Supplementary Material for "Bayesian Inference and Testing of Group Differences in Brain Networks"

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S1. Proofs of Propositions

The supplementary materials contain proofs of Propositions 1, 2 and 3, providing theoretical support for the methodology developed in the article "Bayesian Inference and Testing of Group Differences in Brain Networks".

Proof. **Proposition 1.** Recalling Lemma 2.1 in Durante et al. (2016) we can always represent the conditional probability $p_{\mathcal{L}(\boldsymbol{\mathcal{A}})|y}(\boldsymbol{a})$ separately for each group $y \in \{1,2\}$ as

$$p_{\mathcal{L}(\boldsymbol{\mathcal{A}})|y}(\boldsymbol{a}) = \sum_{h_y=1}^{H_y} \nu_{h_y} \prod_{l=1}^{V(V-1)/2} (\pi_l^{(h_y)})^{a_l}(1-\pi_l^{(h_y)})^{1-a_l}, \quad \boldsymbol{a} \in \mathbb{A}_V,$$

with each $\pi_l^{(h_y)}$ defined as $\text{logit}(\pi_l^{(h_y)}) = Z_l^{(y)} + \sum_{r_y=1}^{R_y} \lambda_{r_y}^{(h_y)} X_{vr_y}^{(h_y)} X_{ur_y}^{(h_y)}$, $l = 1, \dots, V(V-1)/2$ and $h_y = 1, \dots, H_y$. Hence Proposition 1 follows after choosing $\boldsymbol{\pi}^{(h)}, h = 1, \dots, H$ as the sequence of unique component-specific edge probability vectors $\boldsymbol{\pi}^{(h_y)}$ appearing in the above separate factorizations for at least one group y, and letting the group-specific mixing probabilities in (8) be $\nu_{hy} = \nu_{h_y}$ if $\boldsymbol{\pi}^{(h)} = \boldsymbol{\pi}^{(h_y)}$ and $\nu_{hy} = 0$ otherwise. □

Proof. **Proposition 2.** Recalling factorization (8), and letting $\mathbb{A}_V^{-l}$ denote the set containing all the possible network configurations for the node pairs except the lth one, we have that $p_{\mathcal{L}(\mathcal{A})_l|y}(1)$ is equal to

$$\sum_{\mathbb{A}_V^{-l}} \sum_{h=1}^{H} \nu_{hy}\pi_l^{(h)} \prod_{l^* \neq l} (\pi_{l^*}^{(h)})^{a_{l^*}}(1-\pi_{l^*}^{(h)})^{1-a_{l^*}} = \sum_{h=1}^{H} \nu_{hy}\pi_l^{(h)} \sum_{\mathbb{A}_V^{-l}} \prod_{l^* \neq l} (\pi_{l^*}^{(h)})^{a_{l^*}}(1-\pi_{l^*}^{(h)})^{1-a_{l^*}}$$

Then, Proposition 2 follows after noticing that $\prod_{l^* \neq l}(\pi_{l^*}^{(h)})^{a_{l^*}}(1-\pi_{l^*}^{(h)})^{1-a_{l^*}}$ is the joint pmf of independent Bernoulli random variables, and hence the summation over the joint sample space $\mathbb{A}_V^{-l} = \{0,1\}^{V(V-1)/2-1}$, provides $\sum_{\mathbb{A}_V^{-l}} \prod_{l^* \neq l}(\pi_{l^*}^{(h)})^{a_{l^*}}(1-\pi_{l^*}^{(h)})^{1-a_{l^*}} = 1$.

The proof of $p_{\mathcal{L}(\mathcal{A})_l}(1) = \sum_{y=1}^{2} p_{\mathcal{Y}}(y) \sum_{h=1}^{H} \nu_{hy}\pi_l^{(h)}$ follows directly from the above results after noticing that $p_{\mathcal{L}(\mathcal{A})_l}(1) = \sum_{y=1}^{2} p_{\mathcal{Y},\mathcal{L}(\mathcal{A})_l}(y,1) = \sum_{y=1}^{2} p_{\mathcal{Y}}(y) p_{\mathcal{L}(\mathcal{A})_l|y}(1)$. □

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Proof. **Proposition 3.** Recalling the proof of Proposition 1 and factorization (5), we can always represent $\sum_{y=1}^{2}\sum_{\boldsymbol{a}\in\mathbb{A}_V}|p_{\mathcal{Y},\mathcal{L}(\boldsymbol{\mathcal{A}})}(y,\boldsymbol{a})-p^0_{\mathcal{Y},\mathcal{L}(\boldsymbol{\mathcal{A}})}(y,\boldsymbol{a})|$ as

$$\sum_{y=1}^{2}\sum_{\boldsymbol{a}\in\mathbb{A}_V}|p_{\mathcal{Y}}(y)\sum_{h=1}^{H}\nu_{hy}\prod_{l=1}^{V(V-1)/2}(\pi_l^{(h)})^{a_l}(1-\pi_l^{(h)})^{1-a_l}$$
$$-p^0_{\mathcal{Y}}(y)\sum_{h=1}^{H}\nu^0_{hy}\prod_{l=1}^{V(V-1)/2}(\pi_l^{0(h)})^{a_l}(1-\pi_l^{0(h)})^{1-a_l}|,$$

with $\nu^0_{hy}=\nu^0_{h_y}$ if $\boldsymbol{\pi}^{0(h)}=\boldsymbol{\pi}^{0(h_y)}$ and $\nu^0_{hy}=0$ otherwise. Hence $\Pi\{\mathbb{B}_\epsilon(p^0_{\mathcal{Y},\mathcal{L}(\boldsymbol{\mathcal{A}})})\}$ is

$$\int 1(\sum_{y=1}^{2}\sum_{\boldsymbol{a}\in\mathbb{A}_V}|p_{\mathcal{Y},\mathcal{L}(\boldsymbol{\mathcal{A}})}(y,\boldsymbol{a})-p^0_{\mathcal{Y},\mathcal{L}(\boldsymbol{\mathcal{A}})}(y,\boldsymbol{a})|<\epsilon)d\Pi_y(p_{\mathcal{Y}})d\Pi_\pi(\boldsymbol{\pi}^{(1)},\ldots,\boldsymbol{\pi}^{(H)})d\Pi_\nu(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2).$$

Recalling results in Dunson and Xing (2009), a sufficient condition for the above integral to be strictly positive is that $\Pi_y\{p_{\mathcal{Y}}:\sum_{y=1}^{2}|p_{\mathcal{Y}}(y)-p^0_{\mathcal{Y}}(y)|<\epsilon_y\}>0$, $\Pi_\pi\{\boldsymbol{\pi}^{(1)},\ldots,\boldsymbol{\pi}^{(H)}:\sum_{h=1}^{H}\sum_{l=1}^{V(V-1)/2}|\pi_l^{(h)}-\pi_l^{0(h)}|<\epsilon_\pi\}>0$ and $\Pi_\nu\{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2:\sum_{y=1}^{2}\sum_{h=1}^{H}|\nu_{hy}-\nu^0_{hy}|<\epsilon_\nu\}>0$, for every $\epsilon_y>0$, $\epsilon_\pi>0$ and $\epsilon_\nu>0$. The large support for $p_{\mathcal{Y}}$ is directly guaranteed from the Beta prior. Similarly, according to Theorem 3.1 and Lemma 3.2 in Durante et al. (2016), the same hold for the joint prior over the sequence of component-specific edge probability vectors $\boldsymbol{\pi}^{(h)}, h=1,\ldots,H$ induced by priors Π_Z, Π_X and Π_λ in factorization (9). Finally, marginalizing out the testing indicator T, and recalling our prior specification for the mixing probabilities in (12), a lower bound for $\Pi_\nu\{\boldsymbol{\nu}_1,\boldsymbol{\nu}_2:\sum_{y=1}^{2}\sum_{h=1}^{H}|\nu_{hy}-\nu^0_{hy}|<\epsilon_\nu\}$ is

$$\mathrm{pr}(H_0)\Pi_\upsilon\{\boldsymbol{\upsilon}:\sum_{y=1}^{2}\sum_{h=1}^{H}|\upsilon_h-\nu^0_{hy}|<\epsilon_\nu\}+\mathrm{pr}(H_1)\prod_{y=1}^{2}\Pi_{\upsilon_y}\{\boldsymbol{\upsilon}_y:\sum_{h=1}^{H}|\upsilon_{hy}-\nu^0_{hy}|<\epsilon_\nu/2\}.$$

If the true model is generated under independence, the above equation reduces to

$$\mathrm{pr}(H_0)\Pi_\upsilon\{\boldsymbol{\upsilon}:\sum_{h=1}^{H}|\upsilon_h-\nu^0_{h}|<\epsilon_\nu/2\}+\mathrm{pr}(H_1)\prod_{y=1}^{2}\Pi_{\upsilon_y}\{\boldsymbol{\upsilon}_y:\sum_{h=1}^{H}|\upsilon_{hy}-\nu^0_{h}|<\epsilon_\nu/2\},$$

with the Dirichlet priors for $\boldsymbol{\upsilon}$, $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$ ensuring the positivity of both terms. When instead $\nu^0_{h1}\neq\nu^0_{h2}$ for some $h=1,\ldots,H$, the inequality $\mathrm{pr}(H_0)\Pi_\upsilon\{\boldsymbol{\upsilon}:\sum_{y=1}^{2}\sum_{h=1}^{H}|\upsilon_h-\nu^0_{hy}|<\epsilon_\nu\}>0$ is not guaranteed, but $\mathrm{pr}(H_1)\prod_{y=1}^{2}\Pi_{\upsilon_y}\{\boldsymbol{\upsilon}_y:\sum_{h=1}^{H}|\upsilon_{hy}-\nu^0_{hy}|<\epsilon_\nu/2\}$ remains strictly positive for every ϵ_ν under the independent Dirichlet priors for the quantities $\boldsymbol{\upsilon}_1$ and $\boldsymbol{\upsilon}_2$, proving the Proposition. □

References

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