{i_1}\mathbf{1},\ \nabla^2 \hat{r}_0(x_{-i_1})^{-1}\left[y_{-i_1} - y_{i_1}\mathbf{1}\right]\right\rangle \\
&\le \frac{1}{\psi' \circ \hat{H}(x_{-i_1})}\left\langle y_{-i_1} - y_{i_1}\mathbf{1},\ \left[\operatorname{diag}(x_{-i_1}) - x_{-i_1}x_{-i_1}^{\mathrm{T}}\right]\left[y_{-i_1} - y_{i_1}\mathbf{1}\right]\right\rangle \\
&= \frac{1}{\psi' \circ H(x)}\left\langle y - y_{i_1}\mathbf{1},\ \left[\operatorname{diag}(x) - xx^{\mathrm{T}}\right]\left[y - y_{i_1}\mathbf{1}\right]\right\rangle \\
&= \frac{1}{\psi' \circ H(x)}\operatorname*{Var}_{I\sim x}[y_I - y_{i_1}] \\
&= \frac{1}{\psi' \circ H(x)}\operatorname*{Var}_{I\sim x}[y_I].
\end{aligned}
$$

□

Lemma D.2. *Suppose $r_0 = -\psi \circ H$ for some $\psi : [0, \log N] \to \mathbb{R}$ that is strictly increasing, concave, and twice continuously differentiable on $\mathtt{simp}([N])$. Further, suppose that $r_{0:t} = \beta(t) r_0$ for some strictly increasing $\beta : \mathbb{N} \to \mathbb{R}_+$. Then, $[\nabla \hat{r}_0](\mathrm{interior}(\hat{F})) = \mathbb{R}^{[\hat{N}]}$, and the weight vectors produced by the* FTRL$(\mathtt{simp}([N]), G, (r_t)_{t\in\mathbb{Z}_+})$ *algorithm are equivalent to the weights produced by* Hedge *with an implicitly defined learning rate. In particular, the learning rate and weights are the solution to the system of equations*

$$
\begin{aligned}
\eta(t+1) &= \frac{1}{\beta(t)\cdot \psi' \circ H(u(t+1))} \\
u(t+1) &= \left(\frac{\exp\left(-\eta(t+1)\Lambda_i(t)\right)}{\sum_{i'\in[N]}\exp\left(-\eta(t+1)\Lambda_{i'}(t)\right)}\right)_{i\in[N]}.
\end{aligned} \tag{D.3}
$$

Moreover, for any sequence of losses $(\lambda(t))_{t\in\mathbb{N}} \subseteq G$, this system has a unique solution satisfying

$$
\eta(t+1) \in \left[\frac{1}{\beta(t)\cdot\psi'(0)},\ \frac{1}{\beta(t)\cdot\psi'(\log N)}\right].
$$

PROOF OF LEMMA D.2. First, recall that the $\mathrm{FTRL}(\mathtt{simp}([N]),\, G,\, (r_t)_{t\in\mathbb{Z}_+})$ algorithm outputs weights that solve

$$u(t+1) = \operatorname*{arg\,min}_{w\in\mathtt{simp}([N])} \{\langle \Lambda(t),\ w\rangle - \beta(t)\psi(H(w))\}$$
$$= \phi^{-1}\left(\operatorname*{arg\,min}_{\mu\in\hat{F}} \left\{\langle \Lambda_{-i_1}(t) - \mathbf{1}\Lambda_{i_1}(t),\ \mu\rangle - \beta(t)\psi(\hat{H}(\mu))\right\}\right).$$

By Lemma D.1 and Corollary C.1, we know that this means $u(t+1) = \phi^{-1}(\mu)$ for the unique $\mu \in \mathrm{interior}(\hat{F})$ such that

$$\nabla\hat{H}(\mu) = -\frac{\Lambda_{-i_1}(t) - \mathbf{1}\Lambda_{i_1}(t)}{\beta(t)\cdot\psi'(\hat{H}(\mu))}.$$

Thus, by the definition of ϕ and $\hat{H}$,

$$\nabla\hat{H}(u_{-i_1}(t+1)) = -\frac{\Lambda_{-i_1}(t) - \mathbf{1}\Lambda_{i_1}(t)}{\beta(t)\cdot\psi'(\hat{H}(u_{-i_1}(t+1)))} = -\frac{\Lambda_{-i_1}(t) - \mathbf{1}\Lambda_{i_1}(t)}{\beta(t)\cdot\psi'(H(u(t+1)))}. \tag{D.4}$$

It is well known that the unique solution to

$$\nabla\hat{H}(\phi(u)) = \Phi(-X)$$

is given by

$$u_i = \frac{\exp(-X_i)}{\sum_{i'\in[N]}\exp(-X_{i'})}.$$

Therefore, any and all solutions of Eq. (D.4) must also be solutions of Eq. (D.3). Next, we want to show that there is a unique solution, $\eta(t+1)$, to the implicit equation

$$\eta(t+1) = \frac{1}{\beta(t)\cdot\psi'\circ H\left(\left(\frac{\exp\{-\eta(t+1)\Lambda_i(t)\}}{\sum_{i'\in[N]}\exp\{-\eta(t+1)\Lambda_{i'}(t)\}}\right)_{i\in[N]}\right)}. \tag{D.5}$$

On the left hand side, we have $f_1(\eta) = \eta$, which is trivially strictly increasing from 0 to $\frac{1}{\beta(t)\cdot\psi'(\log N)}$ as η increases from 0 to $\frac{1}{\beta(t)\cdot\psi'(\log N)}$. On the right hand side, we have

$$f_2(\eta) = \frac{1}{\beta(t)\cdot\psi'\circ H\left(\left(\frac{\exp\{-\eta(t+1)\Lambda_i(t)\}}{\sum_{i'\in[N]}\exp\{-\eta(t+1)\Lambda_{i'}(t)\}}\right)_{i\in[N]}\right)},$$

which is non-increasing with $f_2(0) = \frac{1}{\beta(t)\cdot\psi'(\log N)}$. Further, by non-negativity of entropy and concavity of ψ, $f_2(\eta) \geq \frac{1}{\beta(t)\cdot\psi'(0)}$. Thus, f_1 and f_2 must intersect at some $\eta \in \left[\frac{1}{\beta(t)\cdot\psi'(0)},\ \frac{1}{\beta(t)\cdot\psi'(\log N)}\right]$, and this intersection is unique by the monotonicity of both functions and the strict monotonicity of f_1.

This guarantees at least one interior point solution to the implicit equation defined in Eq. (D.5). Moreover, since the objective function optimized by the weights output by the $\mathrm{FTRL}(\mathtt{simp}([N]),\, G,\, (r_t)_{t\in\mathbb{Z}_+})$ algorithm is strictly convex, this interior point solution must be the unique optimizer of the objective. Finally, since the sequence of losses was arbitrary and the $\mathrm{FTRL}(\hat{F},\, \hat{G},\, (\hat{r}_t)_{t\in\mathbb{Z}_+})$ algorithm outputs a unique weight vector at each time $t+1$, we conclude that $[\nabla\hat{r}_0](\mathrm{interior}(F)) = \mathbb{R}^{[\hat{N}]}$ as otherwise there would be some loss vector for which the solution to Eq. (D.5) does not exist. □

D.1. Proof of Theorem A.1. Theorem A.1 is an immediate consequence of the combination of Corollary D.1 and Lemmas C.1, D.1 and D.2. In the application of Lemma C.1, we can select $u_*(T) \in \arg\min_{u \in \mathtt{simp}([N])} \langle L(T),\ u\rangle$ such that $H(u_*(T)) = 0$ because at least one arg min occurs at a vertex of the simplex. □

E: PROOFS OF LEMMAS IN Appendix A

E.1. Proof of Lemma A.1. First, observe that

$$H(u) = -\sum_{i \in [N]} u_i \log(u_i) = -\sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}) - \sum_{i \in [N] \setminus \mathcal{I}_0} u_i \log(u_i).$$

To bound the first term, consider the optimization problem

$$\min_{\substack{\langle \mathbf{1},\ u\rangle = 1 \\ \langle \mathbf{1},\ u_{\mathcal{I}_0}\rangle \leq 1}} \sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}),$$

where $u_{\mathcal{I}_0} = \{u_{i_0}\}_{i_0 \in \mathcal{I}_0}$. This is a convex objective with linear constraints, so it can be solved using the Lagrange multiplier method. The Lagrangian is

$$L(u; \alpha, \beta) = \sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}) + \alpha(\langle \mathbf{1},\ u\rangle - 1) + \beta(\langle \mathbf{1},\ u_{\mathcal{I}_0}\rangle - 1),$$

and the dual problem is

$$\max_{\substack{\alpha \in \mathbb{R} \\ \beta \geq 0}} \min_{u \in \mathbb{R}^N} \sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}) + \alpha(\langle \mathbf{1},\ u\rangle - 1) + \beta(\langle \mathbf{1},\ u_{\mathcal{I}_0}\rangle - 1).$$

This gives, for $i_0 \in [N]$ and $i \in [N] \setminus \mathcal{I}_0$,

$$\begin{aligned} \partial_{i_0} L(u; \alpha, \beta) &= \log(u_{i_0}) + 1 + \alpha + \beta, \text{ and} \\ \partial_i L(u; \alpha, \beta) &= \alpha. \end{aligned}$$

Then, at the saddle point, $\alpha = 0$ and $\log u_{i_0} = -\frac{1}{1+\beta}$ for all $i_0 \in \mathcal{I}_0$.

If $\beta = 0$ then $u_{i_0} = \frac{1}{\exp(1)}$ for all $i_0 \in \mathcal{I}_0$. This is only feasible if $N_0 \leq 2$. In this case

$$\sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}) \geq -\sum_{i_0 \in \mathcal{I}_0} \frac{\log(\exp(1))}{\exp(1)} = -\frac{N_0}{\exp(1)}.$$

Otherwise $\beta > 0$, and by the K.K.T. condition, $\langle \mathbf{1},\ u_{\mathcal{I}_0}\rangle = 1$, which implies that $u_{i_0} = \frac{\mathbf{1}}{N_0}$ for all $i_0 \in \mathcal{I}_0$. That is,

$$\sum_{i_0 \in \mathcal{I}_0} u_{i_0} \log(u_{i_0}) \geq -\sum_{i_0 \in \mathcal{I}_0} \frac{\log(N_0)}{N_0} = -\log(N_0).$$

Thus for $N_0 \geq 3$

$$H(u) \leq \log(N_0) - \sum_{i \in [N] \setminus \mathcal{I}_0} u_i \log u_i, \tag{E.1}$$

and for $N_0 \leq 2$

$$H(u) \leq \frac{N_0}{\exp(1)} - \sum_{i \in [N] \setminus \mathcal{I}_0} u_i \log u_i. \tag{E.2}$$

Further, if $\mathcal{I}_0 = \{i_0\}$, since $\log(x) \geq 1 - 1/x$ for all $x \geq 0$,

$$\begin{aligned} H(u) &= -\sum_{i\in[N]} u_i \log(u_i) \\ &= -u_{i_0}\log(u_{i_0}) - \sum_{i\in[N]\setminus\mathcal{I}_0} u_i \log(u_i) \\ &\leq -u_{i_0}\left(1 - \frac{1}{u_{i_0}}\right) - \sum_{i\in[N]\setminus\mathcal{I}_0} u_i \log(u_i) \\ &= (1-u_{i_0}) - \sum_{i\in[N]\setminus\mathcal{I}_0} u_i \log(u_i) \\ &= \sum_{i\in[N]\setminus\mathcal{I}_0} u_i - \sum_{i\in[N]\setminus\mathcal{I}_0} u_i \log(u_i)\,. \end{aligned} \tag{E.3}$$

In order to control the sum over ineffective experts we use the technical result of Lemma E.4, which says that

$$-\sum_{i\in[N]\setminus\mathcal{I}_0} u_i \log(u_i) \leq \frac{1}{(1-p)\exp(1)} \sum_{i\in[N]\setminus\mathcal{I}_0} [u_i]^p \tag{E.4}$$

Combing Eqs. (E.1) to (E.4) gives for $N_0 \geq 1$,

$$H(u) \leq \frac{2}{\exp(1)\log(2)}\log(N_0) + \left(1 + \frac{1}{(1-p)\exp(1)}\right)\sum_{i\in[N]\setminus\mathcal{I}_0} u_i^p.$$

□

E.2. Proof of Lemma A.2. For the first result, observe that from Theorem A.1, $u(t+1)$ is the unique solution to

$$\begin{aligned} \eta(t+1) &= \frac{1}{\sqrt{t+1}\cdot\psi'\circ H(u(t+1))} \\ u(t+1) &= \left(\frac{\exp(-\eta(t+1)L_i(t))}{\sum_{i'\in[N]}\exp(-\eta(t+1)L_{i'}(t))}\right)_{i\in[N]}, \end{aligned}$$

and

$$\eta(t+1) \in \left[\frac{1}{\sqrt{t+1}\cdot[\psi'(0)]}, \frac{1}{\sqrt{t+1}\cdot[\psi'(\log N)]}\right].$$

Now, set $\overline{\eta}(t+1) = \frac{1}{\sqrt{t+1}\cdot\psi'(0)}$. For $i \in [N]\setminus\mathcal{I}_0$, since $\overline{\eta}(t+1) \leq \eta(t+1)$ and $L_{I^*(t)}(t) \leq L_i(t)$ by definition,

$$\begin{aligned} \Big[u_i(t+1)\Big]^p &\leq \left(\frac{u_i(t+1)}{u_{I^*(t)}(t+1)}\right)^p \\ &= \exp\Big\{-p\;\eta(t+1)\Big[L_i(t) - L_{I^*(t)}(t)\Big]\Big\} \\ &\leq \exp\Big\{-p\;\overline{\eta}(t+1)\Big[L_i(t) - L_{I^*(t)}(t)\Big]\Big\} \\ &\leq \min_{i_0\in\mathcal{I}_0}\exp\Big\{-p\;\overline{\eta}(t+1)\left[L_i(t) - L_{i_0}(t)\right]\Big\}. \end{aligned}$$

Thus, using Theorem 2,

$$
\begin{aligned}
\sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\Big[u_i(t+1)^p\Big] &\leq \sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\Big[\min_{i_0\in\mathcal{I}_0} \exp\Big\{-p\;\overline{\eta}(t+1)\,[L_i(t)-L_{i_0}(t)]\Big\}\Big] \\
&\leq \exp\left\{-t\overline{\eta}(t+1)\Delta_0 p + t\overline{\eta}(t+1)^2\frac{p^2}{2}\right\} \\
&= \exp\left\{-t\frac{1}{\sqrt{t+1}\cdot\psi'(0)}\Delta_0 p + t\left(\frac{1}{\sqrt{t+1}\cdot\psi'(0)}\right)^2\frac{p^2}{2}\right\} \\
&\leq \exp\left\{\frac{p^2}{2(\psi'(0))^2}\right\}\exp\left\{-\frac{\Delta_0 p}{\sqrt{2}(\psi'(0))}\sqrt{t}\right\},
\end{aligned}
$$

where in the last inequality we used the fact that $\frac{t}{t+1}\geq\frac{1}{2}$ for $t\in\mathbb{N}$.

For the second result, for each $\alpha\in[0,1]$, we define the *intermediate losses* $\xi^{(\alpha)}(t)=\alpha L(t)+(1-\alpha)\frac{\sqrt{t+1}}{\sqrt{t}}L(t-1)$. We define a new random expert by $\hat{I}^*_\alpha(t)=\arg\min_{i\in[N]}\xi_i^{(\alpha)}(t)$, which is analogous to $I^*(t)$ but for $\xi^{(\alpha)}(t)$. Then, applying Lemma D.2 to the intermediate losses, observe that $v^{(\alpha)}(t+1)$ is the unique solution to

$$
\begin{aligned}
\vartheta^{(\alpha)}(t+1) &= \frac{1}{\sqrt{t+1}\cdot\psi'\circ H(v^{(\alpha)}(t+1))} \\
v^{(\alpha)}(t+1) &= \left(\frac{\exp\left(-\vartheta^{(\alpha)}(t+1)\xi_i^{(\alpha)}(t)\right)}{\sum_{i'\in[N]}\exp\left(-\vartheta^{(\alpha)}(t+1)\xi_{i'}^{(\alpha)}(t)\right)}\right)_{i\in[N]},
\end{aligned}
$$

and

$$
\vartheta^{(\alpha)}(t+1)\in\left[\frac{1}{\sqrt{t+1}\cdot[\psi'(0)]},\ \frac{1}{\sqrt{t+1}\cdot[\psi'(\log N)]}\right].
$$

Next, using that $\ell_i(t)\in[0,1]$ for all $i\in[N]$,

$$
\xi_i^{(\alpha)}(t)=\alpha L_i(t)+(1-\alpha)\sqrt{\frac{t+1}{t}}L_i(t-1)\geq L_i(t)-1.
$$

Then, observe that since $L_i(t)\leq t$ for all $i\in[N]$, $\frac{\sqrt{t+1}}{\sqrt{t}}L_i(t-1)\leq L_i(t-1)+1$ for all $t\in\mathbb{N}$. Thus, for any $i'\in[N]$,

$$
\xi_{i'}^{(\alpha)}(t)=\alpha L_{i'}(t)+(1-\alpha)\sqrt{\frac{t+1}{t}}L_{i'}(t-1)\leq L_{i'}(t)+1.
$$

Combining these two facts gives that for all $\alpha\in[0,1]$,

$$
\xi_i^{(\alpha)}(t)-\xi_{i'}^{(\alpha)}(t)\geq L_i(t)-L_{i'}(t)-2.
$$

Now, for $i \in [N] \setminus \mathcal{I}_0$, taking $\overline{\eta}(t+1) = \frac{1}{\sqrt{t+1} \cdot \psi'(0)}$, and since $\overline{\eta}(t+1) \leq \vartheta(t+1)$ we have

$$\begin{aligned}\left[v_i^{(\alpha)}(t+1)\right]^p &\leq \left(\frac{v_i^{(\alpha)}(t+1)}{v_{\hat{I}_\alpha^*(t)}^{(\alpha)}(t+1)}\right)^p \\ &= \exp\left\{-p\ \vartheta^{(\alpha)}(t+1)\left[\xi_i^{(\alpha)}(t) - \xi_{\hat{I}_\alpha^*(t)}^{(\alpha)}(t)\right]\right\} \\ &\leq \exp\left\{-p\ \overline{\eta}(t+1)\left[\xi_i^{(\alpha)}(t) - \xi_{\hat{I}_\alpha^*(t)}^{(\alpha)}(t)\right]\right\} \\ &\leq \min_{i_0 \in \mathcal{I}_0} \exp\left\{-p\ \overline{\eta}(t+1)\left[L_i(t) - L_{i_0}(t) - 2\right]\right\}.\end{aligned}$$

Thus, again using Theorem 2,

$$\begin{aligned}\sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \sup_{\alpha \in [0,1]} \left[v_i(t+1)^p\right] &\leq \sup_{\pi \in \mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}}\left[\min_{i_0 \in \mathcal{I}_0} \exp\left\{-p\ \overline{\eta}(t+1)\left[L_i(t) - L_{i_0}(t) - 2\right]\right\}\right] \\ &\leq \exp\left\{2p\overline{\eta}(t+1) - t\overline{\eta}(t+1)\Delta_0 p + t\overline{\eta}(t+1)^2 \frac{p^2}{2}\right\} \\ &= \exp\left\{\frac{2p - \Delta_0 pt}{\sqrt{t+1} \cdot \psi'(0)} + \left(\frac{1}{\sqrt{t+1} \cdot \psi'(0)}\right)^2 \frac{p^2 t}{2}\right\} \\ &\leq \exp\left\{\frac{2p}{\psi'(0)} + \frac{p^2}{2(\psi'(0))^2}\right\} \exp\left\{-\frac{\Delta_0 p}{\sqrt{2}(\psi'(0))}\sqrt{t}\right\},\end{aligned}$$

where in the last inequality we again used the fact that $\frac{t}{t+1} \geq \frac{1}{2}$ for $t \in \mathbb{N}$. □

E.3. Proof of Lemma A.3. Substituting the variance bounds of Lemma E.1 Eq. (A.1) using $\beta(t) = \sqrt{t+1}$, ψ increasing and concave, and the fact that $H(u) \leq \log(N)$ gives

$$\begin{aligned}\hat{R}_{\text{FTRL}_H}(t_0) &\leq -\psi(0)\sqrt{t_0+1} + \sum_{t=0}^{t_0}\left[\sqrt{t+1} - \sqrt{t}\right]\psi(\log(N)) + \sum_{t=1}^{t_0} \frac{3}{8\sqrt{t+1} \cdot \psi'(\log(N))}. \\ &= \sqrt{t_0+1}\Big(\psi(\log(N)) - \psi(0)\Big) + \sum_{t=1}^{t_0} \frac{3}{8\sqrt{t+1} \cdot \psi'(\log(N))}.\end{aligned}$$

Then, since

$$\sum_{t=1}^{t_0} \frac{1}{\sqrt{t+1}} \leq \int_0^{t_0} \frac{1}{\sqrt{t+1}} dt = 2\sqrt{t_0+1},$$

we have that

$$\hat{R}_{\text{FTRL}_H}(t_0) \leq \sqrt{t_0+1}\left(\psi(\log(N)) - \psi(0) + \frac{3}{4\psi'(\log(N))}\right).$$

□

E.4. Miscellaneous stochastic and mathematical results. Here we state a few convenient results that will be used repeatedly, but require none of the assumptions of our setting except boundedness. The first two of these lemmas allow us to control the variance of the experts' losses.

LEMMA E.1. *For any* $w \in \texttt{simp}([N])$, $(\ell(t))_{t\in\mathbb{N}} \subseteq [0,1]^N$, *and* $t \in \mathbb{N}$,

$$\operatorname*{Var}_{I\sim w}\left[\left(\sqrt{\frac{t+1}{t}}-1\right)L_I(t-1)-\ell_I(t)\right] \le \frac{9}{16} \quad \textit{and} \quad \operatorname*{Var}_{I\sim w}[\ell_I(t)] \le \frac{1}{4}.$$

PROOF OF LEMMA E.1. Since $\sqrt{t+1}-\sqrt{t} \le \frac{1}{2\sqrt{t}}$ for $t \ge 1$, $\left(\sqrt{\frac{t+1}{t}}-1\right) \in [0, \frac{1}{2t}]$. Combined with $\ell(t) \in [0,1]^{[N]}$ for all $t \in \mathbb{N}$, this gives that for all $i \in [N]$,

$$\left(\sqrt{\frac{t+1}{t}}-1\right)L_i(t-1)-\ell_i(t) \in \left[-1,\ \frac{1}{2}\right].$$

Thus, the result follows since if $a \le X \le b$, then $\operatorname{Var}(X) \le (b-a)^2/4$. □

LEMMA E.2. *For any* $\pi \in \mathscr{P}$, $\hat{\pi} \in \hat{\mathscr{P}}$, *sequence* $(w(t))_{t\in\mathbb{N}}$ *such that* $w(t)$ *is* $\sigma(h(t-1))$*-measurable for all* t, $i_0 \in [N]$, *and* $t \in \mathbb{N}$,

$$\mathbb{E}_{\pi,\hat{\pi}}\sqrt{\operatorname*{Var}_{I\sim w(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim w(t+1)}[\ell_I(t)]}$$
$$\le \frac{9}{4}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i\ne i_0} w_i(t+1)\right]$$

and

$$\mathbb{E}_{\pi,\hat{\pi}}\left[\operatorname*{Var}_{I\sim w(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}}-1\right)L_I(t-1)-\ell_I(t)\right]\operatorname*{Var}_{I\sim w(t+1)}[\ell_I(t)]\right]$$
$$\le \frac{27}{32}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i\ne i_0} w_i(t+1)\right].$$

PROOF OF LEMMA E.2. First, let ν be any distribution such that $\operatorname{Support}(\nu) \subset [-y, 1-y]$ and $x \in [-y, 1-y]$, and suppose $X \sim \alpha\delta_x + (1-\alpha)\nu$ for some $\alpha \in [0,1]$.

Since variance is invariant to shifts, we can suppose $y = 0$ without loss of generality. Define $\mu_\nu = \mathbb{E}_{Z\sim\nu}(Z)$ and $\sigma_\nu^2 = \operatorname{Var}_{Z\sim\nu}(Z)$. Then, using the variance for a mixture distribution,

$$\begin{aligned}\operatorname{Var}(X) &= \alpha x^2 + (1-\alpha)\mu_\nu^2 - (\alpha x + (1-\alpha)\mu_\nu)^2 + (1-\alpha)\sigma_\nu^2 \\ &= \alpha(1-\alpha)x^2 + \alpha(1-\alpha)\mu_\nu^2 - 2\alpha(1-\alpha)x\mu_\nu + (1-\alpha)\sigma_\nu^2 \\ &= \alpha(1-\alpha)(x-\mu_\nu)^2 + (1-\alpha)\sigma_\nu^2\end{aligned}$$

Now,

$$\sup_{x,\nu}\operatorname{Var}(X) = \sup_{x,\mu}\ \sup_{\nu:\mu_\nu=\mu}\ \operatorname{Var}(X).$$

The inner sup is achieved by $\nu(\mu) = \operatorname{Ber}(\mu)$ and has $\sigma^2_{\nu(\mu)} = \mu(1-\mu)$, so that

$$\sup_{x,\nu}\operatorname{Var}(X) = \sup_{\mu}\sup_{x}\alpha(1-\alpha)(x-\mu)^2 + (1-\alpha)\mu(1-\mu).$$

Now, the inner sup is achieved by $x = 0$ when $\mu \geq 1/2$ and by $x = 1$ when $\mu < 1/2$. Due to symmetry we need only consider the case that $\mu \geq 1/2$.

$$\begin{aligned}\sup_{x,\nu} \operatorname{Var}(X) &= \sup_{\mu} \alpha(1-\alpha)\mu^2 + (1-\alpha)\mu(1-\mu) \\ &= \sup_{\mu} \Big[-(1-\alpha)^2\mu^2 + (1-\alpha)\mu\Big].\end{aligned}$$

Since $\mu \in [0,1]$ this is a constrained quadratic maximum. If the unconstrained maximum occurs in interior of the region then it is equal to the constrained maximum. Otherwise the constrained maximum occurs at the boundary.

The unconstrained maximum occurs at $\mu = \frac{1}{2(1-\alpha)}$ with objective value 1/4. This in the interior of the constraint region when $(1-\alpha) > 1/2$; equivalently $\alpha < 1/2$. The boundary values are 0 and $\alpha(1-\alpha)$.

That is,

$$\operatorname{Var}(X) \leq \begin{cases} \alpha(1-\alpha) & : \alpha \geq 1/2 \\ 1/4 & : \alpha < 1/2 \end{cases}. \tag{E.5}$$

Let $w \in \texttt{simp}([N])$ be arbitrary. We can apply Eq. (E.5) to obtain

$$\operatorname*{Var}_{I\sim w}[\ell_I(t)] \leq \frac{1}{4}\mathbb{I}_{[w_{i_0} \leq 1/2]} + (1 - w_{i_0}).$$

Similarly, since $\left(\sqrt{\frac{t+1}{t}} - 1\right) L_I(t-1) - \ell_I(t) \in [-1, \frac{1}{2}]$,

$$\begin{aligned}&\operatorname*{Var}_{I\sim w}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \\ &\qquad \leq \left(\frac{3}{2}\right)^2 \left(\frac{1}{4}\mathbb{I}_{[w_{i_0} \leq 1/2]} + (1 - w_{i_0})\right).\end{aligned}$$

Thus, using Markov's inequality,

$$\begin{aligned}&\mathbb{E}_{\pi,\hat{\pi}} \sqrt{\operatorname*{Var}_{I\sim w(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \operatorname*{Var}_{I\sim w(t+1)}[\ell_I(t)]} \\ &\quad \leq \frac{3}{2}\left(\frac{1}{4}\mathbb{P}_{\pi,\hat{\pi}}[w_{i_0}(t+1) \leq 1/2] + \mathbb{E}_{\pi,\hat{\pi}}[1 - w_{i_0}(t+1)]\right) \\ &\quad \leq \frac{9}{4}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i \neq i_0} w_i(t+1)\right].\end{aligned}$$

Alternatively, using $\operatorname*{Var}_{I\sim w}[\ell_I(t)] \leq 1/4$,

$$\begin{aligned}&\mathbb{E}_{\pi,\hat{\pi}}\left[\operatorname*{Var}_{I\sim w(t+1)}\left[\left(\frac{\sqrt{t+1}}{\sqrt{t}} - 1\right) L_I(t-1) - \ell_I(t)\right] \operatorname*{Var}_{I\sim w(t+1)}[\ell_I(t)]\right] \\ &\quad \leq \frac{9}{16}\left(\frac{1}{4}\mathbb{P}_{\pi,\hat{\pi}}[w_{i_0}(t+1) \leq 1/2] + \mathbb{E}_{\pi,\hat{\pi}}[1 - w_{i_0}(t+1)]\right) \\ &\quad \leq \frac{27}{32}\mathbb{E}_{\pi,\hat{\pi}}\left[\sum_{i \neq i_0} w_i(t+1)\right].\end{aligned}$$

□

Next, we have a result which controls a summation term which appears often in our proofs.

LEMMA E.3. *For any* $\alpha > 0$ *and* $t_0 \geq 1$

$$\sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t}} \exp\left\{-\alpha\sqrt{t}\right\} \leq \frac{2}{\alpha}\exp(-\alpha\sqrt{t_0}).$$

PROOF OF LEMMA E.3.

$$\begin{aligned}\sum_{t=t_0+1}^{T} \frac{1}{\sqrt{t}} \exp\left\{-\alpha\sqrt{t}\right\} &\leq \int_{t_0}^{T} \frac{1}{\sqrt{t}} \exp\left\{-\alpha\sqrt{t}\right\} dt \\ &\leq \int_{t_0}^{\infty} \frac{1}{\sqrt{t}} \exp\left\{-\alpha\sqrt{t}\right\} dt \\ &= \int_{\sqrt{t_0}}^{\infty} 2\exp\left\{-\alpha u\right\} du \\ &= \frac{2}{\alpha}\exp(-\alpha\sqrt{t_0}).\end{aligned}$$

□

Finally, we have a simple fact about logarithms that will be useful when controlling the entropy of weight distributions.

LEMMA E.4. *For* $x \in (0,1]$ *and* $p \in (0,1)$

$$-x\log(x) \leq \frac{1}{(1-p)\exp(1)} x^p.$$

PROOF OF LEMMA E.4. Consider $f(x) = -x^{1-p}\log(x)$. Then, $f(0^+) = f(0)$, $f(1) = 0$, and

$$f'(x) = -(1-p)x^{-p}\log(x) - x^{-p} = -x^{-p}((1-p)\log(x) + 1).$$

Thus, the only critical point of f occurs at $x_0 = \exp(-1/(1-p))$. This is a local max since $\text{sign}(f'(x)) = -\text{sign}(x - x_0)$ for $x \in (0,1)$. Thus, f is maximized on the interval $(0,1)$ at x_0. Hence $f(x) \leq f(x_0) = \frac{1}{(1-p)\exp(1)}$. Multiplying both sides by x^p proves the result. □

F: PROOFS OF LOWER BOUNDS

F.1. Proof of Theorem 3. Our strategy is to define a simple setting with multiple experts (many of them identical), so that we can show the lower bound holds in the asymptotic limit as T and N tends to infinity. Let $\mathcal{Y} = \{(1,0,0),(0,1,0),(0,0,1)\}$, $\hat{\mathcal{Y}} = \texttt{simp}([3])$, and $\ell(\hat{y}, y) = \frac{1}{2}\sum_{i=1}^{3} |\hat{y}_i - y_i|$. Observe that $\ell(\hat{y}, y) \in [0,1]$ for all $\hat{y} \in \hat{\mathcal{Y}}$ and $y \in \mathcal{Y}$. Let $N_0 \leq N \in \mathbb{N}$.

In this setting, consider the distribution

$$\mu_0 = \left(\left(\frac{1}{2}\delta_{(1,0,0)} + \frac{1}{2}\delta_{(0,1,0)}\right)^{\otimes N_0} \otimes \left(\delta_{(0,0,1)}\right)^{\otimes (N-N_0)}\right) \otimes \left(\frac{1}{2}\delta_{(1,0,0)} + \frac{1}{2}\delta_{(0,1,0)}\right),$$

and let $\mathcal{D} = \{\mu_0\}$. Then $\mathscr{P}(\mathcal{D})$ contains a single policy, $\pi_\star$, given by

$$\pi_\star = (h(t) \in \mathcal{H}^t \mapsto \mu_0)_{t\in\mathbb{N}}.$$

Intuitively, each of the effective experts flips a coin to play the first or second element, but the observation is also either the first or second element from an independent coin toss, and the ineffective experts always output the third element.

Now, define the pushforward of the distribution through the loss function by $\mu_0^\ell = \ell_\sharp \mu_0$ to obtain the single loss distribution on the experts. Observe this simplifies to

$$\mu_0^\ell = \mathrm{Ber}(1/2)^{\otimes N_0} \otimes \mathrm{Ber}(1)^{\otimes (N-N_0)}.$$

This singleton policy space satisfies the time-homogeneous convex constraint condition with $\mathcal{I}_0 = [N_0]$, and $\Delta_0 = 1/2$.

Note that any prediction $\hat{y}$ has $\underset{y\sim\mu_0}{\mathbb{E}} \ell(\hat{y}, y) \geq \frac{1}{2}$. For each $i_0 \in \mathcal{I}_0$, let $M_{i_0} = \sum_{t=1}^T \ell(\hat{y}_{i_0}, y)$ be the random variable corresponding to the cumulative loss of the effective expert. Then, $M_{i_0} \overset{\text{iid}}{\sim} \mathrm{Bin}(T, 1/2)$, and

$$\begin{aligned}\inf_{\hat{\pi}\in\hat{\mathscr{P}}_N} \sup_{\pi\in\mathscr{P}(\mathcal{D})} \mathbb{E}_{\pi,\hat{\pi}} \max_{i\in[N]} \sum_{t=1}^T [\ell(\hat{y}(t), y(t)) - \ell(x_i(t), y(t))] \\ \geq \underset{M\sim\mathrm{Bin}(T,1/2)^{\otimes N_0}}{\mathbb{E}} \max_{i_0\in\mathcal{I}_0} \Big(T/2 - M_{i_0}\Big).\end{aligned}$$

Now, since $\frac{2}{\sqrt{T}}(T/2 - M_{i_0})$ are i.i.d. and converge in Wasserstein distance to a $\mathrm{N}(0,1)$ as $T \to \infty$ (from, for example, [4, Theorem 3.1]), and since max is Lipschitz,

$$\lim_{T\to\infty} \underset{M\sim\mathrm{Bin}(T,1/2)^{\otimes N_0}}{\mathbb{E}} \left(\max_{i_0\in\mathcal{I}_0} \frac{1}{\sqrt{T}}(T/2 - M_{i_0})\right) = \frac{1}{2} \underset{Z\sim\mathrm{N}(0,1)^{\otimes N_0}}{\mathbb{E}} \left(\max_{i_0\in\mathcal{I}_0} Z_{i_0}\right).$$

We now turn to the non-asymptotic lower bound of Kamath [8], which states that for all $N_0 \in \mathbb{N}$

$$\frac{\underset{Z\sim\mathrm{N}(0,1)^{\otimes N_0}}{\mathbb{E}} \left(\max_{i_0\in\mathcal{I}_0} Z_{i_0}\right)}{0.23\sqrt{\log N_0}} \geq 1,$$

Now, by the definition of limit, for each N_0 there exists a $t_0(N_0)$ such that for $T \geq t_0(N_0)$

$$\begin{aligned}\underset{M\sim\mathrm{Bin}(T,1/2)^{\otimes N_0}}{\mathbb{E}} \left(\max_{i_0\in\mathcal{I}_0} \frac{1}{\sqrt{T}}(T/2 - M_{i_0})\right) &\geq \left(\frac{0.2}{0.23}\right)\left(\frac{1}{2}\right) \underset{Z\sim\mathrm{N}(0,1)^{\otimes N_0}}{\mathbb{E}} \left(\max_{i_0\in\mathcal{I}_0} Z_{i_0}\right) \\ &\geq \sqrt{(\log N_0)/100}.\end{aligned}$$

Combining these facts, we have that for any $N \in \mathbb{N}$, $N_0 \leq N$ and $T \geq t_0(N_0)$,

$$\inf_{\hat{\pi}\in\hat{\mathscr{P}}_N} \sup_{\pi\in\mathscr{P}(\mathcal{D})} \frac{\mathbb{E}_{\pi,\hat{\pi}} \max_{i\in[N]} \sum_{t=1}^T [\ell(\hat{y}(t), y(t)) - \ell(x_i(t), y(t))]}{\sqrt{(T\log N_0)/100}} \geq 1.$$

□

F.2. Proof of Theorem 4. Fix $N > 0$, $N_0 \leq N$, and $c > 0$ within the respective constraints of either (i) or (ii) of Theorem 4. Let $\mathcal{Y} = \{0,1\}^N$, $\hat{\mathcal{Y}} = [0,1]^N$, and $\ell(\hat{y}, y) = \langle \hat{y},\ y\rangle$, and suppose $T \geq \frac{32\log N}{c^2}$. In order to prove both cases of the D.Hedge lower bound, our approach is first to define a specific example of a $\mathcal{D} \in \mathcal{V}(,(,N), N_0)$. Then, for either case we find a specific policy $\pi \in \mathscr{P}(\mathcal{D})$ which forces D.Hedge to incur at least as much regret as the desired lower bound. It turns out that we do not need anything more complicated than a $\mathcal{D}$ that consists of convex combinations of deterministic experts.

For simplicity, suppose that N_0 is even. (The argument is the same, but with some more housekeeping, when N_0 is odd.) We wish to split $[N]$ up so that $\mathcal{I}_0 = [N_0]$, and thus $[N] \setminus \mathcal{I}_0 = [N] \setminus [N_0]$. To do so, we define a set of distributions on $\mathcal{Y}$ by

$$U = \left\{ \delta_m^{\otimes N_0/2} \otimes \delta_{1-m}^{\otimes N_0/2} \otimes \delta_1^{\otimes (N-N_0)} \text{ s.t. } m \in \{0,1\} \right\} \cup \left\{ \delta_0 \otimes \delta_1^{\otimes (N-1)} \right\},$$

and suppose that each expert $i \in [N]$ predicts $(x)_i = e_i$, the unit vector in direction i. Thus, the set U induces three different expert loss distributions. In each of these, the incurred loss of any expert is assigned either a Dirac measure at 0 or at 1. Thus, the three distributions are defined by which experts incur loss of 0 (with the rest incurring loss of 1). These options are either: a) the first $N_0/2$ incur loss of 0, b) the experts labelled $N_0/2+1$ to N_0 incur loss of 0, and c) only the first expert incurs loss of 0.

Then, we define $\mathcal{D}$ to be the convex hull of U. One can check that any convex combination of the three distributions in U can only lead to an expert in $\mathcal{I}_0$ being optimal in expectation, and additionally note that $\Delta_0 = 1/2$. Consequently, $\mathcal{D} \in \mathcal{V}(,(,N),N_0)$, so it remains to find a $\pi \in \mathscr{P}(\mathcal{D})$ that forces D.Hedge with either parametrization to incur the regret of the theorem.

Before we do this, we first recall the adversarial analysis of D.Hedge by [3, Theorem 2.3]. Similar to that analysis, we will analyze the telescoping series

$$\Psi(t) = \frac{1}{\eta(t+1)} \log(w^{\mathrm{H}}_{I^*(t)}(t+1)) - \frac{1}{\eta(t)} \log(w^{\mathrm{H}}_{I^*(t-1)}(t)),$$

which, for an arbitrary t_0, satisfies

$$\sum_{t=t_0+1}^{T} \Psi(t) = \frac{1}{\eta(T+1)} \log(w^{\mathrm{H}}_{I^*(T)}(T+1)) - \frac{1}{\eta(t_0+1)} \log(w^{\mathrm{H}}_{I^*(t_0)}(t_0+1)).$$

When upper bounding, Cesa-Bianchi and Lugosi used that the first term was negative and kept the second term, but we now wish to use that the second term is positive to obtain

$$\sum_{t=t_0+1}^{T} \Psi(t) \geq \frac{1}{\eta(T+1)} \log(w^{\mathrm{H}}_{I^*(T)}(T+1)). \tag{F.1}$$

Then, we can partition $-\Psi(t)$ into

$$-\Psi(t) = \left(\frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log\left(\frac{1}{w^{\mathrm{H}}_{I^*(t)}(t+1)} \right) + \frac{1}{\eta(t)} \log \left(\frac{\frac{\exp\left\{-\eta(t) L_{I^*(t)}(t)\right\}}{\sum_{i\in[N]} \exp\{-\eta(t) L_i(t)\}}}{\frac{\exp\left\{-\eta(t+1) L_{I^*(t)}(t)\right\}}{\sum_{i\in[N]} \exp\{-\eta(t+1) L_i(t)\}}} \right)$$
$$+ \frac{1}{\eta(t)} \log \left(\frac{\sum_{i\in[N]} \exp\{-\eta(t) L_i(t)\}}{\sum_{i\in[N]} \exp\{-\eta(t) L_i(t-1)\}} \right) + \left[L_{I^*(t)}(t) - L_{I^*(t-1)}(t-1) \right].$$

Observe that

$$\begin{aligned}
&\frac{1}{\eta(t)}\log\left(\frac{\sum_{i\in[N]}\exp\{-\eta(t)L_i(t)\}}{\sum_{i\in[N]}\exp\{-\eta(t)L_i(t-1)\}}\right)\\
&\qquad=\frac{1}{\eta(t)}\log\left(\sum_{i\in[N]}\frac{\exp\{-\eta(t)L_i(t-1)\}}{\sum_{i'\in[N]}\exp\{-\eta(t)L_{i'}(t-1)\}}\exp\{-\eta(t)\ell_i(t)\}\right)\\
&\qquad=\frac{1}{\eta(t)}\log\left(\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\exp\{-\eta(t)\ell_i(t)\}\right)\\
&\qquad=-\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\ell_i(t)\\
&\qquad\qquad+\frac{1}{\eta(t)}\log\left(\exp\left\{\eta(t)\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\ell_i(t)\right\}\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\exp\{-\eta(t)\ell_i(t)\}\right)\\
&\qquad=-\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\ell_i(t)\\
&\qquad\qquad+\frac{1}{\eta(t)}\log\left(\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\exp\left\{\eta(t)\left[-\ell_i(t)+\sum_{i'\in[N]}w_{i'}^{\mathrm{H}}(t)\ell_{i'}(t)\right]\right\}\right).
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
-\Psi(t)&=\left(\frac{1}{\eta(t+1)}-\frac{1}{\eta(t)}\right)\log\left(\frac{1}{w_{I^*(t)}^{\mathrm{H}}(t+1)}\right)+\frac{1}{\eta(t)}\log\left(\frac{\frac{\exp\{-\eta(t)L_{I^*(t)}(t)\}}{\sum_{i\in[N]}\exp\{-\eta(t)L_i(t)\}}}{\frac{\exp\{-\eta(t+1)L_{I^*(t)}(t)\}}{\sum_{i\in[N]}\exp\{-\eta(t+1)L_i(t)\}}}\right)\\
&\qquad-\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\ell_i(t)+\frac{1}{\eta(t)}\log\left(\mathop{\mathbb{E}}_{I\sim w^{\mathrm{H}}(t)}\exp\left\{\eta(t)\left(-\ell_I(t)-\mathop{\mathbb{E}}_{I'\sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right)\\
&\qquad+\left[L_{I^*(t)}(t)-L_{I^*(t-1)}(t-1)\right]\\
&=A(t)+B(t)+C_1(t)+C_2(t)+D(t).
\end{aligned}$$

First, observe that since $\eta(t)$ is decreasing in both cases, $B(t)\geq 0$. Also,

$$\begin{aligned}
\sum_{t=t_0+1}^{T}A(t)&=\sum_{t=t_0+1}^{T}\left(\frac{1}{\eta(t+1)}-\frac{1}{\eta(t)}\right)\log\left(\frac{1}{w_{I^*(t)}^{\mathrm{H}}(t+1)}\right),\\
\sum_{t=t_0+1}^{T}C_1(t)&=-\sum_{t=t_0+1}^{T}\sum_{i\in[N]}w_i^{\mathrm{H}}(t)\ell_i(t),\\
\sum_{t=t_0+1}^{T}C_2(t)&=\sum_{t=t_0+1}^{T}\frac{1}{\eta(t)}\log\left(\mathop{\mathbb{E}}_{I\sim w^{\mathrm{H}}(t)}\exp\left\{\eta(t)\left(-\ell_I(t)-\mathop{\mathbb{E}}_{I'\sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right),\ \text{and}\\
\sum_{t=t_0+1}^{T}D(t)&=L_{I^*(T)}(T)-L_{I^*(t_0)}(t_0).
\end{aligned}$$

Thus, combining these with Eq. (F.1) gives

$$
\begin{aligned}
&-\frac{1}{\eta(T+1)}\log(w^{\mathrm{H}}_{I^*(T)}(T+1)) \\
&\quad\geq -\sum_{t=t_0+1}^{T}\Psi(t) \\
&\quad\geq \sum_{t=t_0+1}^{T}\left(\frac{1}{\eta(t+1)}-\frac{1}{\eta(t)}\right)\log\left(\frac{1}{w^{\mathrm{H}}_{I^*(t)}(t+1)}\right)-\sum_{t=t_0+1}^{T}\sum_{i\in[N]}w^{\mathrm{H}}_i(t)\ell_i(t) \\
&\qquad+\sum_{t=t_0+1}^{T}\frac{1}{\eta(t)}\log\left(\mathop{\mathbb{E}}_{I\sim w^{\mathrm{H}}(t)}\exp\left\{\eta(t)\left(-\ell_I(t)-\mathop{\mathbb{E}}_{I'\sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right) \\
&\qquad+L_{I^*(T)}(T)-L_{I^*(t_0)}(t_0).
\end{aligned}
$$

Rearranging, we see that

$$
\begin{aligned}
&\hat{R}_{\mathrm{H}}(T)-\hat{R}_{\mathrm{H}}(t_0) \\
&\quad\geq \frac{1}{\eta(T+1)}\log(w^{\mathrm{H}}_{I^*(T)}(T+1))+\sum_{t=t_0+1}^{T}\left(\frac{1}{\eta(t+1)}-\frac{1}{\eta(t)}\right)\log\left(\frac{1}{w^{\mathrm{H}}_{I^*(t)}(t+1)}\right) \\
&\qquad+\sum_{t=t_0+1}^{T}\frac{1}{\eta(t)}\log\left(\mathop{\mathbb{E}}_{I\sim w^{\mathrm{H}}(t)}\exp\left\{\eta(t)\left(-\ell_I(t)-\mathop{\mathbb{E}}_{I'\sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right).
\end{aligned}
\tag{F.2}
$$

The way we bound these terms will depend on the specific parametrization and data-generating mechanism chosen for that parametrization.

F.2.1. D.Hedge *with adversarially optimal parametrization.* First, we consider the case of playing D.Hedge with $g(N)=c\sqrt{\log N}$. We define the data-generating mechanism $\pi\in\mathscr{P}(\mathcal{D})$ such that at round t, the distribution on $\mathcal{Y}$ is

$$
\mu_t=\begin{cases}\delta_0^{\otimes(N_0/2)}\otimes\delta_1^{\otimes(N-N_0/2)} & : t \text{ odd}\\ \delta_1^{\otimes(N_0/2)}\otimes\delta_0^{\otimes(N_0/2)}\otimes\delta_1^{\otimes(N-N_0)} & : t\text{ even.}\end{cases}
$$

That is, on even and odd rounds the data alternates between the first half of $\mathcal{I}_0$ incurring loss of 0 and the second half of $\mathcal{I}_0$ incurring loss of 0, with the remaining $N-N_0$ experts always incurring loss of 1. Both of these distributions are actually in U, so they are trivially in $\mathcal{D}$.

Now, due to the deterministic nature of π, we can exactly determine what $w^{\mathrm{H}}(t)$ will look like. In particular, we have that

$$
L_i(t)=\begin{cases}\frac{t-1}{2} & : t\text{ odd, and } i\in[N_0/2]\\ \frac{t+1}{2} & : t\text{ odd, and } i\in[N_0]\setminus[N_0/2]\\ \frac{t}{2} & : t\text{ even, and } i\in[N_0]\\ t & : i\notin[N_0]\,.\end{cases}
\tag{F.3}
$$

Thus, recognizing that $w^{\text{H}}_i(t)$ uses $L_i(t-1)$ and letting $\theta(t) = \exp\left\{-\eta(t)\frac{(t-1)}{2}\right\}$, we can define $w^{\text{H}}_i(t)$ by

$$\begin{cases} [N_0 + (N-N_0)\theta(t)]^{-1} & : t \text{ odd, and } i \in [N_0] \\ \theta(t)[N_0 + (N-N_0)\theta(t)]^{-1} & : t \text{ odd, and } i \notin [N_0] \\ \exp(\eta(t)/2)[N_0 \cosh(\eta(t)/2) + (N-N_0)\theta(t)]^{-1} & : t \text{ even, and } i \in [N_0/2] \\ \exp(-\eta(t)/2)[N_0 \cosh(\eta(t)/2) + (N-N_0)\theta(t)]^{-1} & : t \text{ even, and } i \in [N_0] \setminus [N_0/2] \\ \theta(t)[N_0 \cosh(\eta(t)/2) + (N-N_0)\theta(t)]^{-1} & : t \text{ even, and } i \notin [N_0]. \end{cases}$$

The next thing to observe is that for all t, $I^*(t) \in [N_0]$ a.s., and $w^{\text{H}}_{I^*(t)}(t+1)$ equals

$$\begin{cases} [N_0 + (N-N_0)\theta(t+1)]^{-1} & : t+1 \text{ odd} \\ \exp(\eta(t+1)/2)[N_0 \cosh(\eta(t+1)/2) + (N-N_0)\theta(t+1)]^{-1} & : t+1 \text{ even.} \end{cases} \tag{F.4}$$

Now, let $t_0 = \left\lfloor \frac{16 \log N}{c^2} \right\rfloor$ and suppose $t \geq t_0 + 1$. Then, using $\frac{x}{\sqrt{x+1}} \geq \frac{1}{2}\sqrt{x}$ for $x \geq 1$,

$$\begin{aligned} \theta(t) &\leq \theta(t+1) \\ &= \exp\left\{-\frac{c\sqrt{\log N}}{\sqrt{t+1}}\frac{t}{2}\right\} \\ &\leq \exp\left\{-\frac{c\sqrt{t \log N}}{4}\right\} \\ &\leq \exp\left\{-\frac{c\sqrt{16(\log N)^2}}{4c}\right\} \\ &= \frac{1}{N}. \end{aligned}$$

This gives

$$\frac{1}{N_0 + (N-N_0)\theta(t+1)} \geq \frac{1}{N_0 + (N-N_0)/N} \geq \frac{1}{N_0+1}. \tag{F.5}$$

Also, $\exp\{\eta(t+1)/2\} \geq 1$, so

$$\begin{aligned} \cosh\left(\frac{\eta(t+1)}{2}\right) &= \frac{1}{2}\left[\exp\left\{\frac{c\sqrt{\log N}}{2\sqrt{t+1}}\right\} + \exp\left\{-\frac{c\sqrt{\log N}}{2\sqrt{t+1}}\right\}\right] \\ &\leq \frac{1}{2}\left[\exp\left\{\frac{c^2\sqrt{\log N}}{2\sqrt{16\log N}}\right\} + 1\right] \\ &\leq \exp\left\{c^2/8\right\}. \end{aligned}$$

Thus,

$$\frac{\exp\{\eta(t+1)/2\}}{N_0 \cosh(\eta(t+1)/2) + (N-N_0)\theta(t+1)} \geq \frac{1}{\exp\{c^2/8\} N_0 + 1},$$

which combined with Eq. (F.5) gives that for all $t \geq t_0 + 1$,

$$w^{\text{H}}_{I^*(t)}(t+1) \geq \frac{1}{\exp\{c^2/8\} N_0 + 1}.$$

This observation shows that if $T \geq t_0 + 1$,

$$\frac{1}{\eta(T+1)} \log(w^{\text{H}}_{I^*(T)}(T+1)) \geq -\frac{\sqrt{T+1}}{c\sqrt{\log N}} \Big[c^2/8 + \log(N_0+1)\Big]. \tag{F.6}$$

In order to control the terms of Eq. (F.2), we first observe that

$$\sum_{t=t_0+1}^{T} \left(\frac{1}{\eta(t+1)} - \frac{1}{\eta(t)}\right) \log\left(\frac{1}{w^{\text{H}}_{I^*(t)}(t+1)}\right) \geq 0.$$

Then, we will use Eq. (F.6) to lower bound the first term on the RHS of Eq. (F.2). We now turn to controlling the third term, again supposing $t \geq t_0 + 1$. Notice that if t is odd, then $I \sim w^{\text{H}}(t)$ means $\ell_I(t) \sim \text{Ber}\left(\frac{N_0/2+(N-N_0)\theta(t)}{N_0+(N-N_0)\theta(t)}\right)$. Therefore,

$$\begin{aligned}
&\frac{1}{\eta(t)} \log\left(\mathop{\mathbb{E}}_{I\sim w^{\text{H}}(t)} \exp\left\{\eta(t)\left(-\ell_I(t) - \mathop{\mathbb{E}}_{I'\sim w^{\text{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right) \\
&\quad = \frac{1}{\eta(t)} \log\Bigg(\frac{N_0/2}{N_0+(N-N_0)\theta(t)} \exp\left\{\eta(t)\frac{N_0/2+(N-N_0)\theta(t)}{N_0+(N-N_0)\theta(t)}\right\} \\
&\qquad\qquad + \frac{N_0/2+(N-N_0)\theta(t)}{N_0+(N-N_0)\theta(t)} \exp\left\{-\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right\}\Bigg) \\
&\quad \geq \frac{1}{\eta(t)} \log\Bigg(\frac{N_0/2}{N_0+(N-N_0)\theta(t)} \exp\left\{\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right\} \\
&\qquad\qquad + \frac{N_0/2}{N_0+(N-N_0)\theta(t)} \exp\left\{-\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right\}\Bigg) \\
&\quad = \frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0+(N-N_0)\theta(t)} \cosh\left\{\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right\}\right).
\end{aligned}$$

Now, we observe that $\log(\cosh(x))$ is $\frac{1}{\cosh^2(x_1)}$-strongly convex on $x \in [0, x_1]$. Thus,

$$\log(\cosh(x)) - \log(\cosh(0)) \geq \frac{d}{dy}\log(\cosh(y))\Big|_{y=0} + \frac{x^2}{2\cosh^2(x_1)},$$

so $\cosh(x) \geq \exp\left\{\frac{x^2}{2\cosh^2(x_1)}\right\}$ on this interval. Then, notice that if $t \geq t_0+1$, $\eta(t) \leq c^2/4$. So, $\theta(t) \geq 0$ gives

$$\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)} \leq \frac{\eta(t)}{2} \leq \frac{c^2}{8}.$$

Using this strong-convexity bound on $\cosh(x)$ along with the two inequalities $\theta(t) \leq 1/N$ and $[2\cosh^2(c^2/8)]^{-1} \geq (1/2)\exp\{-c^2/4\}$ results in

$$\begin{aligned}
&\frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0+(N-N_0)\theta(t)} \cosh\left\{\eta(t)\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right\}\right) \\
&\quad \geq \frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0+(N-N_0)\theta(t)}\right) + \frac{\eta(t)}{2\exp\{c^2/4\}}\left(\frac{N_0/2}{N_0+(N-N_0)\theta(t)}\right)^2 \\
&\quad \geq \frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0+(N-N_0)\theta(t)}\right) + \frac{\eta(t)}{2\exp\{c^2/4\}}\left(\frac{N_0}{2N_0+1}\right)^2 \\
&\quad \geq \frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0+(N-N_0)\theta(t)}\right) + \frac{\eta(t)}{18\exp\{c^2/4\}}.
\end{aligned}$$

Finally, using $\log(x) \geq 1 - 1/x$,

$$\frac{1}{\eta(t)} \log\left(\frac{N_0}{N_0 + (N - N_0)\theta(t)}\right) \geq \frac{1}{\eta(t)}\left(1 - \frac{N_0 + (N - N_0)\theta(t)}{N_0}\right) \geq -\frac{N}{N_0}\frac{\theta(t)}{\eta(t)}.$$

Thus, when $t \geq t_0 + 1$ and t is odd,

$$\begin{aligned}(\text{F.7}) \quad & \frac{1}{\eta(t)} \log\left(\mathop{\mathbb{E}}_{I \sim w^{\mathrm{H}}(t)} \exp\left\{\eta(t)\left(-\ell_I(t) - \mathop{\mathbb{E}}_{I' \sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right) \\ & \geq -\frac{N}{N_0}\frac{\theta(t)}{\eta(t)} + \frac{\eta(t)}{18\exp\{c^2/4\}}.\end{aligned}$$

Otherwise, if t is even, then $I \sim w^{\mathrm{H}}(t)$ implies

$$\ell_I(t) \sim \mathrm{Ber}\left(\frac{(N_0/2)\exp\{\eta(t)/2\} + (N - N_0)\theta(t)}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right).$$

So, using $\cosh(x) \geq 1$,

$$\begin{aligned}
&\frac{1}{\eta(t)} \log\left(\mathop{\mathbb{E}}_{I \sim w^{\mathrm{H}}(t)} \exp\left\{\eta(t)\left(-\ell_I(t) - \mathop{\mathbb{E}}_{I' \sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right) \\
&= \frac{1}{\eta(t)} \log\Bigg(\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)} \\
&\qquad\qquad \times \exp\left\{\eta(t)\frac{(N_0/2)\exp\{\eta(t)/2\} + (N - N_0)\theta(t)}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right\} \\
&\qquad + \frac{(N_0/2)\exp\{\eta(t)/2\} + (N - N_0)\theta(t)}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)} \\
&\qquad\qquad \times \exp\left\{-\eta(t)\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right\}\Bigg) \\
&\geq \frac{1}{\eta(t)} \log\Bigg(\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)} \\
&\qquad\qquad \times \exp\left\{\eta(t)\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right\} \\
&\qquad + \frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)} \\
&\qquad\qquad \times \exp\left\{-\eta(t)\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right\}\Bigg) \\
&= \frac{1}{\eta(t)} \log\Bigg(\frac{N_0\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)} \\
&\qquad\qquad \times \cosh\left\{\eta(t)\frac{(N_0/2)\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right\}\Bigg) \\
&\geq \frac{1}{\eta(t)} \log\left(\frac{N_0\exp\{-\eta(t)/2\}}{N_0 \cosh\{\eta(t)/2\} + (N - N_0)\theta(t)}\right).
\end{aligned}$$

Then, using $\log(x) \geq 1 - 1/x$,

$$\begin{aligned}
&\frac{1}{\eta(t)}\log\left(\frac{N_0\exp\{-\eta(t)/2\}}{N_0\cosh\{\eta(t)/2\} + (N-N_0)\theta(t)}\right) \\
&\quad\geq \frac{1}{\eta(t)}\left(1 - \frac{N_0\cosh\{\eta(t)/2\} + (N-N_0)\theta(t)}{N_0\exp\{-\eta(t)/2\}}\right) \\
&\quad= \frac{1}{\eta(t)}\left(\frac{N_0\sinh\{-\eta(t)/2\} - (N-N_0)\theta(t)}{N_0\exp\{-\eta(t)/2\}}\right) \\
&\quad\geq -\frac{1}{\eta(t)}\frac{N}{N_0}\theta(t)\exp\{-\eta(t)/2\} \\
&\quad= -\frac{N}{N_0}\frac{\exp\{-\eta(t)(\frac{t-2}{2})\}}{\eta(t)}.
\end{aligned} \tag{F.8}$$

Combing Eqs. (F.7) and (F.8) and recognizing $\theta(t) \leq \exp\{-\eta(t)(\frac{t-2}{2})\}$ gives us

$$\begin{aligned}
&\frac{1}{\eta(t)}\log\left(\mathop{\mathbb{E}}_{I\sim w^{\mathrm{H}}(t)}\exp\left\{\eta(t)\left(-\ell_I(t) - \mathop{\mathbb{E}}_{I'\sim w^{\mathrm{H}}(t)}[-\ell_{I'}(t)]\right)\right\}\right) \\
&\quad\geq -\frac{N}{N_0}\frac{\exp\{-\eta(t)(\frac{t-2}{2})\}}{\eta(t)} + \frac{\eta(t)}{18\exp\{c^2/4\}}\mathbb{I}_{[t\text{ is odd}]}.
\end{aligned} \tag{F.9}$$

Now, we wish to sum the two terms in Eq. (F.9). First, using that $\frac{t-2}{\sqrt{t}} \geq \frac{3\sqrt{t}}{2}$ when $t \geq 6$ and $t_0 \geq 5$ since $\log N \geq 5/2$, as well as crudely lower bounding t_0 by dividing by 2,

$$\begin{aligned}
&\sum_{t=t_0+1}^{T}\frac{\exp\{-\eta(t)(\frac{t-2}{2})\}}{\eta(t)} \\
&\quad= \sum_{t=t_0+1}^{T}\frac{\sqrt{t}}{c\sqrt{\log N}}\exp\left\{-\frac{c\sqrt{\log N}}{\sqrt{t}}\left(\frac{t-2}{2}\right)\right\} \\
&\quad\leq \sum_{t=t_0+1}^{T}\frac{\sqrt{t}}{c\sqrt{\log N}}\exp\left\{-\frac{3c}{4}\sqrt{t\log N}\right\} \\
&\quad\leq \frac{1}{c\sqrt{\log N}}\int_{t_0}^{\infty}\sqrt{t}\exp\left\{-\frac{3c}{4}\sqrt{t\log N}\right\}dt \\
&\quad= \frac{128\left(t_0\frac{9c^2\log N}{16} + 2\sqrt{t_0}\frac{3c\sqrt{\log N}}{4} + 2\right)\exp\{-(3c/4)\sqrt{t_0\log N}\}}{27c^4(\log N)^2} \\
&\quad\leq \frac{128\left(\frac{16\log N}{c^2}\frac{9c^2\log N}{16} + 2\frac{4\sqrt{\log N}}{c}\frac{3c\sqrt{\log N}}{4} + 2\right)\exp\left\{-(3c/4)\frac{\sqrt{8}\log N}{c}\right\}}{c^4(\log N)^2} \\
&\quad\leq \frac{128\left(16 + \frac{9}{\log N} + \frac{2}{(\log N)^2}\right)}{c^4N^2}.
\end{aligned} \tag{F.10}$$

Then, supposing the worst case where both t_0+1 and T are even, and crudely upper bounding t_0+2 by multiplying by $3/2$,

$$\begin{aligned}\sum_{t=t_0+1}^{T}\eta(t)\mathbb{I}_{[t\text{ is odd}]} &= \sum_{t=(t_0+2)/2}^{(T-1)/2}\eta(2t)\\ &= \sum_{t=(t_0+2)/2}^{(T-1)/2}\frac{c\sqrt{\log N}}{\sqrt{2t}}\\ &\geq \int_{(t_0+2)/2}^{(T-1)/2}\frac{c\sqrt{\log N}}{\sqrt{2t}}dt\\ &= c\sqrt{\log N}[\sqrt{T-1}-\sqrt{t_0+2}]\\ &\geq c\sqrt{(T-1)\log N}-c\sqrt{\log N}\frac{\sqrt{24\log N}}{c}\\ &= c\sqrt{(T-1)\log N}-2\log N\sqrt{6}.\end{aligned} \tag{F.11}$$

Thus, combining Eqs. (F.6), (F.10) and (F.11), we have shown that for $T \geq \frac{16\log N}{c^2}$,

$$\begin{aligned}\hat{R}_{\text{H}}(T) &\geq \hat{R}_{\text{H}}(T)-\hat{R}_{\text{H}}(t_0)\\ &\geq -\frac{\sqrt{T+1}}{c\sqrt{\log N}}\Big[c^2/8+\log(N_0+1)\Big]-\frac{128\Big(16+\frac{9}{\log N}+\frac{2}{(\log N)^2}\Big)}{c^4N_0N}\\ &\qquad +\frac{c\sqrt{(T-1)\log N}}{18\exp\{c^2/4\}}-\frac{2\log N\sqrt{6}}{18\exp\{c^2/4\}}.\end{aligned}$$

Finally, rearranging the restriction on the size N_0 and using $\frac{\sqrt{T-1}}{\sqrt{T+1}}\geq 1/2$, since $\log(N_0+1) < \frac{c^2\log N}{72\exp\{c^2/4\}}-\frac{c^2}{8}$ it holds that

$$\frac{1}{c\sqrt{\log N}}\Big[c^2/8+\log(N_0+1)\Big] < \frac{1}{2}\frac{\sqrt{T-1}}{\sqrt{T+1}}\frac{c\sqrt{\log N}}{18\exp\{c^2/4\}}.$$

Thus, using $N\geq e^9$ and $N_0\geq 1$,

$$\begin{aligned}\hat{R}_{\text{H}}(T) &\geq \frac{c\sqrt{(T-1)\log N}}{36\exp\{c^2/4\}}-\frac{128\Big(16+\frac{9}{\log N}+\frac{2}{(\log N)^2}\Big)}{c^4N_0N}-\frac{2\log N\sqrt{6}}{18\exp\{c^2/4\}}\\ &\geq \frac{c\sqrt{T\log N}}{72\exp\{c^2/4\}}-\frac{1}{3c^2}-\frac{\log N}{3}.\end{aligned}$$

F.2.2. D.Hedge *with stochastically optimal parametrization.* Now, we consider the case of playing D.Hedge with the oracle-informed parameter $g(N,N_0)$. We define the data-generating mechanism $\pi\in\mathscr{P}(\mathcal{D})$ such that for some even t_1, at round t the distribution on $\mathcal{Y}$ is

$$\mu_t=\begin{cases}\delta_0^{\otimes(N_0/2)}\otimes\delta_1^{\otimes(N-N_0/2)} & : t\text{ odd and }t\leq t_1\\ \delta_1^{\otimes(N_0/2)}\otimes\delta_0^{\otimes(N_0/2)}\otimes\delta_1^{\otimes(N-N_0)} & : t\text{ even and }t\leq t_1\\ \delta_0\otimes\delta_1^{\otimes(N-1)} & : t\text{ even and }t>t_1.\end{cases}$$

That is, the data is the same as for D.HEDGE in Appendix F.2.1 up to $t = t_1$, and then afterwards all experts incur loss of 1 except the first expert, which incurs zero loss. Once again, all of these distributions are actually in U, so they are trivially in $\mathcal{D}$.

Since t_1 is even, for $t > t_1$ we expand on Eq. (F.3) to obtain

$$L_i(t) = \begin{cases} \frac{t_1}{2} & : i = 1 \\ \frac{2t-t_1}{2} & : i \in [N_0] \setminus \{1\} \\ t & : i \notin [N_0] \,. \end{cases}$$

Thus, when $t > t_1$,

$$w_1^{\mathrm{H}}(t) = \Big[1 + (N_0 - 1)\exp\{-\eta(t)(t - t_1 - 1)\} + (N - N_0)\exp\{-\eta(t)(t - t_1/2 - 1)\}\Big]^{-1},$$

and for $i \neq 1$, $w_i^{\mathrm{H}}(t)$ equals

$$\begin{cases} \Big[\exp\{-\eta(t)(t_1 - t + 1)\} + N_0 - 1 + (N - N_0)\exp\{-\eta(t)(t_1/2)\}\Big]^{-1} & : i \in [N_0] \setminus \{1\} \\ \Big[\exp\{-\eta(t)(t_1/2 - t + 1)\} + (N_0 - 1)\exp\{\eta(t)(t_1/2)\} + N - N_0\Big]^{-1} & : i \notin [N_0] \,. \end{cases}$$

The next thing to observe is that for $t > t_1$, $w_{I^*(t)}^{\mathrm{H}}(t+1)$ equals

$$\Big[1 + (N_0 - 1)\exp\{-\eta(t+1)(t - t_1)\} + (N - N_0)\exp\{-\eta(t+1)(t - t_1/2)\}\Big]^{-1}. \tag{F.12}$$

Now, define

$$t_2 = \left\lceil 4\left(\frac{\log N}{g(N, N_0)} + t_1\right)^2 \right\rceil.$$

If $t > t_2$, it holds that

$$\begin{aligned} & \sqrt{t} > \frac{2\log N}{g(N, N_0)} + 2t_1 \\ \implies\ & \sqrt{t} > \frac{2\log N_0}{g(N, N_0)} + 2t_1 \\ \implies\ & g(N, N_0)\frac{\sqrt{t}}{2} - g(N, N_0)t_1 > \log N_0 \\ \implies\ & \frac{g(N, N_0)[t - t_1]}{\sqrt{t+1}} > \log(N_0 - 1) \\ \implies\ & (N_0 - 1)\exp\{-\eta(t+1)(t - t_1)\} < 1. \end{aligned} \tag{F.13}$$

Similarly,

$$\begin{aligned} & \sqrt{t} > \frac{2\log N}{g(N, N_0)} + 2t_1 \\ \implies\ & \sqrt{t} > \frac{2\log N}{g(N, N_0)} + t_1 \\ \implies\ & \sqrt{t} - t_1 > \frac{2\log N}{g(N, N_0)} \\ \implies\ & \frac{c[t - t_1/2]}{\sqrt{t+1}} > \log(N - N_0) \\ \implies\ & (N - N_0)\exp\{-\eta(t+1)(t - t_1/2)\} < 1. \end{aligned} \tag{F.14}$$

Combining Eqs. (F.12) to (F.14) shows that when $t > t_2$, since $t_2 > t_1$ by definition, we have

$$w^{\text{H}}_{I^*(t)}(t+1) \geq 1/3.$$

This observation controls the first term of Eq. (F.2). For the second term of Eq. (F.2), we note that by Jensen's inequality,

$$\sum_{t=t_0+1}^{T} \frac{1}{\eta(t)} \log\left(\mathop{\mathbb{E}}_{I\sim w^{\text{H}}(t)} \exp\left\{ \eta(t) \left(-\ell_I(t) - \mathop{\mathbb{E}}_{I'\sim w^{\text{H}}(t)} [-\ell_{I'}(t)] \right) \right\} \right) \geq 0.$$

Define $t_0 = \left\lfloor \frac{16 \log N}{[g(N,N_0)]^2} \right\rfloor$. Now, when $t_0 + 1 \leq t \leq t_1$, $w^{\text{H}}_{I^*(t)}(t+1)$ behaves as in Eq. (F.4). Thus, when $t+1$ is odd, $w^{\text{H}}_{I^*(t)}(t+1) \leq 1/N_0$ since $\theta(t+1) \geq 0$. Otherwise, when $t+1$ is even, we use that since $\log N > 5/2$,

$$\exp\left\{ \frac{\eta(t+1)}{2} \right\} \leq \exp\left\{ \frac{[g(N,N_0)]^2}{8\sqrt{\log N}} \right\} \leq \exp\left\{ [g(N,N_0)]^2/8 \right\},$$

as well as $\theta(t+1) \geq 0$ and $\cosh(x) \geq 1$ to obtain $w^{\text{H}}_{I^*(t)}(t+1) \leq \frac{\exp\{[g(N,N_0)]^2/8\}}{N_0}$. Thus,

$$\begin{aligned} &\sum_{t=t_0+1}^{t_1} \left(\frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \log\left(\frac{1}{w^{\text{H}}_{I^*(t)}(t+1)} \right) \\ &\geq \sum_{t=t_0+1}^{t_1} \left(\frac{1}{\eta(t+1)} - \frac{1}{\eta(t)} \right) \left[\log N_0 - \frac{[g(N,N_0)]^2}{8} \right] \\ &= \frac{\log N_0 - [g(N,N_0)]^2/8}{g(N,N_0)} [\sqrt{t_1+1} - \sqrt{t_0+1}]. \end{aligned} \tag{F.15}$$

Set $t_1 = T/2$, and suppose $T > \frac{32 \log N}{[g(N,N_0)]^2}$ to ensure $t_1 > t_0 + 1$. Then, substituting Eq. (F.15) into Eq. (F.2) gives

$$\begin{aligned} \hat{R}_{\text{H}}(T) &\geq \hat{R}_{\text{H}}(T) - \hat{R}_{\text{H}}(t_0) \\ &\geq -\frac{\sqrt{T+1}}{g(N,N_0)} \log(3) + \frac{\log N_0 - [g(N,N_0)]^2/8}{g(N,N_0)} [\sqrt{t_1+1} - \sqrt{t_0+1}] \\ &\geq -\frac{\sqrt{T+1}}{g(N,N_0)} \log(3) + \frac{\log N_0 - [g(N,N_0)]^2/8}{2g(N,N_0)} \left[\sqrt{T+1} - \frac{\sqrt{32 \log N}}{g(N,N_0)} \right] \\ &\geq \frac{\log N_0}{4g(N,N_0)} \sqrt{T+1} - \frac{3[\log N_0 - [g(N,N_0)]^2/8] \log N}{[g(N,N_0)]^2} \\ &\geq \frac{\log N_0}{4g(N,N_0)} \sqrt{T} - \frac{3 \log N_0}{[g(N,N_0)]^2}, \end{aligned}$$

where we also used $\log N_0 > [g(N,N_0)]^2/4 + 4\log(3)$.

□

G: IMPLEMENTING FTRL-CARE AND META-CARE

The following algorithm efficiently implements FTRL-CARE; its validity follows from Theorem A.1.

Algorithm 1: Implementation of FTRL-CARE

Inputs:

- constants $c_1, c_2 > 0$, number of experts N;
- a function $\texttt{root} : (f, (a, b)) \in (\mathbb{R} \to \mathbb{R}) \times \mathbb{R}^2 \to \texttt{maybe}(\mathbb{R})$ which returns a root of the function f on the interval $[a, b]$ when $f(a)f(b) < 0$, and returns $\texttt{nothing}$ otherwise.

Result: Infinite list of weight vectors, $\{w^{\mathrm{C}}\}_{t\in\mathbb{N}}$

$H = \texttt{Function}\Big(u \mapsto \Big\{\sum_{i\in[N]}[-u_i \log(u_i)]\Big\}\Big)$;

$w = \texttt{Function}\Bigg((\eta, \xi) \mapsto \Bigg\{\Bigg(\frac{\exp(-\eta\xi_i)}{\sum_{i'\in[N]}\exp(-\eta\xi_{i'})}\Bigg)_{i\in[N]}\Bigg\}\Bigg)$;

$L(0) = \texttt{zeroes}(N)$;

$w^{\mathrm{C}}(1) = \texttt{ones}(N)/N$;

for $t \in \mathbb{N}$ **do**

  Receive Data: vector of expert losses from round t, $\ell(t) \in [0, 1]^{[N]}$

  $L(t) = L(t-1) + \ell(t)$;

  $\eta(t+1) = \texttt{root}\Bigg(\texttt{Function}\Big(\eta \mapsto \Big\{\eta - \frac{2c_1\sqrt{c_2 + H(w(\eta, L(t)))}}{\sqrt{t+1}}\Big\}\Big), \Big(\frac{2c_1\sqrt{c_2}}{\sqrt{t+1}}, \frac{2c_1\sqrt{c_2+\log(N)}}{\sqrt{t+1}}\Big)\Bigg)$;

  $w^{\mathrm{C}}(t+1) = w(\eta(t+1), L(t))$

end

Meta-CARE only requires the above implementation of FTRL-CARE and a standard implementation of D.Hedge. The parameters of Meta-CARE can be tuned to optimize the $N_0 = 1$ bound of Theorem 5 and the leading term of Theorem 6, hence improving the universal constants, but it does not affect the order of the bound.

H: SIMULATIONS

In this section, we present a brief simulation analysis of the performance of D.Hedge, FTRL-CARE, and Meta-CARE to provide intuition for how the algorithms differ and to demonstrate the effectiveness of Meta-CARE that we have proved in our analysis. Since the weights of all three algorithms can be completely determined by the expert losses, we specify each scenario using only the loss distributions rather than the distributions on $\mathcal{Y}$ and $\hat{\mathcal{Y}}^N$. In Fig. 1, we plot the expected regret against the number of rounds T for two data-generating mechanisms: the left column ($N_0 = 1$) corresponds to the stochastic setting, where the losses of the first expert are i.i.d. Ber(1/2) and the losses of all the other experts are i.i.d. Ber(1); the right column ($N_0 = 2$) corresponds to an adversarial setting with two effective experts, where on the tth round the loss of the first expert is deterministically t mod 2, the loss of the second expert is deterministally $(t+1)$ mod 2, and the losses of the remaining experts are all deterministically 1. In Fig. 2, we plot the expected regret against the number of experts N for various T. The data-generating mechanism has $N_0 = 2$, and is the same as for the right column of Fig. 1. For both settings, the gap between the expected losses of the best effective and ineffective experts under distributions in the convex hull of those produced by the data-generating mechanism is $\Delta_0 = 1/2$. For all of the simulations, the algorithms are parametrized using $c_{\mathrm{H}} = c_{\mathrm{C},1} = \sqrt{8}$, $c_{\mathrm{C},2} = 1$, and $c_{\mathrm{M}} = 100$. All of the plots display expected regret; for the $N_0 = 1$ case of Fig. 1 this is approximated by averaging over 10 simulations, and for the remaining plots this is exact since the losses are all deterministic. Code to replicate these experiments is available online at `https://github.com/jnegrea/semi-adversarial-public`.

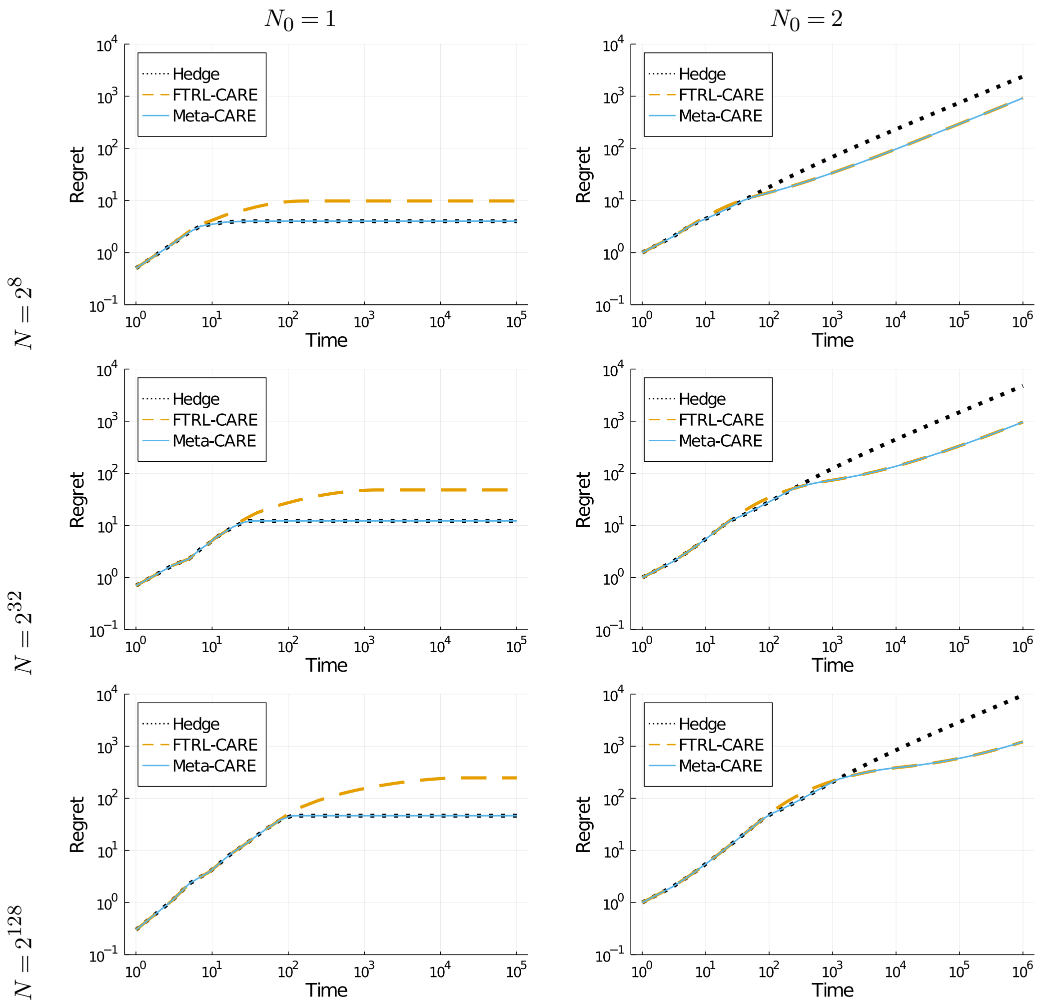

Fig 1: Comparing expected regret as a function of time T, for number of effective experts $N_0 \in \{1,2\}$ and varying total number of experts N. Plots are on a log-log scale; slopes of lines correspond to polynomial powers, and intercepts of lines correspond to log-(constants of proportionality).

Beginning with Fig. 1, for $N_0 = 1$, expected regret levels-off at a higher constant for FTRL-CARE than for D.Hedge. As anticipated by the theory, the period for which adversarial regret is accumulated before the regret levels off increases with N for both D.Hedge and FTRL-CARE, and is longer for FTRL-CARE, leading to higher total expected regret. For $N_0 = 2$, the gap between the expected regret of FTRL-CARE and D.Hedge widens as N increases, corresponding to the $\sqrt{T \log N}$ rate of regret for D.Hedge v.s. the $\sqrt{T \log N_0}$ rate of regret for FTRL-CARE. As anticipated by our theoretical results, there is a phase transition in the regret accumulation for both FTRL-CARE and D.Hedge at roughly the time when the respective expected regrets level off in the $N_0 = 1$ case. In all cases, the expected regret of Meta-CARE closely tracks the better of D.Hedge and FTRL-CARE.

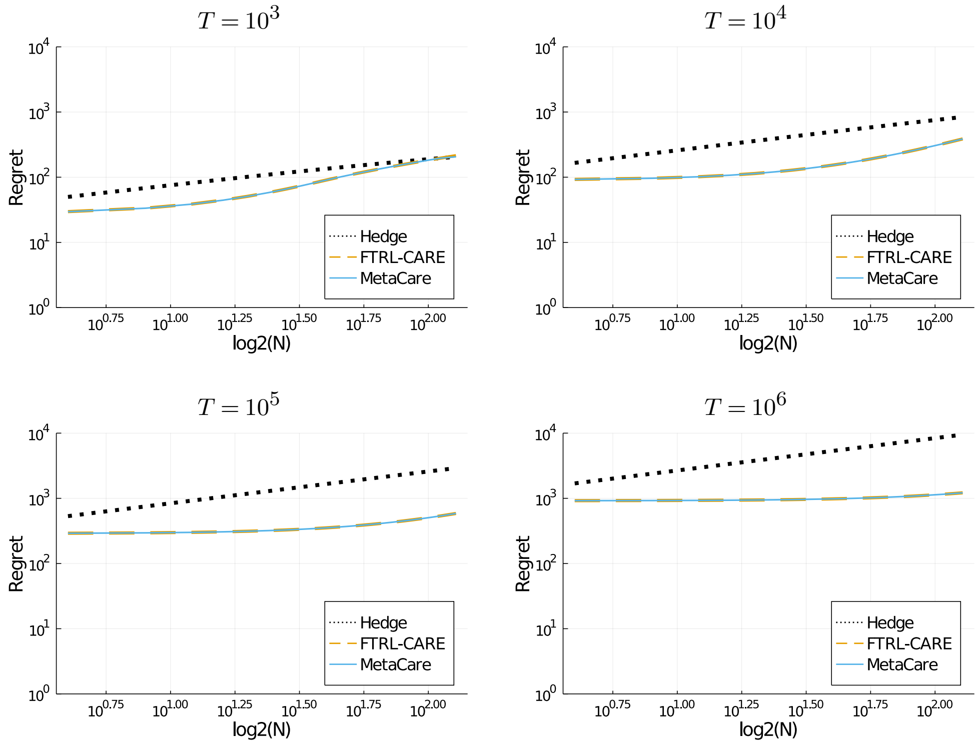

Fig 2: Comparing expected regret as a function of the number of experts N for $N_0 = 2$ effective experts at varying times T. Plots are on a log-log scale; slopes of lines correspond to polynomial powers, and intercepts of lines correspond to log-(constants of proportionality). Note that since the x-axis variable in each case is $\log_2(N)$, the second tick on the x-axis corresponds to $N = 2^{10^{1.00}} = 1024$, and the last tick on the x-axis corresponds to $N = 2^{10^{2.00}} \approx 1.27 \times 10^{30}$.

For Fig. 2, when T is small relative to $\log N$, both FTRL-CARE and D.Hedge have expected regret growing with N according to the adversarial rate, corresponding to a slope of 1/2. When T is large relative to N, so that $\sqrt{T \log N_0} \gg (\log N)^{3/2}/\Delta_0$, the expected regret of FTRL-CARE is approximately constant in N while the expected regret of D.Hedge grows like $\sqrt{\log N}$, as anticipated by our theoretical results. Once again, the expected regret of Meta-CARE closely tracks the better of D.Hedge and FTRL-CARE.

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